Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 28 (2025), No. 1, pp. 205–216
DOI : 10.47974/JDMSC-2138

Coincidence of modified Pell numbers and sums of two Jacobsthal numbers

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Abstract

Let $\{q_r\}_{r \geq 0}$ and $\{J_r\}_{r \geq 0}$ denote Modified Pell and Jacobsthal numbers sequences, respectively. We solve the equation $q_r = J_s + J_t$ using Matveev's fundamental variant of Baker's linear forms theory. Then a reduction procedure of Dujella and Pethö is employed to bring the obtained bounds to a relatively more investigatable domain.

Subject Classification (2020): *11B39, 11D72, 11J70.*

Keywords: *Modified pell sequence, Jacobsthal sequence, Linear forms in logarithms, Reduction method.*

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1. Introduction

The Modified Pell sequence is defined recursively by $q_n = 2q_{n-1} + q_{n-2}$ for $n \geq 2$ with initials $q_0 = q_1 = 1$. The first terms of this sequence are

$$1, 1, 3, 7, 17, 41, 99, 239, 577, ...$$

Diophantine equations that include this sequence and associated sequences attracted a lot of attention. See [1], [3], [6], [12], [14] and [18] for some recent works.

Jacobsthal sequence is defined by $J_{n+1} = J_n + 2J_{n-1}$ for $n \geq 1$ with $J_0 = 0,\ J_1 = 1$. This generates the sequence

$$0, 1, 1, 3, 5, 11, 21, 43, 85, 171, ...$$

Many applications of these important sequences can be found in [15] and [16].

The problem of finding common terms between integer sequences is a very active direction of Diophantine equations. See [2] and [11]. It turns out , with a cursory glance, that the two sequences agree in the first three terms. This motivates the question of determining the common terms between the two sequences. We consider the more general problem of solving the Diophantine equation:

$$q_r = J_s + J_t \tag{1.1}$$

A remarkable direct consequence of the solution will confirm that there are no other intersections exist between these sequences.

The approach is as follows: The implied variables are bounded in terms of a single one via a deep result of Matveev which is a variant of the Baker's theory . The obtained bound is so large and so it is difficult to probe all the possible cases. As a result, we try to obtain certain linear forms so that the reduction lemma of Dujella and Pethö can ba utilized . Lastly, Sage computations are used under the obtained bounds to complete the solution of the equation.

2. Number theoretic backgrounds

2.1 *Modified Pell sequence*

The Pell's Modified sequences is characterised by the identity

$$\Psi(\theta) := \theta^2 - 2\theta - 1 = 0,$$

If $\sigma = 1 + \sqrt{2}$ and $\rho = 1 - \sqrt{2},$ then its Binet form is

$$q_r = \frac{\sigma^r + \rho^r}{\sigma + \rho},\ r \geq 0. \tag{2.1}$$

It is straightforward to show that

$$\sigma^{r-2} < q_r < \sigma^{r+1},\ r \geq 0. \tag{2.2}$$

For more details about modified pell numbers and related sequences we refer to [7], [8], [15] and [19].

2.2 *Jacobsthal sequence*

The Jacobsthal sequence is characterised by the equation

$$G(\theta) = \theta^2 - \theta - 2.$$

The Binet equation is

$$J_r = \frac{2^r - (-1)^r}{3},\ r \geq 1. \tag{2.3}$$

Inductively,

$$2^{r-2} \leq J_r \leq 2^{r-1},\ r \geq 1. \tag{2.4}$$

We refer to [13] and [16] for basic properties.

2.3 *Linear forms in logarithms*

Over $\mathbb{Z}$, let

$$c_0 x^m + c_1 x^{m-1} + \ldots + c_m = c_0 \prod_{i=1}^{m} (x - \kappa^{(i)})$$

be the minimal polynomial of an algebraic number κ with $c_0 \in \mathbb{Z}^+$ and $\kappa^{(i)}$'s are the conjugates of κ. Weil's logarithmic height of κ is

$$\mathfrak{h}(\alpha) := \frac{1}{m}\left(\log c_0 + \sum_{i=1}^{m} \log(\max\{ |\kappa^{(i)}|, 1\}) \right)$$

The function $\mathfrak{h}$ of logarithmic height satisfies the following properties (see [4] for proofs):

$$\mathfrak{h}(\kappa_1 \pm \kappa_2) \leq \mathfrak{h}(\kappa_1) + \mathfrak{h}(\kappa_2) + \log 2;$$
$$\mathfrak{h}(\kappa_1 \kappa_2^{\pm 1}) \leq \mathfrak{h}(\kappa_1) + \mathfrak{h}(\kappa_1);$$
$$\mathfrak{h}(\kappa^s) = |s|\, \mathfrak{h}(\kappa) (s \in \mathbb{Z}).$$

3. Bounding the variables

The purpose of this section is to bound the variables *r*, *s* and *t*. Since Eq.(1.1) is symmetric, we assume that $s \geq t$ throughout the paper. Hence it suffices to bound *s*.

Lemma 3.1: *Let* $s \geq 2$. *Then* $|3\sigma^r 2^{-(s+1)} - 1| < 4.2^{t-s}$.

Proof: By the inequalities (2.2) and (2.4), we have

$$\sigma^{r-2} \leq q_r \leq 2^s \text{ and } 2^{s-3} \leq q_r \leq \sigma^{r+1} \tag{3.1}$$

for $s \geq 1$ and $r \geq 0$. We conclude that

$$(s-3)\frac{\log 2}{\log \sigma} - 1 \leq k \leq \frac{\log 2}{\log \sigma} s + 2. \tag{3.2}$$

For $s \geq 2$, one can assume $k < 2s$. Replacing Jacobsthal and Modified Pell sequences in Eq.(1.1) by Binet forms, one obtain

$$\frac{\sigma^r + \rho^r}{2} = \frac{2^s - (-1)^s}{3} + \frac{2^t - (-1)^t}{3}. \tag{3.3}$$

Therefore

$$\left| \frac{\sigma^r}{2} - \frac{2^s}{3} \right| = \left| \frac{2^t}{3} - \frac{((-1)^s + (-1)^t)}{3} - \frac{\rho^r}{2} \right|. \tag{3.4}$$

Easily, it follows that

$$\left| \frac{\sigma^r}{2} - \frac{2^s}{3} \right| \leq \frac{2^t}{3} + \frac{2}{3} + |\rho^r|.$$

Then

$$\left| \frac{\sigma^r}{2} - \frac{2^s}{3} \right| < \frac{4 \cdot 2^t}{3} \tag{3.5}$$

So

$$|3\sigma^r 2^{-(s+1)} - 1| < 4.2^{t-s}. \tag{3.6}$$

Bugeaud, Mignotte and Siksek used Matveev's theorem [17] to deduce the following fundamental result, see [5].

Theorem 3.2: *Suppose* $\mathbb{A}$ *is a real algebraic number field and* $\kappa_1, \ldots, \kappa_m \in \mathbb{A}$ *are positive. Let* $t_1, \ldots, t_m \in \mathbb{Z}$ *are non zero and satisfy*

$$\Omega := \kappa_1^{t_1}\kappa_2^{t_2}...\kappa_m^{t_m} - 1 \neq 0.$$

If the degree of $\mathbb{A} = d_{\mathbb{A}}$, let

$$H_i \geq \max\{d_{\mathbb{A}}\mathfrak{h}(\kappa_i), |\log \kappa_i|, 0.16\},\ i = 1,...,m.$$

If

$$\beta \geq \max\{|t_1|,...,|t_m|\},$$

then

$$\log|\Omega| > -1.4 \cdot 30^{m+3} \cdot m^{4.5} \cdot d_{\mathbb{A}}^2 \cdot (1+\log d_{\mathbb{A}}) \cdot (1+\log\beta) H_1...H_m. \tag{3.7}$$

Lemma 3.3: *For* $s \geq 2$, *we have* $\left|\frac{3\sigma^r 2^{-(s+1)}}{1+2^{t-s}} - 1\right| < \frac{5}{2^t}$.

Proof: Let $\Omega_1 = \frac{3\sigma^r 2^{-(s+1)}}{1+2^{t-s}} - 1$. If $\Omega_1 = 0$, then $3\sigma^r = 2^{s+1} + 2^{t+1}$. Consider the automorphism ζ such that $\zeta(\sigma) = \rho$. Then $|3\rho^r| = 2^{s+1} + 2^{t+1}$. But $|3\rho^r| < 3$, then $2^{s+1} + 2^{t+1} < 3$. This is impossible. Thus, $\Omega_1 \neq 0$. Taking $\mathbb{A} = \mathbb{Q}(\sigma)$, we get $d_{\mathbb{A}} = 2$. Let

$$m = 3,\ \kappa_1 = 3,\ \kappa_2 = \sigma,\ \kappa_3 = 2,\ t_1 = 1,\ t_2 = r,\ t_3 = -(s+1).$$

Then

$$\mathfrak{h}(\kappa_1) = \log 3;$$

$$\mathfrak{h}(\kappa_2) = \frac{1}{2}\log\sigma;$$

$$\mathfrak{h}(\kappa_3) = \log 2.$$

Let

$$H_1 = 2\log 3,\ H_2 = \log\sigma, \text{ and } H_3 = 2\log 2.$$

Take $\beta = 2s$. By Theorem (3.2)

$$\log|\Omega_1| > -c_1(1+\log 2s),$$

where $c_1 = 1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1+\log 2)(2\log 3)(2\log 2\log\sigma)$. Then

$$\log|\Omega_1| > -3 \times 10^{12}(1+\log 2s). \tag{3.8}$$

Eq. (3.6) gives

$$\log|\Omega_1| < \log 4 + (t-s)\log 2. \tag{3.9}$$

A combination of inequalities (3.8) and (3.9) yields

$$3\times 10^{12}(1+\log 2s) > (s-t)\log 2 - \log 6. \tag{3.10}$$

Thus

$$s\log 2 - 3\times 10^{12}(1+\log 2s) - \log 4 < t\log 2. \tag{3.11}$$

Eq.(3.3) is the same as

$$\frac{\gamma^k}{2} - \frac{2^s(1+2^{t-s})}{3} = -\frac{\rho^k}{2} - \frac{(-1)^s-(-1)^t}{3}. \tag{3.12}$$

So

$$\left|\frac{3\sigma^r 2^{-(s+1)}}{1+2^{t-s}} - 1\right| = \left|\frac{3\cdot 2^{-s}}{1+2^{t-s}}\left(-\frac{\rho^r}{2} - \frac{(-1)^s-(-1)^t}{3}\right)\right|. \tag{3.13}$$

Then

$$\left|\frac{3\sigma^r 2^{-(s+1)}}{1+2^{t-s}} - 1\right| < \frac{5}{2^t}. \tag{3.14}$$

The next result contains an explicit upper bound of s.

Theorem 3.4: *Let* $s \geq 2$. *If s satisfies Eq.(1.1), then* $s < 3\times 10^{28}$.

Proof: Let $\Omega_2 = \frac{3}{1+2^{t-s}}\sigma^r 2^{-(s+1)} - 1$. As before we can show that $\Omega_2 \neq 0$. Using the inequality (3.14), we deduce that

$$\log|\Omega_4| < \log 5 - t\log 2. \tag{3.15}$$

Set

$$\kappa_1 = \frac{3}{1+2^{t-s}},\ \kappa_2 = \sigma,\ \kappa_3 = 2,\ m = 3,\ t_1 = 1,\ t_2 = r,\ t_3 = -(s+1).$$

Setting $\mathbb{A} = \mathbb{Q}(\sigma)$, we find that

$$\mathfrak{h}(\kappa_1) \leq \log 3 + (s-t+1)\log 2;$$

$$\mathfrak{h}(\kappa_2) = \frac{1}{2}\log\sigma;$$

$$\mathfrak{h}(\kappa_3) = \log 2.$$

Take

$$H_1 = 2\log 3 + 2(s-t+1)\log 2,\ H_2 = \log\sigma, \text{ and } H_3 = 2\log 2.$$

Then, for $\beta = 2$ we have

$$\log|\Omega_2| > -c_2(2\log 3 + 2(s-t+1)\log 2)(1+\log 2s),$$

where $c_2 = 1.4\times 30^6 \times 3^{4.5}\times 4\times(1+\log 2)(2\log 2\log\sigma)$.

Eqs.(3.10), (3.11) and (3.15) with routine manipulations give

$$s\log 2 < 1.1\times 10^{13}(1+\log 2s) + 1.2\times 10^{25}(1+\log 2s)^2 + 3. \tag{3.16}$$

Hence,

$$s < 3\times 10^{28}. \tag{3.17}$$

4. Reducing the upper bound

Set $|||\lambda|| := min\{|\lambda - n| : n\in\mathbb{Z}\}$ for any real number λ. The next crucial lemma is proved by Dujella and Pethö in [10] .

Lemma 4.1: *Let* $A,\theta,\lambda > 0, B > 1$ *belong to* $\mathbb{R}$. *suppose that* $\frac{p}{q}$ *is a convergent of* λ *with* $q > 6K$ *where Let K be a positive integer. If* $s,t,\omega > 0$ *and* $\varepsilon := ||\theta q|| - K||\lambda q|| > 0$ *satisfy*

$$0 < |s\lambda - t + \theta| < \frac{A}{B^{\omega}}$$

with $s \le K$, then

$$\omega < \frac{\log\left(\frac{Aq}{\varepsilon}\right)}{\log B}.$$

Lemma 4.2: *Let s and t satisfy Eq.(1.1). If* $s-t \ge 5$, *then* $s-t < 117$.

Proof: Let

$$\Delta_1 = \log(3) + r\log\sigma - (s+1)\log 2.$$

By Eq.(3.6), we have

$$|\Omega_1| = |e^{\Delta_1} - 1| < \frac{4}{2^{s-t}} < \frac{1}{4}, \tag{4.1}$$

So

$$\left|\Delta_1\right| < \frac{1}{2}. \tag{4.2}$$

Then $|\Delta_1| < 2|e^{\Delta_1} - 1|$. So

$$\left|\Delta_1\right| < \frac{8}{2^{s-t}}. \tag{4.3}$$

We observe that $\Delta_1 \neq 0$. It follows that

$$0 < \left|\frac{\log 3}{\log 2} - s + r\left(\frac{\log \sigma}{\log 2}\right) - 1\right| < \frac{8}{2^{s-t}\log 2} < \frac{12}{2^{s-t}}. \tag{4.4}$$

Note that $\left|\frac{\log 3}{\log 2} - s + r\left(\frac{\log \sigma}{\log 2}\right) - 1\right| > \left|\frac{\log 3}{\log 2} - s + r\left(\frac{\log \sigma}{\log 2}\right)\right|$. Moreover, if $\frac{\log 3}{\log 2} - s + r\left(\frac{\log \sigma}{\log 2}\right) = 0$, we deduce from Eq.(4.4) that $1 < \frac{12}{2^{s-t}}$. This is a contradiction since $s - t \geq 5$. Hence,

$$0 < \left|\frac{\log 3}{\log 2} - s + r\left(\frac{\log \sigma}{\log 2}\right)\right| < \frac{12}{2^{s-t}}. \tag{4.5}$$

Let $K = 6 \times 10^{28}$ $(K > 2s > r)$, $\lambda = \frac{\log \sigma}{\log 2}$, $\theta = \frac{\log 3}{\log 2}$, $A = 12$, $B = 2$. Expanding λ we find that $q_{65} > 6K$. Moreover,

$$\varepsilon = \| \theta q_{65} \| - K \| \lambda q_{65} \| > 0.3.$$

Lemma (4.1) entails that $s - t < 117$.

We will use the following result of Legendre, see [9] for details.

Theorem 4.3: *Let* $r \in \mathbb{R}$ *with* $r = [a_0, a_1, \ldots]$ *and let* $p, q \in \mathbb{Z}$. *If*

$$\left|\frac{p}{q} - r\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ *is one of the convergents of the continued fraction of r. Assume that C, h are non negative integers with* $q_h > C$. *Setting* $b = \max_{0 \leq i \leq t}\{a_i\}$, *then*

$$\frac{1}{(b+2)q^2} < \left|\frac{p}{q} - r\right|.$$

Lemma 4.4: *Assume the same conditions as above and let* $t > 5$. *If* $s - t \neq 1$ *and* $s - t < 117$, *then* $t < 122$. *And if* $s - t = 1$ *and* $s - t < 117$, *then* $t \leq 104$.

Proof: Set

$$\Delta_2 = \log\left(\frac{3}{1 + 2^{t-s}}\right) + r\log\sigma - (s+1)\log 2.$$

Eq.(3.14) entails that

$$|\Omega_2| = |e^{\Delta_2} - 1| < \frac{5}{2^t} < \frac{1}{4}. \tag{4.6}$$

We conclude that

$$\left|\Delta_2\right| < \frac{1}{2}. \tag{4.7}$$

Then $|\Delta_2| < 2|e^{\Delta_2} - 1|$. Consequently

$$|\Delta_2| < \frac{10}{2^t}. \tag{4.8}$$

We can show that $\Delta_2 \neq 0$ as before. So

$$0 < \left| \frac{\log\left(\frac{3}{1+2^{t-s}}\right)}{\log 2} - s + r\left(\frac{\log\sigma}{\log 2}\right) \right| < \frac{15}{2^t}. \tag{4.9}$$

Let $K = 6\times 10^{28}$, $\lambda = \frac{\log\sigma}{\log 2}$, $\theta = \frac{\log\left(\frac{3}{1+2^{t-s}}\right)}{\log 2}$, $A = 15$, $B = 2$. Then $q_{65} > 6K$. We investigate ε as follows:

Case I: If $s - t < 117$ with $s - t \neq 1$, one obtain

$$\varepsilon = \|\theta q_{65}\| - K\|\lambda q_{65}\| > 0.01.$$

Then $t < 122$. Hence, $s < 239$ and $r < 478$.

Case II: If $s - t = 1$, then $\varepsilon < 0$. Solving Eq.(1.1) for $s - t = 1$, Eq.(1.1) is the same as

$$q_r = 2^t. \tag{4.10}$$

Since $q_r > \sigma^{r-2}$, then we can assume that $r < 2t$ for $t \geq 3$. From Eq. (3.17), $t < 3\times 10^{28}$. Similar to the previous case, one can show that

$$|\sigma^r 2^{-t} - 1| < \frac{1}{2^t}.$$

For $t \geq 3$, we get

$$\left| r\,\frac{\log\sigma}{\log 2} - t \right| < \frac{4}{2^t} < \frac{1}{4}.$$

As $16t < 2^t$ for $t \geq 7$, then $\frac{4}{2^t} < \frac{1}{2r^2}$. This implies that $\left|\frac{\log\sigma}{\log 2} - \frac{t}{r}\right| < \frac{1}{2r^2}$. So, $\frac{t}{r}$ is a convergent of $\frac{\log\sigma}{\log 2}$, by Legendre's theorem. Emloying $r < K$ and some Sage computations we obtain that

$$b = \max_{0\le i\le 54}\{a_i\} = 100 \text{ and } q_{53} < K < q_{54}.$$

Therefore,

$$\frac{1}{(2+100)K} < \frac{4}{2^t}.$$

Thus

$$2^t < 3\cdot 10^{31}.$$

Hence $t \le 104$.

Arriving at this position, we are ready to explicitly determine the complete set of solutions of Eq.(1.1).

Theorem 4.5: *Let $s \ge t$. If (r,s,t) is a non-negative triple that satisfies Eq.(1.1), then it is one of the five triples*

$$(0,1,0),(0,2,0),(1,1,0),(1,2,0),(2,3,0).$$

Proof: Solving Eq.(4.10) for $t \le 104$ we find no solution and Eq.(1.1) for $t < 122, s < 239$ and $r < 478$ we get the mentioned triples.

The following result is immediate.

Corollary 4.6: *The only numbers which are both modified Pell and Jacobsthal numbers are* $1,1$ *and* 3.

5. Conclusion

Modified Pell numbers that can be expressed as sums of two Jacobsthal numbers are revealed. The approach basically dependens on Matveev's theorem and a reduction result due to Dujella and Pethö to cut down the obtained upper bound to an investigatable level. We found that there are three modified Pell numbers that satisfy the equation. This motivates the investigation of replacing modified Pell numbers by other numbers as Co-balancing, *k*-Fibonacci, *k*-Lucs,....

Declarations

Conflict of interest: The authors declare no competing interests.

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Received September, 2023