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# On $z^S$-ideals of commutative rings

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**Abstract**

Let $R$ be a commutative ring with nonzero unity and $S$ a multiplicatively closed subset (m.c.s) of $R$. An $S$-disjoint ideal of $R$ is called $z^S$-ideal if there exists $s = s_1 \in S$ and whenever $M_S(a) \subseteq M_S(b)$ for some $a \in I$ and $b \in R$, then $sb \in I$, where $M_S(a) = \{Max_S(R) : sa \in M$ for some $s \in S\}$, for each $a \in R$. In the present paper, we introduce and study the concept of $z^S$-ideal which is an extension of the notion of the $z$-ideals. We generalize some well-known results on $z$-ideal to $z^S$-ideal. In addition, many examples, characterizations, and properties of $z^S$-ideal are given. We also investigate the relation between $z$-ideals and $z^S$-ideals of a commutative ring.

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## 1. Introduction

Throughout this article, the term "ring" refers to a commutative ring with nonzero unity. Recently, in 2002, Anderson and Dumitrescu introduced the notion of $S$-noetherian ring which is considered as an extension of the Noetherian concept (see [3]). After that, various authors focused on this generalization and gave the $S$-version of some well-known properties of these rings. After drawing attention, many concepts in commutative algebra were extended to their $S$-version as the $S$-noetherian, for example, we mention the prime ideals, maximal ideals, 2-absorbing ideals, multiplication modules (see [7, 2, 13]). Note that many other concepts in commutative algebra have also been recently generalized and

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well studied, as the concept of $r$-ideals, 2-absorbing semiprimary ideals, etc. (see [10, 12]). In the light of the above, we define and explore the $S$-version of the notion of $z$-ideals. A proper ideal $I$ of a ring $R$ is referred to as a $z$-ideal if for every $x \in I$ and $y \in R$ such that $x$ and $y$ are contained in the same set of maximal ideals, then also $y \in I$. This concept was introduced by "Kohls" in 1957 (see [8]) during the examination of the ring known as $C(X)$, the ring of real-valued continuous functions on a topological space $X$, and it has been expanded, by Mason, to an arbitrary ring in 1973 (see [9]). We mention that, for years, there have been many generalizations on this topic. For example, see [1, 6]. Note that in this article, our focus is on investigating the notion of the $z^S$-ideal of an arbitrary ring, but the case of the ring $C(X)$ can be a very interesting topic to be studied. Presently, we give some notations which will be utilized all through this article and can be found in [4]. For an ideal $I$ of a ring $R$, we denote by $U(R)$, $Max(R)$, $Spec(R)$, and $Min(I)$ the set of all units in $R$, maximal ideals, prime ideals, and minimal ideals over $I$, respectively. For every ideal $K$ of $R$, we define the set $(I:K) = \{r \in R : rK \subseteq I\}$. Clearly, $(I:K)$ is an ideal of $R$. If $K = xR$, we use $(I:x)$ instead of $(I:K)$. For a m.c.s $S$ of a ring $R$, we denote by $I:S = \{x \in R : ux \in I \text{ for some } u \in S\}$. For an element $x$ of R, we define $M_x$ (resp., $\mathcal{M}(x)$) as the intersection (resp., set) of the maximal ideals that contain the element $x$. Recall from [7] that an $S$-disjoint ideal $P$ is termed $S$-prime, if there exists $u \in S$ such that for every $x, y \in R$ if $xy \in I$, we have $ux \in I$ or $uy \in I$. Also, recall from [14] that an $S$-disjoint ideal $M$ of $R$ is called an $S$-maximal ideal if there exists $s \in S$ such that for each $S$-disjoint ideal $N$ containing $M$ we have $sN \subseteq M$. By $Max_S(R)$ (resp., $Spec_S(R)$) we mean the set of the $S$-maximal (resp., $S$-prime) ideals of $R$. Let $x$ be an element of $R$, we define $\mathcal{M}_S(x) = \{M \in Max_S(R) : sx \in M \text{ for some } s \in S\}$. Now, we will give the principal definition of this article. An $S$-disjoint ideal of $R$ is said to be $z^S$-ideal if there exists $s = s_I \in S$ such that if $\mathcal{M}_S(a) \subseteq \mathcal{M}_S(b)$, where $a \in I$ and $b \in R$, then $sb \in I$, (see Proposition 2.5 for an equivalent definition). Among other results of this article, Section 2 considers the fundamental properties of $z^S$-ideals. It is well known that the assertion "$I$ is a $z$-ideal" does not imply, in general, that "$IR_S$ is a $z$-ideal" even if $I \cap S = \emptyset$. also "$IR_S$ is a $z$-ideal" does not also imply that "$I$ is a $z$-ideal". So, a question that naturally arises here is, what condition on $I$ to get $IR_S$ is a $z$-ideal of $R_S$. For this reason, we prove in Theorem 2.17 that $I$ is a $z^S$-ideal of $R \Leftrightarrow IR_S$ is a $z$-ideal of $R_S$, and there exists $t \in S$ such that $I:S = I:t$. In section 3, we introduce the concept of $S$-radical ideals. In fact, an $S$-disjoint ideal $I$ is called $S$-radical if there exists $s \in S$ such that $\sqrt{I:S} = I:s$. We show that

the $z^S$-ideals are $S$-radicals, (Proposition 3.4). We turn of this paper, by the section 4, with a study of the direct and inverse image of a $z^S$-ideal by a ring homomorphism. In fact, we prove that if $f$ is a surjective homomorphism from $R$ to $R'$ and $S'$ be a m.c.s of $R'$, then the inverse image of a $z^{S'}$-ideal of $R'$ by $f$ is a $z^S$-ideal of $R$, where $S = f^{-1}(S') = \{r \in R : f(r) \in S'\}$. (see Proposition 4.3). Moreover, for the direct image, we start by defining the $S$-strict homomorphism. A ring homomorphism $f : R \to R'$ is called $S$-strict if it is surjective and if $a \in R \setminus S^*$ then $f(a) \in R' \setminus f(S)$. Then, it was shown that for a $z^S$-ideal $I$ and a $S$-strict homomorphism $f : R \to R'$ with $S \cap ker(f) = \varnothing$, we have $f(I)$ is a $z^{f(S)}$-ideal of $R'$ (see Proposition 4.6). Throughout this paper, $R$ means a ring and $S$ a m.c.s of the ring $R$.

## 2. Basic properties of $z^S$-ideals

We start with the following definition.

**Definition 2.1 ([14]):** Let $M$ be an $S$-disjoint ideal of $R$, then $M$ is called an $S$-maximal ideal if there exists $u \in S$ such that for each $S$-disjoint ideal $N$ containing $M$ we have $uN \subseteq M$.

**Example 2.2:** Let $R = \mathbb{Z} \times \mathbb{Z}$ and $S = \mathbb{Z} \times \mathbb{Z}^*$. Clearly, $S$ is a m.c.s and every $S$-disjoint ideal is $S$-maximal. Indeed, every $S$-disjoint ideal $I$ of $R$ is of the form $n\mathbb{Z} \times (0)$ for some integer $n$. Take $s = (n, 0) \in S$ and assume that there exists an $S$-disjoint ideal $J$ of $R$ such that $I \subseteq J$. We can write $J = m\mathbb{Z} \times (0)$ for some integer $m$. Then we can easily get $sJ \subseteq I$. For example, we can see that $4\mathbb{Z} \times (0)$ is an $S$-maximal ideal that is not maximal.

It was proved that if $Q$ is $S$-prime, then there exists $s \in S$ such that $(Q : s') \subseteq (Q : s)$ for each $s' \in S$ (or, equivalently, $Q : S = Q : s$ for some $s \in S$), that is to say, $(Q : s)$ is the maximal element of the family $\{(Q : s')\}_{s' \in S}$, ([11], Lemma 2.16(i)). Since every $S$-maximal ideal is $S$-prime ([14], Proposition 10), the previous property is also true for the $S$-maximal ideal. As noted in [14], for $Q \in Spec_S(R)$ (resp., $M \in Max_S(R)$), we denote this $s \in S$ by $s_Q$ (resp., $s_M$). Also, we will show that this $s \in S$ is not necessarily unique even for the $S$-maximal ideals (see Remark 2.10) and the $s_Q$ or $s_M$ will be signified one of the $s \in S$ such that $(Q : s)$ is the maximal element of $\{(Q : s')\}_{s' \in S}$.

**Notation 2.3:** Let $x$ be an element of a ring $R$. We denote by $\mathcal{M}_S^*(x) = \{(M : s_M) : M \in \mathcal{M}_S(x)\}$ and $M_x^{*S} = \cap \mathcal{M}_S^*(x)$. Also, by $H_S(R)$ we mean the set of all prime ideals $P$ maximal with the property $P \cap S = \varnothing$. Furthermore,

we denote by $H_S(x)$ the set of elements of $H_S(R)$ containing $x$, and $H_{S,x} = \cap H_S(x)$.

**Lemma 2.4:** For every $a \in R$, $H_S(a) = \mathcal{M}_S^*(a)$, and so $M_a^{*S} = H_{S,a}$.

***Proof:*** First, note that $Max(R_S) = \{PR_S : P \in H_S(R)\}$. Let $P \in H_S(a)$, it is enough to show that $P \in \mathcal{M}_S(a)$. Let $J$ be an $S$-disjoint ideal containing $P$, then $PR_S \subseteq JR_S$. Because $PR_S \in Max(R_S)$ and $JR_S$ is proper, we have $PR_S = JR_S$, then $P : S = J : S$, it results that $J \subseteq P : S = P$. Conversely, let $P \in \mathcal{M}_S^*(a)$, then $P = (M : s_M)$ for some $M \in \mathcal{M}_S(a)$. Clearly, $a \in P$, Since $P \cap S = \varnothing$, $PR_S \subseteq QR_S$, for some $Q \in H_S(R)$, then $M \subseteq Q$. Because $M \in Max_S(R)$, we have $s_M Q \subseteq M$, thus $Q = P$, and $P \in H_S(a)$.

**Proposition 2.5:** The following are equivalent for an $S$-disjoint ideal $I$ of $R$:

1. $\exists s = s_I \in S$ such that if $M_a^{*S} = M_b^{*S}$, where $a \in I$ and $b \in R$, then $sb \in I$.
2. $\exists s = s_I \in S$ such that if $\mathcal{M}_S^*(a) = \mathcal{M}_S^*(b)$, where $a \in I$ and $b \in R$, then $sb \in I$.
3. $\exists s = s_I \in S$ such that if $\mathcal{M}_S(a) = \mathcal{M}_S(b)$, where $a \in I$ and $b \in R$, then $sb \in I$.
4. $\exists s = s_I \in S$ such that if $\mathcal{M}_S(a) \subseteq \mathcal{M}_S(b)$, where $a \in I$ and $b \in R$, then $sb \in I$.
5. $\exists s = s_I \in S$ such that if $\mathcal{M}_S^*(a) \subseteq \mathcal{M}_S^*(b)$, where $a \in I$ and $b \in R$, then $sb \in I$.
6. $\exists s = s_I \in S$ such that $sM_a^{*S} \subseteq I$, for all $a \in I$.
7. $\exists s = s_I \in S$ such that $sH_{S,a} \subseteq I$, for all $a \in I$.

***Proof:*** 1) $\Rightarrow$ 2). It is readily seen that $\mathcal{M}_S^*(a) = \mathcal{M}_S^*(b) \Rightarrow M_a^{*S} = M_b^{*S}$.

2) $\Rightarrow$ 3). It is clear.

3) $\Rightarrow$ 4). If $\mathcal{M}_S(a) \subseteq \mathcal{M}_S(b)$, where $a \in I$ and $b \in R$, then $\mathcal{M}_S(ab) = \mathcal{M}_S(b)$. Indeed, let $M \in \mathcal{M}_S(ab)$, then $s'ab \in M$ for some $s' \in S$. Used ([11], Lemma 2.16(i)), either $s_M a \in M$ or $s_M b \in M$. Therefore, $M \in \mathcal{M}_S(a) \cup \mathcal{M}_S(b) = \mathcal{M}_S(b)$. The converse is clear. Now, by hypothesis we have $sb \in I$, as desired.

4) $\Rightarrow$ 5). If $\mathcal{M}_S^*(a) \subseteq \mathcal{M}_S^*(b)$, then $\mathcal{M}_S(a) \subseteq \mathcal{M}_S(b)$. To see this, take $Q \in \mathcal{M}_S(a)$, then $(Q : s_Q) \in \mathcal{M}_S^*(a)$ and thus $(Q : s_Q) = (N : s_N)$, for some $N \in \mathcal{M}_S(b)$. Therefore, $s'b \in N$ for some. But $s_Q N \subseteq Q$, then $s_Q s'b \in Q$, and $Q \in \mathcal{M}_S(b)$.

5) $\Rightarrow$ 6). Let $a \in I$ and $x \in M_a^{*S}$. First, we prove that $\mathcal{M}_S^*(a) \subseteq \mathcal{M}_S^*(x)$. For, let $(Q : s_Q) \in \mathcal{M}_S^*(a)$, where $Q \in \mathcal{M}_S(a)$, then $x \in (Q : s_Q)$ and so $s_Q x \in Q$, thus $Q \in \mathcal{M}_S(x)$, this implies that $(Q : s_Q) \in \mathcal{M}_S^*(x)$. Now, using hypothesis, we conclude that $sx \in I$. Therefore, $sM_a^{*S} \subseteq I$, as desired.

6) $\Rightarrow$ 7) $\Rightarrow$ 1). Using the previous lemma.

**Definition 2.6:** An $S$-disjoint ideal $I$ of a ring $R$ is called $z^S$-ideal if it satisfies one of the equivalent conditions of Proposition 2.5.

**Remark 2.7:**

(i) Clearly, the $S$-disjoint maximal ideals are $S$-maximal. Also, the $S$-maximal ideals are a $z^S$-ideals.

(ii) If $S \subseteq U(R)$, then the set of $z$-ideals is equal to the set of $z^S$-ideal. Indeed, it is not difficult to prove that

$$Max_S(R) = Max(R) \text{ and so } \mathcal{M}_S(a) = \mathcal{M}(a), \text{ for each } a \in R.$$

**Example 2.8:** In Example 2.2, we prove that every $S$-disjoint ideal of $R$ is a $S$-maximal ideal, and so a $z^S$-ideal. Then, according to [5], $4\mathbb{Z} \times (0)$ is an $z^S$-ideal distinct from being a $z$-ideal.

**Example 2.9:** Let $D$ be a principal ideal domain (PID), $\mathbb{P}(D)$ a system of irreducible elements, $p \in \mathbb{P}(D)$, and $S = \{p^n : n \in \mathbb{N}\}$ a m.c.s of $D$. Then:

1. $I = aD$ is $S$-disjoint, where $a = p^\alpha q_1^{n_1} \cdots q_s^{n_s}$, if and only if $s \geq 1$.
2. $Max_S(D) = \{p^\alpha qD : \alpha \in \mathbb{N}, q \in \mathbb{P}(D), \text{ and } p \neq q\}$.
3. The $z^S$-ideal of $D$ are of the form $aD$, where $a = p^\alpha q_1 \cdots q_s$, $s \geq 1$, $\alpha \in \mathbb{N}$, and the $q_i$ are in $\mathbb{P}(D) \setminus \{p\}$.

***Proof:***

1. Obvious.
2. Let $q \in \mathbb{P}(D)$ such that $q \neq p$, and $M = p^\alpha qD$. By 1., we have $M \cap S = \varnothing$. Now, assume that $M \subseteq J$ for some $S$-disjoint ideal $J$. Based on a irreducibility argument we can easily show that $J = p^\beta qD$ and $\beta \leq \alpha$, therefore $p^\alpha J \subseteq M$. Also, one can easily show that the $S$-maximal ideals are of the previous form.
3. First of all, by 2., we can easily see that for $a = p^\alpha q_1^{n_1} \cdots q_s^{n_s} \in D$ we have $\mathcal{M}_S(a) = \{p^\beta q_i D : \beta \in \mathbb{N}, \text{ and } 1 \leq i \leq s\}$. Also, for every $S$-maximal

ideal $M = p^{\alpha}qD$, we can suppose that $s_M = p^{\alpha}$, and so $(M : s_M) = qD$. Then, $M_a^{*S} = q_1 \cdots q_s D$.

Now, for the "if" part, let $I = p^{\alpha}q_1 \cdots q_s D$, where the $q_i \in \mathbb{P}(D) \setminus \{p\}$, and $\alpha \in \mathbb{N}$. Clearly, $I \cap S = \emptyset$. Take a nonzero element $a \in I$, then $a = p^{\alpha}q_1 \cdots q_s d$ for some $d \in D$. by the above, $M_a^{*S} = q_1 \cdots q_s rD$ for some $r \in D$, hence, $p^{\alpha}M_a^{*S} = p^{\alpha}q_1 \cdots q_s rD \subseteq I$, and consequently, it proves $I$ is a $z^S$-ideal of $D$.

For the "only if" part, let $I = dD$ be a nonzero $z^S$-ideal of $D$, where $d = p^{\alpha}q_1^{n_1} \cdots q_s^{n_s}$, the $q_i \in \mathbb{P}(D) \setminus \{p\}$, $\alpha \in \mathbb{N}$, the $n_i \in \mathbb{N}^*$, and $s \geq 1$, because $I \cap S = \emptyset$. We will show that each $n_i = 1$. Since $d \in D$, we have $s_I M_d^{*S} \subseteq I$, then $s_I q_1 \cdots q_s \in I$, and consequently, $n_i = 1$ for each $1 \leq i \leq s$.

**Remark 2.10:** Now, we will show that the $s_I$ or the $s_M$, as in the definition of the $z^S$-ideal and the $S$-maximal ideal, are not unique in general. Like in the previous example, let $M = p^{\alpha}qD$ be an $S$-maximal ideal. It is clear that $M : p^{\alpha} = qD$, also if $\alpha' \geq \alpha$, we have $(M : p^{\alpha}) = (M : p^{\alpha'})$. Now, one can see that for each integer $j$, we obtain $(M : p^j) \subseteq (M : p^{\alpha}) = qD$.

Now, we focus on the intersection of $z^S$-ideals.

**Proposition 2.11:** Then the following statements hold:

1. If $S$ is finite, then the intersection of $z^S$-ideals is also a $z^S$-ideal.
2. If $I_1, \cdots, I_n$ are $z^S$-ideals, then $I = \cap_{i=1}^{n} I_i$ is a $z^S$-ideal.

***Proof:*** It is straightforward.

**Definition 2.12 ([7]):** A m.c.s $S$ is called strongly-multiplicatively set if for every family $(s_{\alpha})_{\alpha \in \Lambda}$ of elements of $S$ we have

$$(\cap_{\alpha \in \Lambda} s_{\alpha} R) \cap S \neq \emptyset$$

**Proposition 2.13:** If $S$ is a strongly-multiplicatively closed subset, then the intersection of $z^S$-ideals is a $z^S$-ideal.

***Proof:*** Let $(I_{\alpha})_{\alpha \in \Lambda}$ be a family of $z^S$-ideals of $R$ and $I = \cap_{\alpha \in \Lambda} I_{\alpha}$. For each $\alpha \in \Lambda$, there exists $s_{\alpha} \in S$ such that whenever $a \in I_{\alpha}$ we have $s_{\alpha}M_a^{*S} \subseteq I_{\alpha}$. By hypothesis, we have $(\cap_{\alpha \in \Lambda} s_{\alpha} R) \cap S \neq \emptyset$. Pick an element $t$ of this intersection, and let $a \in I$. For each $\alpha \in \Lambda$ we have $s_{\alpha}M_a^{*S} \subseteq I_{\alpha}$, then $tM_a^{*S} \subseteq I_{\alpha}$. Thus $tM_a^{*S} \subseteq I$. Therefore, $I$ is a $z^S$-ideal of $R$.

**Example 2.14:** The assumption "S is a strongly-multiplicatively closed subset" is not necessary for the previous proposition. Indeed, let $S = \{2^i : i \in \mathbb{N}\}$. Then $S$ is a m.c.s of $\mathbb{Z}$ but not even anti-Archimedean. In fact, suppose that $(\cap_{i\in\mathbb{N}} 2^i\mathbb{Z}) \cap S \neq \varnothing$ and let $j \in \mathbb{N}$ such that $2^j \in \cap_{i\in\mathbb{N}} 2^i\mathbb{Z}$. Now, take $j_0 > j$, then by hypothesis, $2^j \in 2^{j_0}\mathbb{Z}$, which is a contradiction. Also, put $I_p = p\mathbb{Z}$, where $p$ is a prime number other than 2. By the preceding example, we have $I_p$ is a $z^S$-ideal. Furthermore, $\cap_{p\geq 3;prime} p\mathbb{Z} = (0)$, and $(0)$ is a $z^S$-ideal of $\mathbb{Z}$, since $M_0^{*S} = \cap_{M\in Max_S(R)} (M : s_M) \subseteq \cap_{p\geq 3;prime} p\mathbb{Z} = (0)$.

It was shown that, if $I_1$ and $I_2$ are $z$-ideals, $I_1I_2$ is a $z$-ideal $\Leftrightarrow$ $I_1I_2 = I_1 \cap I_2$ (see [9]). The following proposition explores a result similar to the one mentioned.

**Proposition 2.15:** For two $z^S$-ideals $I$ and $J$, we have: $IJ$ is a $z^S$-ideal of $R \Leftrightarrow s(I \cap J) \subseteq IJ$ for some $s \in S$.

***Proof:*** ($\Rightarrow$) By hypothesis, there exists $s \in S$ such that $sM_a^{*S} \subseteq IJ$, for all $a \in IJ$. Let $a \in I \cap J$, then $a^2 \in IJ$. This signifies that $sM_{a^2}^{*S} \subseteq IJ$. But $M_{a^2}^{*S} = M_a^{*S}$, hence $sa \in IJ$.

($\Leftarrow$) Since $I$ (resp., $J$) is a $z^S$-ideal, let $s_I \in S$ (resp., $s_J \in S$) such that for each $a \in I$ (resp., $a \in J$) we have $s_IM_a^{*S} \subseteq I$ (resp., $s_JM_a^{*S} \subseteq J$), and let $a \in IJ$, then $s_Is_JM_a^{*S} \subseteq I \cap J$, then $uM_a^{*S} \subseteq IJ$ with $u = ss_Is_J$. Therefore, $IJ$ is a $z^S$-ideal.

Remember that the saturation $S^*$ of $S$ is characterized by $S^* = \{x \in R : \frac{x}{1}$ is invertible in $S^{-1}R\}$. One can easily see that $S^*$ is a m.c.s, and $x \in S^* \Leftrightarrow \exists\, y \in R$ such that $xy \in S$.

**Proposition 2.16:** $I$ is a $z^S$-ideal of $R \Leftrightarrow I$ is a $z^{S^*}$-ideal.

***Proof:*** It is clear, since $R_S = R_{S^*}$, and the two rings have the same prime spectrum.

**Question 1:** Let $S_1 \subseteq S_2$ be two m.c.s and $I$ is a $z^{S_1}$-ideal of $R$ such that $I \cap S_2 = \varnothing$, it is true that $I$ is a $z^{S_2}$-ideal?

In general, the assertion "$I$ is a $z$-ideal" does not imply that "$IR_S$ is a $z$-ideal" even if $I \cap S = \varnothing$. For example, clearly $(0)$ is a $z$-ideal of $\mathbb{Z}$. Take a nonzero prime ideal $P$, and $S = A \setminus P$, then the nonzero ideal $PR_P$ represent the unique $z$-ideal of $R_P$. Also, "$IR_S$ is a $z$-ideal" does not also imply that "$I$ is a $z$-ideal". For example, let $(R, M)$ be a local ring, and assume that $P \subset M$ is a prime ideal. Take $S = R \setminus P$, then $PR_P$ is the

unique $z$-ideal of $R_P$ however $P$ is not a $z$-ideal of $R$. (see also Example 2.19). But we have

**Theorem 2.17:** *$I$ is a $z^S$-ideal of $R \Leftrightarrow IR_S$ is a $z$-ideal of $R_S$, and $I:S = I:t$ for some $t \in S$.*

***Proof:*** For the "only if" part, by assumption, let $s = s_I \in S$ such that for every $a \in I$, $sM_a^{*S} \subseteq I$. Let $a \in I:S$ and let $s \in S$, such that $as \in I$. Then, $s_I a \in s_I M_a^{*S} = s_I M_{sa}^{*S} \subseteq I$, thus $a \in I:s_I$. Now, we demonstrate that $IR_S$ is a $z$-ideal of $R_S$. For, let $x \in IR_S$, thus $x = \frac{a}{s}$ for some $a \in I$ and $s \in S$. Clearly $\mathcal{M}_{R_S}(\frac{a}{s}) = \{PR_S : P \in H_S(a)\}$. let $y = \frac{b}{u} \in \bigcap_{P \in H_S(a)} PR_S$, then $s_I b \in s_I \bigcap_{P \in H_S(a)} P = s_I H_{S,a} = s_I M_a^{*S} \subseteq I$. Consequently, $y = \frac{s_I b}{s_I u} \in IR_S$.

For the "if" part, let $a \in I$. One can easily show that $\mathcal{M}_{R_S}(\frac{a}{1}) = \{PR_S : P \in H_S(a)\}$. Since $IR_S$ is a $z$-ideal of $R_S$, $\bigcap_{P \in H_S(a)} PR_S \subseteq IR_S$, it results that, $(\bigcap_{P \in H_S(a)} P)\, R_S \subseteq \bigcap_{P \in H_S(a)} PR_S \subseteq IR_S$. Then $\bigcap_{P \in H_S(a)} P \subseteq I : S = I : s$, and so $sM_a^{*S} \subseteq I$.

**Corollary 2.18:**

1. *If $S$ consists of nonzerodivisors and there exists $s \in S$ such that $IR_S \cap R = (I:s)$. Then $I$ is a $z^S$-ideal of $R \Leftrightarrow IR_S$ is a $z$-ideal of.*
2. *If $S$ is finite, then $I$ is a $z^S$-ideal of $R \Leftrightarrow IR_S$ is a $z$-ideal of $R_S$.*
3. *If $P$ is $S$-prime, then $P$ is a $z^S$-ideal of $R \Leftrightarrow PR_S$ is a $z$-ideal of $R_S$.*

***Proof:*** According to the previous theorem, it suffices to prove that there exists $s \in S$ such that $I:S = I:s$, that is evident in all cases.

**Example 2.19:** Let $P \in Spec(R)$ and $S = R \setminus P$, then P is a $z^S$-ideal. Indeed, since $PR_P \in Max(R_S)$, then it is a $z$-ideal. Also, since $P$ is an $S$-prime ideal, then $P$ is a $z^S$-ideal. For example, let $(R, M)$ be an integral local domain which is not a field and $S = R \setminus (0)$, then (0) is a $z^S$ -ideal of $R$ that is not a $z$-ideal.

**Remark 2.20:** Using the $S$-maximal ideals, we can give a new description of maximal ideal of the ring $R_S$. In fact, we can prove that $Max(R_S) = \{MR_S : M \in Max_S(R)\}$.

## 3. S-Radical and $z^S$-ideals

From [14], recall that we denote by

$$\sqrt[S]{I} = \{x \in R : ux^m \in I \text{ for some } u \in S \text{ and an integer } m\}.$$

It is clear that $\sqrt[S]{I}$ is an ideal of $R$ containing $I$ and $\sqrt{I}$. Note that, as mention in [14], we have $\sqrt[S]{I} = \sqrt{(IR_S)^c}$, where $(IR_S)^c$ is the inverse image of $IR_S$ by the natural homomorphism $\pi : R \to R_S$.

Also, one can easily see that $\sqrt[S]{I} = \sqrt{I : S}$.

**Definition 3.1:** An $S$-disjoint ideal $I$ of a ring $R$ is called $S$-radical if there exists $u \in S$ such that $u\sqrt[S]{I} \subseteq I$, or, equivalently, $\sqrt{I : S} = I : u$.

**Example 3.2:** Let $R = \mathbb{Z}[X]$ and $S = \{2^n : n \in \mathbb{N}\}$.

(i) Put $I = 4XR$. Clearly, $I$ is a non radical $S$-disjoint ideal. However, $I$ is an $S$-radical ideal of R. Indeed, let $f \in \sqrt[S]{I}$, then $2^n f^m \in I$ for some $n, m \in \mathbb{N}$, thus $2^n f^m = 4Xh$ for some $h \in R$. It follows that $2^n f(0)^m = 0$, thus $f(0) = 0$, and so $f = Xf_1$ for some $f_1 \in R$. Consequently, $4f \in I$, and $4\sqrt[S]{I} \subseteq I$.

(ii) Put $I = X^3R$, we have $X \in \sqrt[S]{I}$, but for each $n \in \mathbb{N}$, $2^n X \notin I$. Hence, $I$ is not $S$-radical.

**Remark 3.3:** For a finite m.c.s $S$, every $S$-disjoint radical ideal is $S$-radical.

It is well-known that the $z$-ideals are necessarily radical. Now,

**Proposition 3.4:** Every $z^S$-ideal is $S$-radical.

***Proof:*** If $L$ is a $z^S$-ideal, then by Theorem 2.17, $LR_S$ is a $z$-ideal of $R_S$, and $L : S = L : u$ for some $u \in S$. Since every $z$-ideal is radical, then $LR_S = \sqrt{LR_S} = \sqrt{L}R_S$. It follows that $L : S = \sqrt{L} : S$. Thus, $\sqrt{L:S} = \sqrt{\sqrt{L}:S} = \sqrt{L} : S = L : S = L : u$. Consequently, $L$ is $S$-radical ideals.

The converse of the previous result is not in general true.

**Example 3.5:** Let $M, Q \in Spec(R)$ such that $Q \subset M$, and $S = R \setminus M$. Clearly, $Q$ is $S$-radical. Also, note that $QR_M$ is not a $z$-ideal of $R_M$, because $MR_M$ is the unique $z$-ideal of $R_M$. Now, using Theorem 2.17, $Q$ is not $z^S$-ideal.

**Example 3.6:**

(i) Let $R = \mathbb{Z} \times \mathbb{Z}$ and $S = \mathbb{Z} \times \mathbb{Z}^*$. We showed that each $S$-disjoint ideal is a $z^S$-ideal, and so $S$-radical, by the previous proposition.

(ii) In Example 3.2, we proved that $I$ is not $S$-radical, hence it is not $z^S$-ideal.

**Example 3.7:** If $D$ is a PID, and $S=\{p^n : n\in\mathbb{N}\}$, then the concept of $z^S$-ideals and that of $S$-radical ideals coincides. In fact, as shown in Example 2.9, an $S$-radical ideal should be of the form $I=aD$, where $a=p^{\alpha}q_1^{n_1}\cdots q_s^{n_s}$, the $q_i\in\mathbb{P}(D)\setminus\{p\}$, the $n_i\geq 1$, and $s\geq 1$. Now, assume that $n_j\geq 2$ for some $j\in\{1,\cdots,s\}$, then clearly $q_1\cdots q_s\in\sqrt{I:S}$, but for every $s\in S$, $sq_1\cdots q_s\notin I$, which is a contradiction. Thus, the $S$-radical ideal of $D$ are of the form $aD$, where $a=p^{\alpha}q_1\cdots q_s$, $s\geq 1$, $\alpha$ an integer and the $q_i\in\mathbb{P}(D)$ not equals to $p$, as the $z^S$-ideals.

**Proposition 3.8:** Every $S$-disjoint ideal minimal over a $z^S$-ideal is also a $z^S$-ideal.

***Proof:*** Since $P$ is minimal over $I$, then $PR_S\in Min(IR_S)$. Furthermore, $I$ is a $z^S$-ideal, using Theorem 2.17, $IR_S$ is a $z$-ideal of $R_S$. The conclusion is drawn from ([9], Theorem 1.1) and Theorem 2.17.

**Corollary 3.9:** *If* $(0)$ *is a* $z^S$*-ideals, then every minimal prime S-disjoint ideal is so.*

**Corollary 3.10:** *Let* $J$ *and* $P$ *are two S-disjoint ideals where* $P\in Spec(R)$ *and* $J\cap P$ *is a* $z^S$*-ideal, then one of the two ideals is a* $z^S$*-ideal.*

***Proof:*** If $P$ is not a $z^S$-ideal, pick $Q\in Min(J\cap P)$ such that $J\cap P\subseteq Q\subseteq P$. By the previous proposition, $J\subseteq Q\subset P$. Hence, $J=J\cap P$ is a $z^S$-ideal of $R$.

Now, we give some properties of the $S$-radical ideals concept. In fact, a relation between $S$-radical ideals and radical ideals is given.

**Proposition 3.11:** If $I$ is $S$-radical, then,

1. $\exists t\in S$ such that $(I:t)$ is a radical ideal.
2. $\exists t\in S$ such that $I:S=I:t$.
3. $IR_S$ is a radical ideal of $R_S$.

***Proof:*** Clear.

**Proposition 3.12:**

1. If $R$ is an integral domain (or $S$ consists of nonzerodivisors) and $JR_S\cap R=(J:u)$. Then $J$ is $S$-radical $\Leftrightarrow$ $J_S$ is radical in $R_S$.
2. If $S$ is finite, then $J$ is $S$-radical $\Leftrightarrow$ $J_S$ is radical in $R_S$.

***Proof:*** The "only if" part of 1. and 2. is ensured by the previous proposition.

1. Assume that $JR_S$ is radical. We will show that $\sqrt{J:S} = J:u$. Indeed, clearly, $\sqrt{J:S} \subseteq \sqrt{JR_S} \cap R = JR_S \cap R = J:u$, by hypothesis, and we are done.
2. It is obvious.

**Proposition 3.13:** *I* is *S*-radical in $R \Leftrightarrow I$ is $S^*$-radical.

***Proof:*** By ([14], Proposition 5) and the fact that $V_S(I) = V_{S^*}(I)$.

**Question 2:** If $S_1 \subseteq S_2$ be two m.c.s and $I$ is an $S_1$-radical ideal with $I \cap S_2 = \varnothing$, it is true that $I$ is an $S_2$-radical ideal?

## 4. Direct and inverse image of $z^S$-ideals

For $s \in S$, we denote by $\overline{s} = \pi(s)$ where $\pi : R \to R/I$ is the canonical homomorphism. We denote by $\overline{S} = \{\overline{u} \mid u \in S\}$. Clearly, if *I* is *S*-disjoint, then $\overline{S}$ is a m.c.s of $R/I$. Now, we will focus on the study of the inverse image of a $z^S$-ideal by a ring homomorphism. It is well known that *M* is a maximal ideal of a ring $R \Leftrightarrow R/M$ is a field. In the following, we will give the *S*-version of this property. But before that, we will recall from [14], the definition of the S-version of the field, namely *S*-field. In matter of fact, *R* is called an *S*-field, if the zero ideal is *S*-maximal.

**Proposition 4.1:** If *M* is an *S*-disjoint ideal of *R*. Then

$$M \text{ is } S-\text{maximal ideal} \Leftrightarrow R/M \text{ is } \overline{S}-\text{field}.$$

***Proof:*** " $\Rightarrow$ " We prove that $(\overline{0})$ is $\overline{S}$-maximal. For that, let $\mathcal{J} = I/M$ be an $\overline{S}$-disjoint ideal of $R/M$, then *I* is an *S*-disjoint ideal. Since $M \subseteq I$, then by hypothesis, $s_M I \subseteq M$. Therefore, $\overline{s_M}(I/M) = (\overline{0})$, as desired.

" $\Leftarrow$ " Clear.

**Lemma 4.2:** *Let* $\varphi : R \to R'$ *be a ring isomorphism and* $S'$ *a m.c.s of* $R'$. *Then* $S = \varphi^{-1}(S') = \{\varphi^{-1}(s') : s' \in S'\}$ *is a m.c.s of R and*

$$R \text{ is an } S-\text{field} \Leftrightarrow R' \text{ is an } S'-\text{field}$$

***Proof:*** It is straightforward.

**Proposition 4.3:** If $g$ is a surjective homomorphism from $R$ to $R'$ and $S'$ is a m.c.s of $R'$. Then the inverse image of a $z^{S'}$-ideal of $R'$ by $g$ is a $z^{S}$-ideal of $R$, where $S = g^{-1}(S') = \{r \in R : g(r) \in S'\}$.

***Proof:*** Clearly, $S$ is a m.c.s of $R$. First, we will prove that the inverse image of an $S'$-maximal ideal of $R'$ by $g$ is an $S$-maximal ideal of $R$. Consider $\pi : R' \to R'/I$ the canonical homomorphism, where $I$ an ideal of $R'$, then $\pi \circ g$ is a surjective homomorphism with $g^{-1}(I)$ as kernel. Therefore, $R/g^{-1}(I) \simeq R'/I$ by the following isomorphism

$$\begin{array}{rccc} \varphi : & R/g^{-1}(I) & \to & R'/I \\ & \tilde{a} & \mapsto & \overline{g(a)} \end{array}$$

If $M \in Max_{S'}(R')$, then according to the previous proposition, $R'/M$ is a $S'$-field. Using the previous lemma, we have $R/g^{-1}(M)$ is a $\varphi^{-1}(S')$-field. But it is easy to see that $\varphi^{-1}(S') = \tilde{S}$, this mean that $R/g^{-1}(M)$ is a $\tilde{S}$-field, that is, $g^{-1}(M) \in Max_S(R)$. Now, let $J$ be a $z^{S'}$-ideal of $R'$. Clearly, $g^{-1}(J) \cap S = \varnothing$. Let $a \in g^{-1}(J)$ and $b \in R$ such that $\mathcal{M}_S(a) \subseteq \mathcal{M}_S(b)$. We have $\mathcal{M}_{S'}(g(a)) \subseteq \mathcal{M}_{S'}(g(b))$. Indeed, let $M \in \mathcal{M}_{S'}(g(a))$, Since $g$ is surjective, there exists then $s' = g(s) \in S'$ such that $g(s)\, g(a) \in M$, hence $sa \in g^{-1}(M)$, and so $g^{-1}(M) \in \mathcal{M}_S(a)$. By assumption, $ub \in g^{-1}(M)$ for some $u \in S$, this implies that $g(ub) = g(u)\, g(b) \in M$, and so $M \in \mathcal{M}_{S'}(g(b))$. Since $g(a) \in J$ and $J$ is a $z^{S'}$-ideal of $R'$, $s_J g(b) \in J$, then $s_1 b \in g^{-1}(J)$, where $s_1 \in S$ and $g(s_1) = s_J$, then we are done.

Finally, we will pay attention to the study of the direct image of a $z^{S}$-ideal by an homomorphism. As in the previous proposition, we will examine first the $S$-maximal ideals. But even when $S = \{1\}$ or $S \subseteq U(R)$, it is well-known that the direct image of a maximal ideal is not in general maximal. For this reason, in the following, we will look for an additional condition to achieve our aim.

Let $g : R \to R'$ be a ring homomorphism and $S$ a m.c.s of $R$. It is clear that if $S \cap Ker(f) = \varnothing$, then $f(S)$ is a m.c.s of $R'$. In the subsequent part, we mean by $S$ a m.c.s of $R$ disjoint with $ker(g)$.

**Definition 4.4:** A ring homomorphism $g : R \to R'$ is called $S$-strict if it is surjective and if $x \in R \setminus S^* \Rightarrow g(x) \in R' \setminus g(S)$.

**Lemma 4.5:** If $g : R \to R'$ is a surjective ring homomorphism. Then g is S-strict $\Leftrightarrow$ For each ideal I of R, if $I \cap S = \varnothing$, then $g(I) \cap g(S) = \varnothing$.

***Proof:*** $\Rightarrow$ Assume that $g(I) \cap g(S) \neq \varnothing$, then there exists $x \in I$ such that $g(x) \in g(S)$. Hence $x \in S^*$, by hypothesis, it follows that $\exists y \in R$ such that $xy \in S$, so $xy \in S \cap I$, as desired.

$\Leftarrow$ Let $x \in R \setminus S^*$ and tack $I = xR$. Assume that $I \cap S \neq \varnothing$ and take $y \in I \cap S$, so $y = xr$ for some $r \in R$, it results that $x \in S^*$, absurd. By assumption, $g(I) \cap g(S) = \varnothing$, and so $g(x) \in R' \setminus g(S)$.

**Proposition 4.6:** If $g : R \to R'$ is a $S$-strict and $I$ a $z^S$-ideal of $R$, then $f(I)$ is a $z^{f(S)}$-ideal of $R'$.

***Proof:*** Since $I \cap S = \varnothing$, then by the previous lemma, we have $g(I) \cap g(S) = \varnothing$. Now, let $a' \in g(I)$ and $b' \in R'$ such that $\mathcal{M}_{g(S)}(a') \subseteq \mathcal{M}_{g(S)}(b')$. Write $a' = g(a)$ and $b' = g(b)$ for some $a \in I$ and $b \in R$. We will show that $\mathcal{M}_S(a) \subseteq \mathcal{M}_S(b)$. For that, let $M \in \mathcal{M}_S(a)$. There exists $s \in S$ such that if $Q$ is an $S$-disjoint ideal that containing $M$, then $sQ \subseteq M$. We have $M \cap S = \varnothing$, then by hypothesis, $g(M) \cap g(S) = \varnothing$. Furthermore, if $J$ is $g(S)$-disjoint and $g(M) \subseteq J$, then $M \subseteq g^{-1}(J)$. Also, it is easy to prove that $g^{-1}(J) \cap S = \varnothing$, hence $sg^{-1}(J) \subseteq M$, this implies that $g(s)J \subseteq g(M)$. It results that $g(M) \in Max_{g(S)}(R')$. Moreover, $ua \in M$ for some $u \in S$, then $g(u)\, g(a) \in g(M)$. According to the above, there exists $t \in S$ such that $g(t)\, g(b) \in g(M)$, thus $tb \in g^{-1}(g(M))$. But we have $M \subseteq g^{-1}(g(M))$ and $g^{-1}(g(M)) \cap S = \varnothing$, then $sg^{-1}(g(M)) \subseteq M$, thus $stb \in M$, this means that $M \in \mathcal{M}_S(b)$, as required. Since $a \in I$, we have $s_1 b \in I$, and $g(s_1)\, g(b) \in g(I)$. Consequently, $g(I)$ is a $z^{g(S)}$-ideal of $R'$.

**Lemma 4.7:** $\forall\, x \in R \setminus S^*,\ \exists N \in Max_S(R)$ *such that* $x \in N$.

***Proof:*** Put $I = xR$, so $I$ is $S$-disjoint. It is well-known that there exists an $S$-disjoint ideal $P \in Spec(R)$ such that $I \subseteq P$. Let

$$\mathcal{F} = \{P \in Spec(R) : I \subseteq P \text{ and } P \cap S = \varnothing\}.$$

$(\mathcal{F}, \subseteq)$ is inductive subset. Hence, $(\mathcal{F}, \subseteq)$ admits a maximal elements denoted by $N$, that is an $S$-maximal ideal. Indeed, $N \cap S = \varnothing$ and $I \subseteq N$. Assume that there exists an $S$-disjoint ideal $J$ of $R$ such that $N \subseteq J$, so there exists an $S$-disjoint prime ideal $P_0$ such that $J \subseteq P_0$, this implies that $I \subseteq P_0$, and so $P_0 \in \mathcal{F}$. Since $N \subseteq P_0$, then by the maximality of $N$ in $(\mathcal{F}, \subseteq)$ we have $P_0 = N$, and $J = N$.

**Proposition 4.8:**

$$x \in Jac_S(R) \Leftrightarrow \forall\, s \in S, \forall\, y \in R, s + xy \in S^*,$$

where $Jac_S(R) = \bigcap_{N \in Max_S(R)} (N : s_N)$.

***Proof:*** ($\Rightarrow$) If $x \in Jac_S(R)$, and suppose that there exists $y \in R$ and $s \in S$ such that $s + xy \notin S^*$. By the previous lemma, $s + xy \in M$ for some $M \in Max_S(R)$. However $x \in (M : s_M)$, this implies that $s_M x \in M$, hence $s_M s + s_M xy \in M$. It follows that $s_M s \in M \cap S$, a contradiction.

($\Leftarrow$) Assume that $x \notin Jac_S(R)$. There exists $N \in Max_S(R)$ such that $x \notin (N : s_N)$, this yields that $s_N x \notin N$, and so $x \notin N$, then $N \subset N + xR$ and $s_N(N + xR)$ not in $N$, then $S \cap (N + xR) \neq \emptyset$. Pick $s \in S \cap (N + xR)$, then $s = m + xa \in S$ for some $m \in N$ and $a \in R$, thus $m = s - xa \in N \cap S^*$, and so $N \cap S \neq \emptyset$, a contradiction.

Now, by the following remark, we give a class of examples of $S$-strict ring homomrphism.

**Remark 4.9:** If $\pi : R \to R/I$ is the canonical homomorphism, then

$$\pi \text{ is } S - strict \Leftrightarrow I \subseteq Jac_S(R).$$

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