

Characterization of ideals of Q -algebras related to its G -part

Ananya Anantayasethi [†]
Algebra and Applications Research Unit
Mahasarakham University
Mahasakham 44150
Thailand

Jörg Kopitz ^{*}
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia 14476
Bulgaria

Abstract

In this work, we study the G -part of Q -algebras A , i.e. the set $G(A) = \{w \in A \mid 0w = w\}$. We show that $G(A)$ is an abelian group, whenever $G(A)$ is an ideal and characterize all Q -algebras A such that $G(A)$ is an ideal of A . Moreover, we show that, up to isomorphism, there is only one Q -algebra A with $G(A) = A$, which is only possible if $|A| = 2^k$ ($k \in \mathbb{N}$).

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1. Introduction

J. Neggers et al. have established a new concept within the theory of BCK-algebras in 2001. The idea in the background was a generalization of three concepts: BCH- algebra, BCI-algebra and BCK-algebra. This new concept, which gives a roof of all these three concepts, is called Q -algebra [13]. Let us consider a non-empty set, which we call A . Further, we fix an

[†] E-mail: ananya.a@msu.ac.th

^{*} E-mail: kopitz@math.bas.bg (Corresponding Author)

element on A , denoted by 0 . Let $\bullet$ be a binary operation on the set A . If the operation $\bullet$ satisfies the next three properties:

$$(Q_1) \quad \zeta \bullet \zeta = 0;$$

$$(Q_2) \quad \zeta \bullet 0 = \zeta;$$

$$(Q_3) \quad (\zeta \bullet \nu) \bullet \beta = (\zeta \bullet \beta) \bullet \nu \text{ for elements } \zeta, \nu \text{ and } \beta \text{ in } A,$$

then we call the triple $(A; \bullet, 0)$ a Q -algebra. In the praxis, we will omit the symbol $\bullet$ by convenient reasons. Many authors have already discovered important facts about Q -algebras. Let list us here several important publications: [1, 3, 6, 7, 11]. In these papers were studied, for example, pseudo Q -algebras. Some of the most important concepts in Q -algebras are the ideals, the G -part, and the homomorphisms. The concept of an ideal in Q -algebras was developed by Neggers et al. in [13]. An ideal of a Q -algebra A is a subset I of A , $I \neq \emptyset$, which satisfies the next 2 conditions (Λ_1) and (Λ_2) :

$$(\Lambda_1) \quad 0 \in I;$$

$$(\Lambda_2) \quad \omega \in A \text{ and } \nu \in I \text{ with } \omega\nu \in I \text{ implies } \omega \in I.$$

The set $\{0\}$ and A are obviously ideals of a Q -algebra A .

Example 1.1: Let $A = \{0, a, x, y, z\}$. Let $\bullet$ be the operation on A defined as following:

$\bullet$	0	a	x	y	z
0	0	a	z	z	x
a	a	0	x	x	z
x	x	z	0	0	a
y	c	z	0	0	a
z	z	x	a	a	0

It is a practice to examine that $(A; \bullet, 0)$ is a Q -algebra. There is only one non-trivial ideal of A , namely $\{0, a\}$.

This concept of ideal bases on the concept of a BCK-ideal (see [8]). In [2], S. Ahn and his coauthor K. So study important properties of ideals of Q -algebras. The authors of [2] give namely some necessary conditions for an arbitrary subset of a Q -algebra to be an ideal. Another concept of an ideal in a Q -algebra was given in [12]. The authors introduce the concept

of a Q -ideal in that paper. Further, J. Neggers et al. consider also a modification, namely the concept of an implicative ideal [13]. In [1], H. K. Abdullah and M. Taeh mention the concept of a prime ideal in Q -algebras. In particular, Abdullah and Taeh show that an ideal, which consists of all elements of the Q -algebra, exempt of one element, is always a so-called prime ideal. In 2016, Y. B. Yun et al. generalized the given idea of Q -algebras to the new concept of pseudo Q -algebras and introduced the notation of a pseudo ideal of a pseudo Q -algebra [10]. In [5], the notion of a PN - Q -ideal in an algebra so-called PN - Q is established, also a relation to Q -ideals in Q -algebras is given. Very recently in [14], the authors have provided an interesting concept for ideals in so-called True-False-structures, which are related to BCK-algebras. Finally, let us look at the concept of ϕ - r ideal of a commutative ring, given in [15]. In a commutative ring R , if we realize the meaning of the transformation ϕ on the set of ideals of R and the meaning of the notation $Ann(a)$, for an element a of R , then we can point out an analogues between the ideals of Q -algebras and that new notation of a ϕ - r ideal for commutative rings.

In [13], J. Neggers and his co-authors introduced the notion so called the G -part of a Q -algebra. For a Q -algebra A , we denote the set G -part by $G(A)$ and

$$G(A) = \{\omega \in A : 0\omega = \omega\}.$$

This concept bases on the concept of the BCI G -part of a BCI -algebra introduced in [9]. Obviously, we have $G(A) \neq \emptyset$. Indeed, by (Q_1) , $00 = 0$, then $0 \in G(A)$. From Example 1.1, we get that $G(A) = \{0, a\}$. The authors of [13] determined all Q -algebras of order 2 and order 3, respectively, such that the set G -part achieves the property of ideal. Moreover, the authors showed that $G(A) \neq A$ when $|A| = 3$. More specifically, the G -part is an ideal if and only if the G -part is a singleton set, i.e. $G(A) = \{0\}$. Further, some properties of the G -part and ideals for any Q -algebra were obtained. For example:

Proposition 1.2: ([13]) Let A be a Q -algebra. Let v, ω and ζ be elements in $G(A)$. If, then $\omega = \zeta$. Hence, $G(A)$ satisfies the left cancellation property.

Let us note that there is an error information about the elements of $G(A)$ in [13], namely [13, Proposition 3.4]: “Let ω be an element in a Q -algebra A . We get that $\omega \in G(A)$ iff $0\omega \in G(A)$.” is incorrect since the authors of [13] used Proposition 1.2 ([13, Corollary 3.3]) incorrectly. The following example illustrate the mistake:

Example 1.3: Let $A = \{0, a, x, z\}$. The binary operation $\bullet$ defines on A as following table:

$\bullet$	0	a	x	z
0	0	a	0	0
a	a	0	a	a
x	x	a	0	0
z	z	a	0	0

It is a practice to declare the properties (Q_1) , (Q_2) and (Q_3) . Thus, $(A; \bullet, 0)$ is a Q -algebra and $G(A) = \{0, a\}$. Since $0b = 0 \in G(A)$, but $b \notin G(A)$, it follows that the converse of [13, Proposition 3.4] is invalid. On the other hand, we can characterize the elements in $G(A)$ of the form $0x$, $x \in A$:

Let us mention here, later on we will denote the symbol A as a Q -algebra A unless otherwise specified.

Proposition 1.4: For any $d \in A$ we get that $0d \in G$ if and only if $(0d)d = 0$.

Proof: Let $d \in A$ and $0d \in G(A)$. Then $0(0d) = 0d$. It follows that $0 = (0d)(0d) = (0(0d))d = (0d)d$. Thus, $(0d)d = 0$. Assume now for the converse direction that $(0d)d = 0$. Then by (Q_1) , we get that $0d = ((0d)(0d))d$. By (Q_3) we can calculate that $((0d)(0d))d = ((0d)d)(0d) = 0(0d)$. Therefore, $0(0d) = 0d$. Hence, $0d \in G(A)$.

The concept of the G -part of a Q -algebra was extended to pseudo Q -algebras in [4]. Additional, in [4] the authors considered the concept of a p -radical $B(A)$ and show that $B(A)$ is an ideal. It appears the question for any Q -algebra, under which conditions, the set G -part is an ideal? The answer of that question will one of the main objectives of the present paper. On the other hand, we know that a homomorphism on (pseudo) Q -algebras maps ideals to ideals (see [4]). In the present paper, we will also characterize, up to isomorphism, all Q -algebras A such that $G(A)$ is the ideal of A .

2. The Sets G -part $G(A)$

First, we will prove some properties of $G(A)$. The following lemma shows that the elements in set $G(A)$ commute, i.e. $\omega v = v\omega$, whenever ω and v belong to $G(A)$.

Lemma 2.1: Let ω and v be elements of $G(A)$. Then we get that $\omega v = v\omega$.

Proof: Let $\omega, v \in G(A)$. It follows that $0\omega = \omega$ and $0v = v$. Then by (Q_3) , we get that $\omega v = (0\omega)v = (0v)\omega = v\omega$.

Proposition 2.2: Let $\omega, v, \beta \in G(A) \setminus \{0\}$. Then the next three conditions are valid:

- (i) $\omega \neq v$ implies $\omega v \notin \{\omega, v, 0\}$.
- (ii) If $\omega v = \beta$, then $\omega\beta = v$ and $v\beta = \omega$.
- (iii) For any $0 \neq \rho \in A$, $\rho\omega \neq \rho$.

Proof:

- (i) Assume now that $\omega v \in \{0, \omega, v\}$. If $\omega v = 0$, then by $(Q_1), (Q_3)$, and Lemma 2.1, we get $\omega = 0$ $\omega = (vv)$ $\omega = (v\omega)$ $v = (\omega v)$ $v = 0$ $v = v$, a contradiction. If $\omega v = \omega$, then similarly we get $v = 0$ $v = (\omega\omega)$ $v = (\omega v)$ $\omega = \omega$ $\omega = 0$, a contradiction. If $\omega v = v$, then we get a contradiction since $\omega = 0$ $\omega = (vv)$ $\omega = (v\omega)$ $v = (\omega v)$ $v = vv = 0$.
- (ii) Suppose that $\omega v = \beta$. Then we get $\omega = 0$ $\omega = (vv)$ $\omega = (v\omega)$ $v = (\omega v)$ $v = \beta$ $v = v\beta$ and $v = 0$ $v = (\omega\omega)$ $v = (\omega v)$ $\omega = \beta$ $\omega = \omega\beta$.
- (iii) Let $0 \neq \rho \in A$. Assume that $\rho\omega = \rho$. Then by (Q_1) and (Q_3) , we get $0 = \rho$ $\rho = (\rho\omega)$ $\rho = (\rho\rho)$ $\omega = 0$ $\omega = \omega$, this contradicts to $\omega \neq 0$. Thus, for any $0 \neq \rho \in A$ we have $\rho\omega \neq \rho$.

Note that $G(A)$ can form a Q -algebra. We will give a necessary condition that $G(A)$ is a Q -algebra.

Proposition 2.3: Let $|A| \geq 3$. If $|A|$ is odd, then $G(A) \neq A$.

Proof: Assume that $|A|$ is odd. We may assume that $A = \{0 = x_1, x_2, \dots, x_{2k+1}\}$ for some non-negative integer k . We will prove by a contradiction. Assume now that $G(A) = A$. Then we get that $x_2 x_i \notin \{0, x_2, x_i\}$, $i \in \{3, 4, \dots, 2k+1\}$, by Proposition 2.2 (i). Moreover, we have that $x_2 x_3, x_2 x_4, \dots, x_2 x_{2k+1}$ are all pairwise different by Proposition 1.2. Let $i \in \{3, 4, \dots, 2k+1\}$ and we may assume that $x_2 x_i = x_{i+1}$. Then by Proposition 2.2(ii), we get that $x_2 x_{i+1} = x_i$. It follows that $x_2 x_{2k+1} = x_2 x_m$ for some $m \neq 2k+1$. This contradicts to Proposition 1.2.

Corollary 2.4: Let $|G(A)| \neq 1$. If $G(A)$ is a Q -algebra, then $|G(A)|$ is even.

Form Lemma 2.1, we know that the elements in set $G(A)$ commute. It appears that a Q -algebra A is a group whenever $A = G(A)$. Moreover, in

the case that $G(A)$ is an ideal of A , there follows that set $G(A)$ itself forms an abelian group.

Proposition 2.5: If $G(A)$ is an ideal, we get that $G(A)$ is an abelian group.

Proof: Suppose here now $G(A)$ is an ideal of A . Let v and ω be elements in $G(A)$. By (Q_1) and (Q_3) , we get that $(v\omega)v = (vv)\omega = 0\omega = \omega$. There follows $(v\omega)v \in G(A)$. Since $(v\omega)v \in G(A)$, $v \in G(A)$ and $G(A)$ is ideal, then $v\omega \in G(A)$. Therefore, $G(A)$ is closed. Now let v, ω and c belong to $G(A)$. Then by (Q_3) and Lemma 2.1, we get that $(v\omega)c = (\omega v)c = (\omega c)v = v(\omega c)$. Therefore, the associative law is satisfied. Since $0v = v = v0$ for all $v \in G(A)$, then 0 is an identity element. Moreover, v is an inverse of itself, since $vv = 0$. Altogether, with Lemma 2.1, $G(A)$ is an abelian group.

Observe that Proposition 1.2 is a consequence of Proposition 2.5. Since A is an ideal of A , we can characterize all Q -algebras A with $G(A) = A$.

Corollary 2.6: $G(A) = A$ if and only if A is an abelian group.

Corollary 2.6 suggests that several groups form Q -algebras. Let ζ be an element of a group. The order of ζ in a group denoted by $|\zeta|$.

Proposition 2.7: Let $e \in A$ and $+$ be a binary operation defined on A . Then $(A; +)$ is an abelian group with an element e as identity and $|v| = 2$ for all $e \neq v \in X$ if and only if $(A; +, e)$ is a Q -algebra with $G(A) = A$.

Proof: $(\Rightarrow)$ Let $\omega \in A$. Then $\omega + \omega = e$, i.e. (Q_1) is satisfied. Since e is an identity element of a group A , then $\omega + e = \omega$. Therefore, (Q_2) is satisfied. Moreover, we have $G(A) = A$ since $e + \omega = \omega$. Now let ρ_1, ρ_2 and ρ_3 be elements in A . Then $(\rho_1 + \rho_2) + \rho_3 = \rho_1 + (\rho_2 + \rho_3) = \rho_1 + (\rho_3 + \rho_2) = (\rho_1 + \rho_3) + \rho_2$. Therefore, (Q_3) is satisfied. Hence, $(A; +, e)$ is a Q -algebra.

$(\Leftarrow)$ Let $(A; +, e)$ be a Q -algebra with $G(A) = A$. Then by Corollary 2.6, $(A; +, e)$ is an abelian group. Let $v \in A$. Then we get that $e = v + v$ by (Q_1) . Therefore, the order of v is two, i.e. $|v| = 2$ whenever $v \neq e$.

3. Ideals of Form $G(A)$

From Example 1.1, we have $G(A) \subseteq A$ and $G(A)$ is an ideal of A . But in Example 1.3, $G(A) \subseteq A$ and $G(A)$ is not an ideal of A . It appears the question under which conditions $G(A)$ is an ideal of A . The next lemma present us the fact of the cardinality of $G(A)$ such that $G(A)$ is not an ideal of A of order greater or equal 3.

Lemma 3.1: Let $|A| = n, n \geq 3$. If $G(A)$ is an ideal of A , $|G(A)| \neq n-1$.

Proof: Suppose now that $G(A)$ is an ideal of A . We assume now that $|G(A)| \neq n-1$. From (Λ_1) , we have $0 \in G(A)$. Now let $G(A) = \{0 = x_1, x_2, \dots, x_{n-1}\}$. It follows that $0x_i = x_i$ for all $i \in \{2, 3, \dots, n-1\}$. Since $x_2, x_3, \dots, x_{n-1} \in G(A)$ but $x_n \notin G(A)$, we have $x_n x_i = x_n$ for all $i \in \{2, 3, \dots, n-1\}$, otherwise $x_n \in G(A)$, since $G(A)$ is an ideal, a contradiction. Let $x_k \in \{x_2, x_3, \dots, x_{n-1}\}$. Then $x_k \neq 0$ and $0 = x_n x_n = (x_n x_k)(x_n x_k) = (x_n(x_n x_k))x_k = (x_n x_n)x_k = 0x_k$. Thus, $0x_k = 0$, a contradiction. Therefore, $|G(A)| \neq n-1$, this complete the proof.

On Q-algebras, in order to give a characterization with condition that the G -part is an ideal, we will formulate an equivalent characterization of the concept of ideal.

Proposition 3.2: Let K be a non-empty subset of A . Then a subset K is an ideal of A if and only if $(A \setminus K)K \subseteq A \setminus K$.

Proof: Let a subset K be an ideal of A . Let $x \in A \setminus K$ and $y \in K$. Suppose now that $xy \in K$. Then by (Λ_2) , we get $x \in K$, a contradiction. This shows that $(A \setminus K)K \subseteq A \setminus K$.

Conversely, suppose that $(A \setminus K)K \subseteq A \setminus K$. Let $x \in A$ and $y \in K$ such that $xy \in K$. Assume that $x \in A \setminus K$. Then $xy \in (A \setminus K)K \subseteq A \setminus K$, i.e. $xy \notin K$, a contradiction. Hence, $x \in K$. Therefore, K is an ideal of A .

Proposition 3.2 gives an alternative equivalent definition of ideal.

Proposition 3.3: $G(A)$ is an ideal of A if and only if $G(A)$ is a group and for all $x_1, x_2 \in A$ and all $y \in G(A)$, $x_1 y = x_2 y$ implies $x_1 = x_2$ or $x_1, x_2 \notin G(A)$.

Proof: Assume $G(A)$ is an ideal of A . Then by Proposition 2.5 $G(A)$ is an abelian group. Let $x_1, x_2 \in A$ and $y \in G(A)$ with $x_1 y = x_2 y$. Suppose that $x_1 \in G(A)$ or $x_2 \in G(A)$. Then $y = 0y = (x_1 x_1)y = (x_1 y)x_1 = (x_2 y)x_1 = (x_2 x_1)y$ and similarly, $y = 0y = (x_2 x_2)y = (x_2 y)x_2 = (x_1 y)x_2 = (x_1 x_2)y$. This provides $(x_2 x_1)y = y \in G(A)$, $(x_1 x_2)y = y \in G(A)$. By (Λ_2) , $x_2 x_1$ and $x_1 x_2$ belong to $G(A)$. If $x_1 \in G(A)$, we get $x_2 x_1 \in G(A)$ and then by (Λ_2) , $x_2 \in G(A)$. Similarly, $x_2 \in G(A)$ implies $x_1 \in G(A)$. Thus, $x_1 y = x_2 y$ implies $x_1 = x_2$ since $G(A)$ is a group by Proposition 2.5.

Conversely, suppose that $G(A)$ is a group such that $x_1 y = x_2 y$ implies $x_1 = x_2$ for all $x_1, x_2 \in A$ and all $y \in G(A)$ whenever $x_1 \in G(A)$ or $x_2 \in G(A)$. Let $x \in A \setminus G(A)$ and $y \in G(A)$. Assume now $xy \in G(A)$. Then there exists $a \in G(A)$ such that $ay = xy$. This provides $a = x$, i.e. $a \in A \setminus$

$G(A)$, a contradiction. Hence, xy belongs to $A \setminus G(A)$. Thus, $(A \setminus G(A))G(A) \subseteq X \setminus G(A)$. By Proposition 3.2 we get $G(A)$ is an ideal of A .

Next, we will turn to the case that the cardinality of the Q -algebra A is 4.

Proposition 3.4: Let $|A| = 4$ and $G(A) \neq \{0, A\}$. Then $G(A)$ is an ideal if and only if for all $x \neq z \in A$ and $y \in G(A)$, $xy \neq zy$.

Proof: Let $A = \{0, a, b, c\}$.

($\Rightarrow$) Let $G(A)$ be an ideal of A . Then $|G(A)| \neq 3$, by Lemma 3.1. Without loss of generality let $G(A) = \{a, 0\}$. Suppose now that there are elements $x, z \in A$ with $x \neq z$ and $xy = zy$ for some $y \in G(A)$. If $y = 0$, then $x0 = z0$. Thus, $x = x0 = z0 = z$, a contradiction. If $y = a$, we get $xa = za$. Then $aa = 0 \neq a = 0a$ implies that there is $x_0 \in \{x, z\} \setminus \{0, a\}$. Then it is easy to verify that $0 = (x_0 a)(xa) = (x_0(xa))a$ provides $xa \notin G(A)$, i.e. $xa, za \in \{b, c\}$. This implies $x, z \in \{b, c\}$. Hence, there is $x_1 \in \{b, c\}$ such that $x_1 = x_1 a$. So, we have $0 = (x_1 a)(x_1 a) = (x_1(x_1 a))a = 0a = a$, a contradiction.

($\Leftarrow$) Let z and x be elements in A with $z \neq x$, $y \in G(A)$ and $xy \neq zy$. Suppose that $|G(A)| \neq 2$. There follows $|G(A)| = 3$ and then we let $G(A) = \{a, b, 0\}$. Then $a = 0a$ and $b = 0b$. Since b and a are elements in $G(A)$, then the assumption implies that $0a, aa, ba$, and ca as well as $0b, ab, bb$, and cb are pairwise different. There follows that $ba, ca \in \{c, b\}$, $ab, cb \in \{c, a\}$. Since a and b belong to $G(A)$, then by Lemma 2.1, we get that $ab = ba$. Therefore, $ba = ab \in \{c, b\} \cap \{c, a\}$, i.e. $ab = ba = c$. Therefore, $ca = b$ and $cb = a$. Since $0 = cc = (ba)c = (bc)a$, we have $bc = a$, otherwise $(bc)a \neq 0$. Then we can calculate that $0c = (bb)c = (bc)b = ab = c$. Therefore, c belongs to $G(A)$, a contradiction. Therefore, $|G(A)| \neq 3$ and $G(A) = \{0, x\}$ for some $x \in \{c, b, a\}$. It is not complicated to examine that $G(A)$ is a group with an identity 0. Thus, using Proposition 3.3, we can conclude that $G(A)$ is an ideal.

Theorem 3.5: Let $|A| = 4$. Then $G(A)$ is an ideal of A if and only if $G(A)$ satisfies one of the followings:

- (i) $|G(A)| \in \{1, 4\}$.
- (ii) $|G(A)| = 2$ and $\omega y \neq \nu y$ for all $y \in G(A)$, $\omega \neq \nu \in A$.

Proof: ($\Rightarrow$) Assume $G(A)$ is ideal. Then by Lemma 3.1 we get $|G(A)| \neq |A| - 1 = 4 - 1 = 3$. Therefore, $|G(A)| \in \{1, 2, 4\}$. If $|G(A)| \in \{1, 4\}$, then (i)

is satisfied. Now let $|G(A)| = 2$. Since $G(A)$ is an ideal, using Proposition 3.4, we get (ii).

($\Leftarrow$) If the cardinality of $G(A)$ is equal to 1 or 4, then $\{0\} = G(A)$ or $A = G(A)$. Then it is obvious that $G(A)$ is an ideal. If $G(A)$ satisfies contradiction (ii), then by Proposition 3.4, $G(A)$ is an ideal of A .

4. Homomorphisms

Let $(A_1; \bullet, 0)$ and $(A_2; \bullet', 0')$ be Q -algebras. A homomorphism from A_1 to A_2 is a function $\Upsilon: A_1 \rightarrow A_2$ which satisfies the property $\Upsilon(\omega \bullet v) = \Upsilon(\omega) \bullet' \Upsilon(v)$ for all $\omega, v \in A_1$. We say Υ is an isomorphism if Υ is bijective and we will say that A_1 is isomorphic to A_2 , denoted by $A_1 \cong A_2$. It is easy to see that $\Upsilon(0) = 0'$. We will observe that $\Upsilon(G(A_1)) = \{\Upsilon(v) : v \in G(A_1)\}$, the homomorphic image of $G(A_1)$, is a subset of $G(A_2)$.

Proposition 4.1: Let A_1 and A_2 be Q -algebras and let $\Upsilon: A_1 \rightarrow A_2$ be a homomorphism. If $v \in G(A_1)$, then $\Upsilon(v) \in G(A_2)$.

Proof: Let $v \in G(A_1)$. Then $0v = v$. There follows $\Upsilon(v) = \Upsilon(0v) = \Upsilon(0)\Upsilon(v) = 0'\Upsilon(v)$. Therefore, $\Upsilon(v) \in G(A_2)$.

Supposing $G(A_1)$ of a Q -algebra A_1 is a group and considering the restriction of Υ to $G(A)$ as a homomorphism from $G(A)$ to $G(B)$, Proposition 4.1 proves:

Corollary 4.2: Let $\Upsilon: A_1 \rightarrow A_2$ be a Q -algebra homomorphism. Then $\Upsilon(G(A_1))$ is a subgroup of $G(A_2)$ whenever $G(A_1)$ is a group.

Let A be a Q -algebra with the cardinality m , for some $m \in \{1, 2, \dots\}$. If $G(A) = A$, then by Corollary 2.6, we get that A is a finite abelian group. Moreover, each element, except 0, in A has order 2. It is well-known that a group G with identity ρ and $\omega^2 = \rho$ for all $\omega \in G$ has the order 2^k for some $k \in \{1, 2, \dots\}$. It turns out that A has the order 2^k for some positive integer k . By the Fundamental Theorem of Finite Abelian Groups, A is isomorphic to the direct product $\mathbb{Z}_2^k$. This proves:

Proposition 4.3: Let A_1 and A_2 be finite Q -algebras of order m . If $|G(A_1)| = |G(A_2)| = m$, then $A_1 \cong A_2$.

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