

The structure of the unit group of the group algebra $K_qSL(2, \mathbb{Z}_7)$

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Abstract

This article focusus on characterizing the units of the group algebra $K_qSL(2, \mathbb{Z}_7)$, where $SL(2, \mathbb{Z}_7)$ is a special linear group of 2×2 matrices over the finite field $\mathbb{Z}_7$.

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1. Introduction

Let $\mathcal{U}(\mathcal{K}_q G)$ denote the units of the group algebra $\mathcal{K}_q G$ of G and the finite field $\mathcal{K}_q$ with characteristic p and $q = p^k$. A central problem in a study of group algebras involves determining the units in semi simple group algebras (see [3, 13, 14, 16]). The classification of these unit groups is particularly important for applications such as see [4] [13] (which remains a major challenge in group algebra theory), and the constructing the convolutional codes within the framework of group algebras [7].

Significant research has been dedicated to characterizing the units of the semisimple group algebra $\mathcal{K}_q G$ (see [2, 3, 11, 12, 14, 15, 16, 18, 19, 22]). A key contribution in this area comes from Bakshi et al. [3], who examined the units of semisimple group algebra (SSG) of all metabelian groups. Consequently, much of the subsequent research has shifted towards the units of group algebras of non-metabelian groups. It has been established that Sharma et. al., constructed the units of group algebra of $SL(2, \mathbb{Z}_3)$ and S_4 . The units of the SSG of these groups had been characterized in [10] and [9], respectively. Later, Mittal [14] characterized the units of group algebras for non-metabelian groups of order up to 72. Further contributions include the work of Sharma et al. [17], who characterized the units of the SSG of $SL(2, \mathbb{Z}_5)$. Consequently, Arvind [1] characterized the units of the SSG of $SL(3, \mathbb{Z}_2)$. Sivaranjani et al. also made significant contributions by studying the unit groups of SSG for the special linear groups $SL(2, 8)$ and $SL(2, 9)$ in [20], as well as identifying the units of the SSG of $GL(2, 5)$ in [21]. Furthermore, Jasim et al. computed the unit group for $SL(2, 49)$, utilizing induced characters of the group's cyclic subgroups, as discussed in [8].

In this paper, our main aim is to deduce the units of the SSG of the group $SL(2, \mathbb{Z}_7)$. We remark that as the size of the group increases, it becomes more and more challenging to uniquely identify the decompositions. We also refer to [15, 16] for some other very recent works in this direction.

Let $\mathcal{U}(\mathcal{K}_q SL(2, \mathbb{Z}_7))$ denotes the units of $\mathcal{K}_q SL(2, \mathbb{Z}_7)$, where $\mathcal{K}_q$ is a finite field of characteristic p . To determine $\mathcal{U}(\mathcal{K}_q SL(2, \mathbb{Z}_7))$, our main task is to deduce the Wedderburn decomposition (WD) of $\mathcal{K}_q SL(2, \mathbb{Z}_7)$ (since it is trivial to obtain the units from the WD).

In this paper, Section 2 covers the necessary preliminaries for this study. In Section 3, we present a characterization of the units of $\mathcal{K}_q SL(2, \mathbb{Z}_7)$, stated in Theorem 3.1. Lastly, Section 4 provides the concluding remarks.

2. Preliminaries

In this article, we denote $J(\mathcal{K}_q G)$ as the Jacobson radical of $\mathcal{K}_q G$ and the order of $SL(n, \mathbb{Z}_t)$ is,

$$\frac{1}{t-1}(t^n - 1)(t^n - t) \cdots (t^n - t^{n-1}). \quad (1)$$

Next, we will review some notations and results from [6]. Let s represent the exponent of G , and η be a primitive s^{th} root of unity over $\mathcal{K}$. Now, define the set $T_{G, \mathcal{K}} = \{t \mid \sigma(\eta) = \eta^t, \text{ where } \sigma \in \text{Gal}(\mathcal{K}(\eta) : \mathcal{K})\}$.

Since $\text{Gal}(\mathcal{K}(\eta) : \mathcal{K})$ is cyclic, in particular, for any $\sigma \in \text{Gal}(\mathcal{K}(\eta) : \mathcal{K})$, $\exists$ a positive integer s such that the action of σ on the element η can be expressed as $\sigma(\eta) = \eta^s$. This relationship indicates a specific way that the elements in the extension field $\mathcal{K}(\eta)$ are related to those in the base field $\mathcal{K}$, reinforcing the cyclic structure of the Galois group.

Now, considering any p -regular element $a \in G$, where $p \nmid o(a)$, we define the element γ_a as follows:

$$\gamma_a = \sum_{h \in C_a} h,$$

where C_g is the conjugacy class of g . This definition constructs γ_g as a sum of all elements in the conjugacy class, effectively capturing the behavior of g under conjugation within the group.

The cyclotomic $\mathcal{K}$ -class associated with γ_g is

$$S\mathcal{K}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G, \mathcal{K}}\},$$

where $T_{G, \mathcal{K}}$ is a specific set of integers that relates to the cyclotomic structure of the group. This class essentially consists of elements formed by varying g across all the powers g^t , where t runs through the indices in the set $T_{G, \mathcal{K}}$.

Finally, the theorem that follows characterizes the set $T_{G, \mathcal{K}}$, which encapsulates the relationships between the elements of the group G and their representations in the finite field $\mathcal{K}$. This characterization is crucial for understanding how the properties of the group and its elements interact with the field's structure, especially in the context of cyclotomic classes and representations.

Theorem 2.1: [14, Theorem 2.3] *Given that $\mathcal{K}$ is a finite field and that $\gcd(d, s) = 1$, let its order be a prime power of d . $e = \text{order}_s(d)$ is the multiplicative order of d modulo s . Following that, we have*

$$T_{G,\mathcal{K}} = \{1, d, \dots, d^{e-1}\} \bmod s.$$

Proposition 2.1: [6, Proposition 1.2] *There is an one-to-one correspondence between the simple components of $\frac{\mathcal{K}G}{J(\mathcal{K}G)}$ and the cyclotomic K_q -classes in G .*

Theorem 2.2: [6, Theorem 1.3] *Assume that G has t cyclotomic $\mathcal{K}$ -classes and $\text{Gal}(\mathcal{K}(\theta) : \mathcal{K})$ is a cyclic group, then $|S_i| = [K_i : \mathcal{K}]$ with appropriate index ordering if $S_1, S_2, \dots, S_t$ are the cyclotomic $\mathcal{K}$ -classes of G and $K_1, K_2, \dots, K_t$ are the simple components of $Z\left(\frac{\mathcal{K}G}{J(\mathcal{K}G)}\right)$.*

Proposition 2.2: [13, Proposition 3.6.11] *Let G' be the commutator subgroup of G and let $\mathcal{K}G$ be a semisimple group algebra, then*

$$\mathcal{K}G \simeq \mathcal{K}(G / G') \oplus \Delta(G, G').$$

Proposition 2.3: [13, Proposition 3.6.7] *Let $N \trianglelefteq G$ and let $\mathcal{K}G$ be a SSG, then*

$$\mathcal{K}G \simeq \mathcal{K}(G / N) \oplus \Delta(G, N).$$

Proposition 2.4: [5, Proposition 1] *Let $\mathcal{K}G$ denotes semisimple group algebra with characteristic p for $\mathcal{K}$. It can be expressed as*

$$\mathcal{K}G \cong \bigoplus_{i=1}^r M_{n_i}(\mathcal{K}_i),$$

As a result, $p \nmid n_i$.

Proposition 2.5: [1, Corollary 3.8] *For each $\mathcal{K}$, let $\mathcal{K}G$ be an SSG with characteristic p . $M(n, \mathcal{K})$ is one of the Wedderburn components of $\mathcal{K}G$ if there exists an irreducible representation of degree n over $\mathcal{K}$.*

3. Main result

Let G denotes $SL(2, \mathbb{Z}_7)$. In this section, we characterize the units of $\mathcal{K}_q G$ for $p \nmid o(G)$ such that $\mathcal{K}_q G$ is semisimple. We know that $o(G)$ is 336. By Maschke's theorem, $\mathcal{K}_q G$ is semisimple for $p \neq 2, 3, 7$. In the following theorem, we discuss the WD of the $\mathcal{K}_q G$.

Theorem 3.1: *The WD of $\mathcal{K}_q G$ is given by:*

- (1) *for $p^k \equiv \{1, 23, 25, 65, 71, 79, 95, 113, 121, 127, 137, 151\} \bmod 168$, we have*
- $$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus (\mathbf{M}(3, \mathcal{K}_q))^2 \oplus (\mathbf{M}(4, \mathcal{K}_q))^2 \oplus (\mathbf{M}(6, \mathcal{K}_q))^3 \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2.$$

- (2) for $p^k \equiv \{11, 29, 37, 43, 53, 67, 85, 107, 109, 149, 155, 163\} \pmod{168}$, we have $\mathcal{K}_q G \simeq \mathcal{K}_q \oplus (\mathbf{M}(3, \mathcal{K}_q))^2 \oplus (\mathbf{M}(4, \mathcal{K}_q))^2 \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(6, \mathcal{K}_{q^2}) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2$.
- (3) for $p^k \equiv \{17, 31, 41, 47, 55, 73, 89, 97, 103, 143, 145, 167\} \pmod{168}$, we have $\mathcal{K}_q G \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(4, \mathcal{K}_{q^2}) \oplus (\mathbf{M}(6, \mathcal{K}_q))^3 \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2$.
- (4) for $p^k \equiv \{5, 13, 19, 59, 61, 83, 101, 115, 125, 131, 139, 157\} \pmod{168}$, we have $\mathcal{K}_q G \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(4, \mathcal{K}_{q^2}) \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(6, \mathcal{K}_{q^2}) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2$.

Proof: Using (1), we know that $o(G)$ is 336. Also, it has 11 conjugacy classes. The representatives, size and order of conjugacy classes are tabulated below.

Representative	Size	Order
$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1
$g_2 = \begin{pmatrix} 0 & 6 \\ 1 & 5 \end{pmatrix}$	24	14
$g_3 = \begin{pmatrix} 0 & 4 \\ 5 & 5 \end{pmatrix}$	24	14
$g_4 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$	1	2
$g_5 = \begin{pmatrix} 0 & 6 \\ 1 & 2 \end{pmatrix}$	24	7
$g_6 = \begin{pmatrix} 0 & 4 \\ 5 & 2 \end{pmatrix}$	24	7
$g_7 = \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix}$	42	4
$g_8 = \begin{pmatrix} 0 & 6 \\ 1 & 4 \end{pmatrix}$	42	8
$g_9 = \begin{pmatrix} 0 & 6 \\ 1 & 3 \end{pmatrix}$	42	8
$g_{10} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$	56	3
$g_{11} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$	56	6

It is evident from the table above that the exponent of G is 168. Since $\mathcal{K}_q G$ is semisimple, the WD is given by (cf. [13]) that

$$\mathcal{K}_q G \simeq \bigoplus_{i=1}^r \mathbf{M}(n_i, \mathcal{K}_{q^i}),$$

Also, G is a perfect one, i.e., the derived subgroup G' of G is G itself. Consequently, the quotient group $G/G' \simeq \{1\}$ (identity of the group G). Then

$$\mathcal{K}_q G \simeq \mathcal{K}_q \bigoplus_{i=1}^{r-1} \mathbf{M}(n_i, \mathcal{K}_{q^i}), \quad (2)$$

Next, we observe that the set $T_{G, \mathcal{K}_q}$ in our case can be explicitly written as

$$\begin{aligned} T_{G, \mathcal{K}_q} = \{ & 1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 53, 55, 59, 61, 65, 67, 71, \\ & 73, 79, 83, 85, 89, 95, 97, 101, 103, 107, 109, 113, 115, 121, 125, 127, 131, \\ & 137, 139, 143, 145, 149, 151, 155, 157, 163, 167 \} \bmod 168. \end{aligned}$$

Case 1: For $p^k \equiv \{1, 23, 25, 65, 71, 79, 95, 113, 121, 127, 137, 151\} \bmod 168$, we observe that the order of the cyclotomic $\mathcal{K}_q$ -class of γ_g is 1 for every $g \in G$ (i.e., $|S\mathcal{K}_q(\gamma_g)| = 1 \quad \forall g \in G$). Using this fact, we proceed to rewrite further. (2) as

$$\mathcal{K}_q G \simeq \mathcal{K}_q \bigoplus_{i=1}^{10} \mathbf{M}(n_i, \mathcal{K}_q) 335 = \sum_{i=1}^{10} n_i^2, n_i \geq 2. \quad (3)$$

There are 203 possible choices for n_i 's. To find it uniquely, consider $N \trianglelefteq G$ generated by $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ and $G/N \simeq PSL(3, 2)$. We know that $PSL(3, 2) \simeq SL(3, 2)$. The WD of $SL(3, 2)$ is ([1, theorem 3.1])

$$\mathcal{K}_q(SL(3, 2)) \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_q)^2 \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \mathbf{M}(8, \mathcal{K}_q). \quad (4)$$

Using this and proposition 2.3, we further obtain from (3) that

$$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_q)^2 \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \mathbf{M}(8, \mathcal{K}_q) \bigoplus_{i=1}^5 \mathbf{M}(n_i, \mathcal{K}_q), \quad (5)$$

with

$$168 = \sum_{i=1}^5 n_i^2, n_i \geq 2. \quad (6)$$

The possible choices of n_i 's that fulfill (6) are

$(2, 3, 3, 5, 11), (2, 3, 5, 7, 9), (3, 3, 5, 5, 10), (3, 5, 6, 7, 7),$ and $(4, 4, 6, 6, 8)$.

Next, to find it uniquely, we apply proposition 2.4. We claim that $M(5, \mathcal{K}_q)$ will never be a Wedderburn component of $\mathcal{K}_q G$ in this case. Suppose $M(5, \mathcal{K}_q)$ is a Wedderburn component of $\mathcal{K}_q G$. Since $5^6 \bmod 168 \equiv 1$, we can take 5 as a particular value of q in this case. At this point, proposition 2.4 implies that 5 must not divide any of the n_i 's. Therefore, our claim is true. Hence, $M(5, \mathcal{K}_q)$ is not a Wedderburn component of $\mathcal{K}_q G$. It is now straight-forward to note that the only possibility of n_i 's left is $(4, 4, 6, 6, 8)$ since all other possibilities contain 5. Hence, we utilize (5) to get the required Wedderburn decomposition as

$$\mathcal{K}_q G \cong \mathcal{K}_q \oplus (\mathbf{M}(3, \mathcal{K}_q))^2 \oplus (\mathbf{M}(4, \mathcal{K}_q))^2 \oplus (\mathbf{M}(6, \mathcal{K}_q))^3 \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2.$$

Case 2: $p^k \equiv \{11, 29, 37, 43, 53, 67, 85, 107, 109, 149, 155, 163\} \pmod{168}$. The cyclotomic $\mathcal{K}_q$ classes of γ_s are

$$S\mathcal{K}_q(\gamma_{s_i}) = \{\gamma_{s_i}\}, \text{ for } 1 \leq i \leq 7, S\mathcal{K}_q(\gamma_{s_8}) = \{\gamma_{s_8}, \gamma_{s_9}\},$$

$$S\mathcal{K}_q(\gamma_{s_{10}}) = \{\gamma_{s_{10}}\}, \mathcal{K}_q(\gamma_{s_{11}}) = \{\gamma_{s_{11}}\}.$$

By prop 2.1 and 2.2,

$$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus_{i=1}^8 \mathbf{M}(n_i, \mathcal{K}_q) \oplus \mathbf{M}(n_9, \mathcal{K}_{q^2}) 335 = \sum_{i=1}^8 n_i^2 + 2n_9^2, n_i \geq 2. \quad (7)$$

To find the unique value of n_i 's, we apply propositions 2.4 and 2.5. We observe that the WD of $\mathcal{K}_q(SL(3, 2))$ in this case is same as in (4 (see [1])). Therefore, (7) further derives that

$$\begin{aligned} \mathcal{K}_q G \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_q)^2 \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \\ \mathbf{M}(8, \mathcal{K}_q) \oplus_{i=1}^3 \mathbf{M}(n_i, \mathcal{K}_q) \oplus \mathbf{M}(n_4, \mathcal{K}_{q^2}), \end{aligned} \quad (8)$$

with

$$168 = \sum_{i=1}^3 n_i^2 + 2n_4^2, n_i \geq 2. \quad (9)$$

The choices of n_i 's fulfilling (9) are given by

$$(2, 5, 11, 3), (3, 3, 10, 5), (3, 5, 6, 7), (4, 4, 8, 6), (5, 5, 10, 3) \text{ and } (6, 6, 8, 4).$$

Further, by employing proposition 2.4, we note that the choice $(2, 5, 11, 3)$ is out of race since for $p = 11$ and $k = 1$, $p^k \equiv 1 \pmod{168}$. Thus,

we are remaining with 5 choices out of which only one is correct. Next, to pinpoint the correct option, we demonstrate that $\mathbf{M}(4, \mathcal{K}_q)$ is consistently serve as a Wedderburn component in this scenario. Specifically, we set $p=11$. To show this, we consider the following elements a and b in $GL(4, \mathcal{K}_{11})$:

$$a = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -5 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -1 & -5 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & -1 \end{pmatrix},$$

where $\mathcal{K}_{11} = \{0, 1, \dots, 10\}$. Similar to case 1, we can observe that the subgroup generated by these elements in $GL(4, \mathcal{K}_{11})$ is isomorphic to $SL(2, \mathbb{Z}_7)$. This means that there exists a one-one map injective homomorphism between $SL(2, \mathbb{Z}_7)$ and $GL(4, \mathcal{K}_{11})$. As a result, Proposition 2.5 ensures that $M(4, \mathcal{K}_{11})$ will consistently appear as a Wedderburn component of $\mathcal{K}_q G$. It is now straight-forward to note that the only possibility of n_i 's is $(4, 4, 8, 6)$ since no other possibility contains 4 in first three coordinates. Thus, it follows from (8) that

$$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus (\mathbf{M}(3, \mathcal{K}_q))^2 \oplus (\mathbf{M}(4, \mathcal{K}_q))^2 \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(6, \mathcal{K}_{q^2}) \\ \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2.$$

Case 3: $p^k \equiv \{17, 31, 41, 47, 55, 73, 89, 97, 103, 143, 145, 167\} \pmod{168}$. The cyclotomic $\mathcal{K}_q$ classes of γ_g are

$$S\mathcal{K}_q(\gamma_{g_i}) = \{\gamma_{g_i}\}, \text{ for } i = 1, 4, S\mathcal{K}_q(\gamma_{g_2}) = \{\gamma_{g_2}, \gamma_{g_3}\},$$

$$S\mathcal{K}_q(\gamma_{g_5}) = \{\gamma_{g_5}, \gamma_{g_6}\}, \mathcal{K}_q(\gamma_{g_i}) = \{\gamma_{g_i}\} \text{ for } 7 \leq i \leq 11.$$

By using proposition 2.1 and theorem 2.2, (2) gives

$$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus \bigoplus_{i=1}^6 \mathbf{M}(n_i, \mathcal{K}_q) \oplus \mathbf{M}(n_7, \mathcal{K}_{q^2}) \oplus \mathbf{M}(n_8, \mathcal{K}_{q^2}) \quad (0)$$

with

$$335 = \sum_{i=1}^6 n_i^2 + 2(n_7^2 + n_8^2), n_i \geq 2. \quad (11)$$

We observe that the WD of $\mathcal{K}_q(SL(3, 2))$ in this case is (see [1])

$$\mathcal{K}_q(SL(3, 2)) \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \mathbf{M}(8, \mathcal{K}_q).$$

Using this and proposition 2.3, we further obtain from (10) and (11) that

$$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \mathbf{M}(8, \mathcal{K}_q) \oplus_{i=1}^3 \mathbf{M}(n_i, \mathcal{K}_q) \oplus \mathbf{M}(n_4, \mathcal{K}_{q^2}) \quad (12)$$

with

$$168 = \sum_{i=1}^3 n_i^2 + 2n_4^2, \quad n_i \geq 2. \quad (13)$$

The total possibilities of n_i 's satisfying (13) are

$$(2, 5, 11, 3), (3, 3, 10, 5), (3, 5, 6, 7), (4, 4, 8, 6), (5, 5, 10, 3), \text{ and } (6, 6, 8, 4).$$

To uniquely deduce n_i 's we follow a different approach than the one in cases 1 and 2. Suppose that $\mathbf{M}(2, \mathcal{K}_q)$ is a Wedderburn component of $\mathcal{K}_q G$. This means there exists an onto $\mathcal{K}_q$ -algebra homomorphism between $\mathcal{K}_q G$ and $\mathbf{M}(2, \mathcal{K}_q)$. Let Ψ be restriction of this homomorphism to G , i.e., $\Psi: G \rightarrow GL(2, \mathcal{K}_q)$. This means that G must have an irreducible representation of degree 2, but one can note that there is no such non-trivial homomorphism from $SL(2, 7)$ to $GL(2, \mathcal{K}_{17})$ (we take $p=17$ as a particular case). Therefore, $\mathbf{M}(2, \mathcal{K}_q)$ can not be a Wedderburn component of $\mathcal{K}_q G$. Similarly, we can show that $\mathbf{M}(n, \mathcal{K}_q)$ can not be a Wedderburn components of $\mathcal{K}_q G$ for $n=3, 4, 5$. Consequently, the required choice of n_i 's satisfying (13) is $(6, 6, 8, 4)$. Hence, (12) implies that the WD in this case is

$$\mathcal{K}_q G \cong \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(4, \mathcal{K}_{q^2}) \oplus (\mathbf{M}(6, \mathcal{K}_q))^3 \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2.$$

Case 4: $p^k \equiv \{5, 13, 19, 59, 61, 83, 101, 115, 125, 131, 139, 157\} \pmod{168}$. The cyclotomic $\mathcal{K}_q$ classes of γ_s are

$$S\mathcal{K}_q(\gamma_{s_1}) = \{\gamma_{s_1}\}, S\mathcal{K}_q(\gamma_{s_4}) = \{\gamma_{s_4}\}, \mathcal{K}_q(\gamma_{s_7}) = \{\gamma_{s_7}\},$$

$$S\mathcal{K}_q(\gamma_{s_{10}}) = \{\gamma_{s_{10}}\}, \mathcal{K}_q(\gamma_{s_{11}}) = \{\gamma_{s_{11}}\}, \mathcal{K}_q(\gamma_{s_2}) = \{\gamma_{s_2}, \gamma_{s_3}\},$$

$$S\mathcal{K}_q(\gamma_{s_5}) = \{\gamma_{s_5}, \gamma_{s_6}\}, \mathcal{K}_q(\gamma_{s_8}) = \{\gamma_{s_8}, \gamma_{s_9}\}.$$

By using proposition 2.1 and theorem 2.2, (2) gives

$$\mathcal{K}_q G \simeq \mathcal{K}_q \oplus_{i=1}^4 \mathbf{M}(n_i, \mathcal{K}_q) \oplus_{i=5}^7 \mathbf{M}(n_i, \mathcal{K}_{q^2}) \quad (14)$$

with

$$335 = \sum_{i=1}^4 n_i^2 + 2(n_5^2 + n_6^2 + n_7^2), \quad n_i \geq 2. \quad (15)$$

In this case, we know from [1] that

$$\mathcal{K}_q(SL(3, 2)) \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \mathbf{M}(8, \mathcal{K}_q).$$

Using this and proposition 2.3, we further obtain from (14) and (15) that

$$\begin{aligned} \mathcal{K}_q G \simeq \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus \\ \mathbf{M}(8, \mathcal{K}_q) \oplus \mathbf{M}(n_1, \mathcal{K}_q) \oplus_{i=2}^3 \mathbf{M}(n_i, \mathcal{K}_{q^2}) \end{aligned} \quad (16)$$

with

$$168 = n_1^2 + 2 \sum_{i=2}^3 n_i^2, \quad n_i \geq 2. \quad (17)$$

There are only two possibilities of n_i 's fulfilling (17) given by (8, 4, 6) and (10, 3, 5). On taking $p = 5$, we note using proposition 2.4 that (10, 3, 5) can not be the choices of n_i 's. Therefore, (16) derives that

$$\begin{aligned} \mathcal{K}_q G \cong \mathcal{K}_q \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(4, \mathcal{K}_{q^2}) \oplus \mathbf{M}(6, \mathcal{K}_q) \oplus \\ \mathbf{M}(6, \mathcal{K}_{q^2}) \oplus \mathbf{M}(7, \mathcal{K}_q) \oplus (\mathbf{M}(8, \mathcal{K}_q))^2. \end{aligned}$$

Corollary 3.1: *The units of $\mathcal{K}_q G$ is given by:*

Conditions on p^k	Unit group
$p^k \equiv \begin{Bmatrix} 1, 23, 25, 65, 71, 79, 95, \\ 113, 121, 127, 137, 151 \end{Bmatrix} \pmod{168}$	$\mathcal{K}_q^* \oplus (GL(3, \mathcal{K}_q))^2 \oplus (GL(4, \mathcal{K}_q))^2 \oplus$ $(GL(6, \mathcal{K}_q))^3 \oplus GL(7, \mathcal{K}_q) \oplus (GL(8, \mathcal{K}_q))^2$
$p^k \equiv \begin{Bmatrix} 11, 29, 37, 43, 53, 67, 85 \\ 107, 109, 149, 155, 163 \end{Bmatrix} \pmod{168}$	$\mathcal{K}_q^* \oplus GL(3, \mathcal{K}_q)^2 \oplus GL(4, \mathcal{K}_q)^2 \oplus GL(6, \mathcal{K}_q) \oplus$ $GL(6, \mathcal{K}_{q^2}) \oplus GL(7, \mathcal{K}_q) \oplus (GL(8, \mathcal{K}_q))^2$

Contd...

$p^k \equiv \left\{ \begin{matrix} 17, 31, 41, 47, 55, 73, 89, \\ 97, 103, 143, 145, 167 \end{matrix} \right\} \bmod 168$	$\mathcal{K}_q^* \oplus GL(3, \mathcal{K}_{q^2}) \oplus GL(4, \mathcal{K}_{q^2}) \oplus$ $(GL(6, \mathcal{K}_q))^3 \oplus GL(7, \mathcal{K}_q) \oplus (GL(8, \mathcal{K}_q))^2$
$p^k \equiv \left\{ \begin{matrix} 5, 13, 19, 59, 61, 83, 101, \\ 115, 125, 131, 139, 157 \end{matrix} \right\} \bmod 168$	$\mathcal{K}_q^* \oplus GL(3, \mathcal{K}_{q^2}) \oplus GL(4, \mathcal{K}_{q^2}) \oplus GL(6, \mathcal{K}_q) \oplus$ $GL(6, \mathcal{K}_{q^2}) \oplus GL(7, \mathcal{K}_q) \oplus (GL(8, \mathcal{K}_q))^2$

4. Conclusion

We have examined the units of the SSG of $SL(2, \mathbb{Z}_7)$. To accomplish this, we calculated the decomposition by employing various results from group algebra theory and a key result from representation theory. It is evident that as the group's size increases, it becomes more challenging to uniquely determine the decomposition, given the numerous potential choices for the Wedderburn components.

References

- [1] N. Arvind and S. Panja, "The unit group of $K_q SL(3, 2)$, $p > 11$," arXiv:2106.07261.
- [2] Y. Bai, Y. Li, and J. Peng, "Unit groups of finite group algebras of abelian groups of order 17 to 20," *AIMS Math.*, vol. 6, pp. 7305–7317 (2021).
- [3] G. K. Bakshi, S. Gupta, and I. B. S. Passi, "The algebraic structure of finite metabelian group algebras," *Comm. Algebra*, vol. 43, pp. 2240–2257 (2015).
- [4] A. Bovdi and J. Kurdics, "Lie properties of the group algebra and the nilpotency class of the group of units," *J. Algebra*, vol. 212, pp. 28–64 (1999).
- [5] C. Dietzel and G. Mittal, "Summands of finite group algebras," *Czech. Math. J.*, vol. 71, pp. 1011–1014 (2021).
- [6] R. Ferraz, "Simple components of center of $\Phi G/J(\Phi G)$," *Comm. Algebra*, vol. 36, pp. 3191–3199 (2008).
- [7] T. Hurley, "Convolutional codes from units in matrix and group rings," *Int. J. Pure Appl. Math.*, vol. 50, pp. 431–463 (2009).
- [8] N. S. Jasim, H. H. Luaibi, and R. N. Majeed, "Computations for the special linear group $(2, 49)$," *J. Inter. Math.*, vol. 24, pp. 1677–1683 (2021).

- [9] M. Khan, R. K. Sharma, and J. Srivastava, "The unit group of FS_4 ," *Acta Math. Hungar.*, vol. 118, pp. 105–113 (2008).
- [10] S. Maheshwari and R. K. Sharma, "The unit group of the group algebra $K_qSL(2, Z_3)$," *J. Algebra Comb. Discrete Appl.*, vol. 3, pp. 1–6 (2015).
- [11] N. Makhijani and R. K. Sharma, "The unit group of algebra of circulant matrices," *Int. J. Group Theory*, vol. 3, pp. 13–16 (2014).
- [12] N. Makhijani, R. K. Sharma, and J. B. Srivastava, "The unit group of some special semisimple group algebras," *Quaest. Math.*, vol. 39, pp. 9–28 (2016).
- [13] C. P. Milies and S. K. Sehgal, *An Introduction to Group Rings*, Springer, Dordrecht (2002).
- [14] G. Mittal and R. K. Sharma, "On unit group of finite group algebras of non-metabelian groups up to order 72," *Math. Bohemica*, vol. 146, pp. 1–27 (2021).
- [15] G. Mittal and R. K. Sharma, "Computation of Wedderburn decomposition of group algebras from their subalgebra," *Bull. Korean Math. Soc.*, vol. 59, pp. 781–787 (2022).
- [16] G. Mittal and R. K. Sharma, "Unit group of semisimple group algebras of some non-metabelian groups of order 120," *Asian-European J. Math.*, vol. 15, 2250059 (2022).
- [17] R. K. Sharma and G. Mittal, "On the unit group of a semisimple group algebra $K_qSL(2, Z_5)$," *Math. Bohemica*, vol. 147, pp. 1–10 (2022).
- [18] R. K. Sharma, J. B. Srivastava, and M. Khan, "The unit group of KS_3 ," *Acta Math. Acad. Paedagog. Nyhazi (N.S.)*, vol. 23, pp. 129–142 (2007).
- [19] R. K. Sharma and P. Yadav, "Unit group of algebra of circulant matrices," *Int. J. Group Theory*, vol. 2, pp. 1–6 (2013).
- [20] S. N. U. Sivaranjani, E. Nandakumar, G. Mittal, and R. K. Sharma, "Units of the semisimple group algebras $F_qSL(2, 8)$ and $F_qSL(2, 9)$," *Annales Math. et Info.*, vol. 59, pp. 117–130 (2023).
- [21] S. N. U. Sivaranjani, E. Nandakumar, G. Mittal, and R. K. Sharma, "On the unit group of the semisimple group algebra $K_qGL(2, Z_5)$," *J. Inter. Math.*, vol. 27, pp. 121–134 (2024).
- [22] G. Tang, Y. Wei, and Y. Li, "Unit groups of group algebras of some small groups," *Czech. Math. J.*, vol. 64, pp. 149–157 (2014).

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