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Different types of odd harmonious labeling of super subdivision of various graphs

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Abstract

If the vertices of a graph $\Gamma(\alpha, \beta)$, with $\alpha = |V(\Gamma)|$ and $\beta = |E(\Gamma)|$, can be labelled with unequal integers within the set $[0, 2\beta - 1]$, so that the edges labels generated by the sum of the labels of the end vertices modulo 2β are distinct odd numbers from the set $[1, 2\beta - 1]$. Then, we regard the graph $\Gamma(V, E)$ to be a semi-odd harmonic graph. An odd harmonious graph satisfies the additional condition that modular asthmatic is not done. A strong odd harmonious graph is defined as an odd harmonious graph whose vertices can be labelled with unequal integers from the set $[0, \beta]$. In this study, we introduce semi-odd harmonious labeling, which is an extension of odd harmonious labeling. We establish the existence of semi-odd harmonious labeling for the super subdivision of the following graphs: Y–tree, the graph $T_{g,\, g-2}$, the graph $P_g \odot K_1$, as well as the twig and crown graph. We demonstrate odd harmonious labeling for the super subdivision of the subsequent graphs: cycle, dragon graph $P_m(C_g)$, wheel graph, friendship graph Fr_g, dumbbell graph $Db_{m,g}$, fan graph and the helm graph. The presence of strong odd harmonious labeling for the super subdivision is shown in the final section.

Subject Classification: *(2010) 05C78, 05C90.*

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1. Introduction

Only graphs lacking isolated vertices, loops, or numerous edges will be taken into consideration in this paper. If a graph $\Gamma=(V,E)$ has α vertices and β edges, it is referred to as a (α,β)-graph. We follow Grose [1] for appropriate terminology and notation. Consider ξ and σ are two positive integers, we refer $[\xi,\sigma]$ to be an interval of integers a, where $\xi \le a \le \sigma$. Through our paper, the variable $j \in [1,t]$.

For many mathematical issues employed in structural models, graph theory plays a crucial role. Due to its widespread application in many fields, graph labeling has grown to be an important and rich area of research in graph theory. For instance, identifying several coding methods for protecting database management systems and communication networks, ensuring the integrity and secrecy of messages sent between group members. For further inspiring applications of graph labeling, see [2].

Numerous works organise coding using particular types of graphs with various labelling schemes. Uma Maheswari et al., [3], introduced super mean labelling coding with Fibonacci web graph. Sun flower graphs SF_n were used by Prasad et al. [4] to create a method of encoding secret communications. Additionally, by utilising GMJ coding techniques, any labelling graph can be transformed into a code (see [4] and the references therein).

A mapping that, subject to certain constraints transforms graph elements (vertices, edges, or both) into non-negative numbers is called a labeling of a graph. Rosa [5] first proposed the concept of graph labeling at the beginning of the 1960s. Following this research, many graph labeling strategies have been investigated. Edge graceful labeling was first used in 1985 by Lo [6]. Elsonbaty and Daoud [7] first suggested an edge even graceful labelling of a graph in 2017. Following that, more graphs with an edge even graceful labeling were investigated [8–10]. Zeen El Deen et al. [11, 12] established the concept of an edge $\delta-$graceful labeling of a graph G, for any positive integer δ. In [13, 14] other types of graph labelling were studied. The reader can learn about all various sorts of labeling in a lively examination of graph labeling in Gallian [15].

Harmonious graphs naturally appear in the study by Graham and Solane [16], they obtained some graphs are harmonious. An injective function $\Phi: V(\Gamma) \to [0,\beta-1]$ is called *harmonious labeling* of a (α,β)-graph provided by the induced mapping $\Phi^*: E(\Gamma) \to [0,\beta]$, given by:

$\Phi^*(uv) = [\Phi(u) + \Phi(v)] \bmod \beta,$ is a bijective function. Harmonious graph refers to a graph that allows for harmonious labeling.

It was Liang and Bai [17], who first proposed the idea of an odd harmonious labeling. If there is an injective $\Phi : V(\Gamma) \to [0, 2\beta - 1]$ such that the induced mapping $\Phi^* : E(\Gamma) \to [0, 2\beta - 1],$ is a bijective function, identified by: $\Phi^*(uv) = [\Phi(u) + \Phi(v)]$ then Γ is said to be odd harmonious.

The authors of [17] showed that there are various requirements for odd labelling of graphs to exist. Numerous studies concerning the odd harmonious labelling of various types of graphs have been published; see[18, 19]. In their investigation of various path and cycle related graphs, grid graphs, and plus graphs, Jeyanthi et al. [20–22], pushed the odd harmonious labeling of graphs away. Additionally, Jeyanthi et al. [23] studied the odd harmonious labeling of super subdivision of various uncomplicated graphs. Beatressl et al. first described even-odd harmonious graphs in [24]. The authors in [25] create more odd harmonious graphs, by combine the concepts of odd harmonious labelling with the opposite skew product.

Definition 1.1: The part consists of two vertices in complete bipartite graph $K_{2,t}$ is termed as 2- vertices part of $K_{2,t}$, while the section consists of t vertices is known as t–vertices section of $K_{2,t}$.

Definition 1.2: A super subdivision of a graph Γ, symbolised by $SSD_{(2,t)}(\Gamma)$, for any positive integer t is a distinct graph created from Γ by substituting each edge xy of the graph Γ by a complete bipartite graph $K_{2,t}$, so that the end vertices of each edge xy in Γ are combined with the two vertices of 2- vertices section of $K_{2,t}$ once eliminating the edge xy from Γ.

Illustration: Let C_g be a cycle with $V(C_g) = \{u_h, h \in [1, g]\}$. The graph $SSD_{(2,t)}(C_g)$ contains a vertex set $V[SSD_{(2,t)}(C_g)] = \{u_h, v_h^j, h \in [1, g]\}$. So, $SSD_{(2,t)}(C_g)$ includes $\alpha = gt + g$ and $\beta = 2t\,g$, see Figure 1.

Subdivision of graphs have many application in topological indices [26], Zagreb indices and Zagreb coindices [27].

2. Semi-odd harmonious labeling

Semi-odd harmonious labelling, a brand-new classification that is more inclusive than the odd harmonious labelling, is presented in this

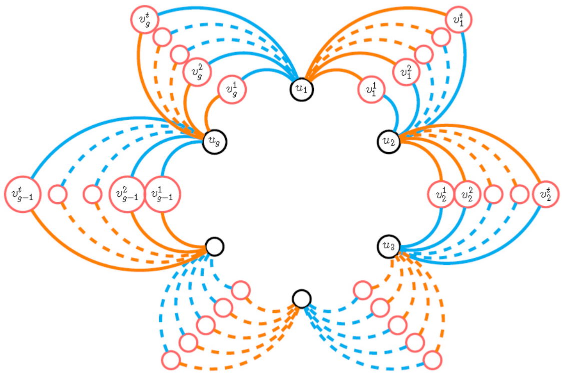

Figure 1

A cycle C_g and its $SSD_{(2,t)}(C_g)$.

section. As a result, semi-odd harmonic graphs include both odd and strong odd harmonious graphs.

Definition 2.1: An injective function $\Phi : V(\Gamma) \rightarrow \{0,1,2,3,\cdots,2\beta-1\}$ is referred to as semi-odd harmonious labeling of the graph $\Gamma(V,E)$ if the induced mapping $\Phi^* : E(\Gamma) \Rightarrow \{1,3,5,\cdots,2\beta-1\}$, given by: $\Phi^*(uv) = [\Phi(u) + \Phi(v)] \mod 2\beta$, is a bijective function. The graph that allows for semi-odd harmonic labelling will be referred to as a semi-odd harmonious graph.

A Y-tree, denoted by Y_g, is a graph procured from a path by supplementing an edge to a vertex of a path adjacent to an endpoint, where $V(Y_g) = \{u_h,\ h \in [1,g]\}$ and $E(Y_g) = \{u_h u_{h+1},\ h \in [1,\ g-2]\} \cup \{u_{g-2}u_g\}$.

Theorem 2.2: *For integer $g \geq 2$, and the Y_g tree, the super subdivision graph $SSD_{(2,t)}[Y_g]$ is semi-odd harmonious graph.*

Proof: The graph $SSD_{(2,t)}[Y_g]$ consists of $V[SSD_{(2,t)}(Y_g)] = \{u_h,\ h \in [1,\ g]\} \cup \{v_h^j,\ h \in [1,\ g-1]\}$. and edge set $E[SSD_{(2,t)}(Y_g)] = \{u_h v_h^j,\ u_{h+1} v_h^j,\ h \in [1,\ g-1]\}$. So, the graph $SSD_{(2,t)}(Y_g)$ includes $\alpha = gt + g - t$ and $\beta = 2tg - 2t$.

We specify the semi-odd harmonious labeling $\Phi : V[SSD_{(2,t)}(Y_g)] \rightarrow \{0,1,2,\cdots,\ 4tg-4t-1\}$ as follows:

$\Phi(u_1) = 4tg - 6t$,

$\Phi(u_h) = 2t\,h - 2t\,,\quad h \in [2, g-1],$

$\Phi(u_g) = 2t\,g - 2t.$

$\Phi(v_h^j) = 2[t\,h + j - t] - 1\,,\quad h \in [1, g-2]\,,$

$\Phi(v_{g-1}^j) = 2[g\,t - t + j] - 1\,.$

Considering the labelling pattern described above, the induced edge labels $\Phi^* : E[SSD_{(2,t)}(Y_g)] \to \{1, 3, 5, \cdots, 4tg - 4t - 1\}$ are

$\Phi^*(u_1 v_1^j) = 2t(2g-3) + 2j - 1,$

$\Phi^*(u_h v_h^j) = [\Phi(u_h) + \Phi(v_h^j)] \bmod 2\beta$

$= 4t(h-1) + 2j - 1\,,\qquad h \in [2, g-2]\,,$

$\Phi^*(u_{n-2} v_{g-1}^j) = 4t(g-2) + 2j - 1\,,$

$\Phi^*(u_{h+1} v_h^j) = [\Phi(u_{h+1}) + \Phi(v_h^j)] \bmod 2\beta$

$= 2t(2h-1) + 2j - 1\,,\qquad h \in [1, g-2]\,,$

$\Phi^*(u_g v_{g-1}^j) = [\Phi(u_g) + \Phi(v_{g-1}^j)] \bmod 2\beta = [4t\,g + 2j - 4t - 1] \bmod [4tg - 4t\,] = 2j - 1\,.$

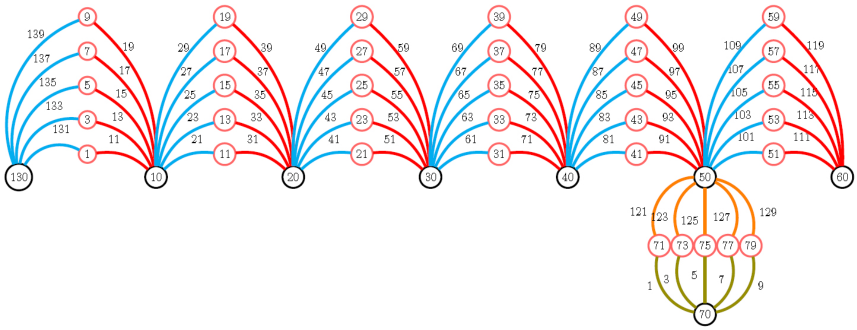

Figure 2

$SSD_{(2,5)}[Y_8]$ of the Y_8 tree with its semi-odd harmonious labeling.

The proof is complete because all of the edges' labels are odd, distinct, and located in the codomain $\{1,3,\cdots,4tg-4t-1\}$.

Illustration: The super subdivision $SSD_{(2,5)}[Y_8]$ of the Y_8 tree with its semi-odd harmonious labeling, where $t=5$ are depicted in Figure 2.

The graph $T_{g,g-2}$ gotten from a path $P_g=\{u_h, h\in[1,g]\}$ by connecting precisely one pendant edge to each inner vertex of the path P_g, then $V(T_{g,g-2}) = \{u_h,\ h\in[1,g]\} \cup \{v_h,\ h\in[1,\ g-2]\}$ and $E(T_{g,g-2}) = \{u_h u_{h+1},\ h\in[1,\ g-1],\} \cup \{u_{h+1}v_h,\ h\in[1,\ g-2]\}$.

Theorem 2.3: *For integer $g\geq 4$ and a graph $T_{g,g-2}$, the super subdivision graph $SSD_{(2,t)}[T_{g,g-2}]$ is semi-odd harmonious graph.*

Proof: The graph $SSD_{(2,t)}[T_{g,g-2}]$ is formed of the vertex set $V[SSD_{(2,t)}(T_{g,g-2}] = \{u_h,\ h\in[1,\ g]\} \cup \{u_h^j,\ h\in[1,\ g-1]\} \cup \{v_h,\ v_h^j,\ h\in[1,\ g-2]\}$. That being said, the graph $SSD_{(2,t)}\ (T_{g,g-2})]$ includes $\alpha = (2g-2)\ (t+1)\ -t$ and $\beta=2t\ (2g-3)$, see Figure 3

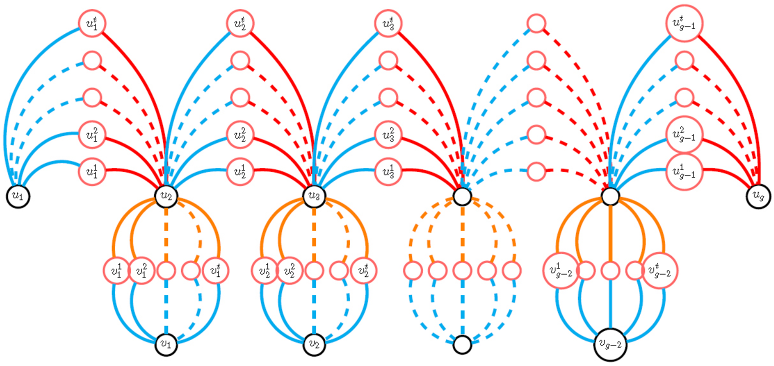

Figure 3:
$SSD_{(2,t)}[T_{g,g-2}]$

We introduce the vertex labeling function $\Phi : V[SSD_{(2,t)}\ (T_{g,g-2})] \to \{0,1,2,\ \cdots,\ 8tg\ -12t\ -1\}$ as follows:

$$\Phi(u_h)=\begin{cases}(2t+1)(h-1) & h=1,3,5,\cdots,g-1,\ \text{and } g \text{ is even;} \\ & \text{or } h=1,3,5,\cdots,g,\ \text{and } g \text{ is odd;} \\ h(2t+1)-2 & h=2,4,6,\cdots,g-2,\ \text{and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-1, \text{and } g \text{ is odd.}\end{cases}$$

$$\Phi(v_h)=\begin{cases}(2t+1)(h+1)+4t-2 & h=1,3,5,\cdots,g-3, \text{ and } g \textit{ is } \text{even;} \\ & \text{or } h=1,3,5,\cdots,g-2, \text{ and } g \textit{ is } \text{odd;} \\ h(2t+1)+2 & h=2,4,6,\cdots,g-4, \text{and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-3, \text{and } g \text{ is odd.}\end{cases}$$

When g is even

$\Phi(u_g) = (g-2)\,(2t+1)\,-4t+2$ and $\Phi(v_{g-2}) = g(2t+1)$,

$$\Phi(u_h^j)=\begin{cases}(6t-1)(h-1)+4j-3 & h=1,3,5,\cdots,g-1, \text{and } g \text{ is even;} \\ & \text{or } h=1,3,5,\cdots,g-2, \text{and } g \text{ is odd;} \\ 2t(3h-4)+4j-h-1 & h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-1, \text{and } g \text{ is odd.}\end{cases}$$

$$\Phi(v_1^j)=8(n-2)t+4j-1$$

$$\Phi(v_h^j)=\begin{cases}(6t-1)(h-1)+4(j-t)-1 & h=3,5,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ h(6t-1)+4(j-t)-3 & h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd.}\end{cases}$$

Considering the labelling pattern indicated above, the induced edge labels Φ^* are

$$\Phi^*(u_h u_h^j)=[\Phi(u_h)+\Phi(u_h^j)] \bmod 2\beta = 8t\,h-8t+4j-3, \quad h\in[1,g-1],$$

$$\Phi^*(u_{h+1}u_h^j)=\begin{cases}4t(2h-1)+4j-3 & h=1,3,5,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=1,3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 8t(h-1)+4j-1 & h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,}\end{cases}$$

$$\Phi^*(u_g u_{g-1}^j)=4t(2g-5)+4j-1, \qquad g \text{ is even,}$$

$$\Phi^*(v_1 v_1^j)=[\Phi(v_1)+\Phi(v_1^j)] \bmod (2\beta)=[8gt-8t+4j-1] \bmod (8tg-12t)=4t+4j-1,$$

$$\Phi^*(v_h v_h^j)=\begin{cases}8t\,h-4t+4j-1 & h=2,3,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=2,3,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 8t\,g+4j-4t-1 & h=g-2, \text{ and } g \text{ is even,}\end{cases}$$

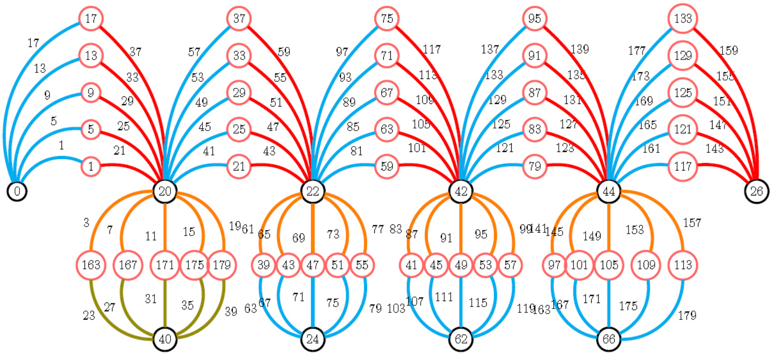

Figure 4

$SSD_{(2,5)}[T_{6,4}]$ **with its semi-odd harmonious labeling.**

$$\Phi^*(v_h^j u_{h+1}) = \begin{cases} 8t(h-1)+4j-1 & h=1,2,3,\cdots,g-3, \text{ and } g \text{ is even; or } h=1,2,3,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 4t(2h-1)+4j-3 & h=2,4,\cdots,g-2, \text{ and } g \text{ is even; or } h=2,4,\cdots,g-3, \text{ and } g \text{ is odd.} \end{cases}$$

As a result, the edges' labels are all odd, distinct, and located in the codomain $\{1,3,\cdots,8tg-12t-1\}$, which completes the proof.

Illustration: The super subdivision $SSD_{(2,5)}[T_{6,4}]$ with its semi-odd harmonious labeling is represented in Figure 4 where $t=5$.

Theorem 2.4: *For integer $g \geq 3$ and a comb graph $P_g \odot K_1$, the super subdivision $SSD_{(2,t)}[P_g \odot K_1]$ of is semi-odd harmonious graph.*

Proof: Let $P_g \odot K_1$ be the comb graph with $V(P_g \odot K_1) = \{u_h, v_h, h \in [1,g]\}$ and $E(P_g \odot K_1) = \{u_h u_{h+1}, h \in [1, g-1]\} \cup \{u_h v_h, h \in [1,g]\}$. The graph $SSD_{(2,t)}[P_n \odot K_1]$ includes $V[SSD_{(2,t)}(P_n \odot K_1)] = \{u_h, v_h, v_h^j, h \in [1, g]\} \cup \{u_h^j, h \in [1, ng-1]\}$ and $E[SSD_{(2,t)}(P_g \odot K_1)] = \{u_h u_h^j, u_{h+1} u_h^j, h \in [1, g-1]\} \cup \{u_h v_h^j, v_h v_h^j, h \in [1, g]\}$. So, the graph $SSD_{(2,t)}(P_g \odot K_1)$ contains $\alpha = 2gt + 2g - t$ and $\beta = 4tg - 2t$.

We clarify the vertex labeling function $\Phi : V[SSD_{(2,t)}(P_g \odot K_1)] \rightarrow \{0,1,2, \cdots, 8gt - 4t - 1\}$ as follows:

$$\Phi(u_h)=\begin{cases} 2t\,h+h+2t-1 & h=1,3,5,\cdots,g-1 \text{ and } g \text{ is even; or } h=1,3,5,\cdots,g, \text{ and } g \text{ is odd;} \\ 2t\,h+4t+h-2 & h=2,4,6,\cdots,g, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(v_h)=\begin{cases} 2t\,h+h-2t-1 & h=1,3,5,\cdots,g-1, \text{ and } g \text{ is even; or } h=1,3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 2t(h+2)+h-4 & h=2,4,6,\cdots,g, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(v_g)=2gt++g+t1, \quad g \text{ is odd,}$$

$$\Phi(u_1^j)=8(g-1)t+4j-1,$$

$$\Phi(u_h^j)=\begin{cases} 2t(3h-5)+4j-h & h=3,5,\cdots,g-1, \text{ and } g \text{ is even; or } h=3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 2t(3h-4)+4j-h-1 & h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(v_h^j)=\begin{cases} (6t-1)(h-1)+4j-3 & h=1,3,5,\cdots,g-1, \text{ and } g \text{ is even; or } h=1,3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ (6t-1)(h-2)+4j-1 & h=2,4,6,\cdots,g, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(v_g^j)=2t(3g-5)-(g+2)+4j, \text{ if } g \text{ is odd.}$$

Using the labelling pattern outlined above, the induced edge labels Φ^* are

$$\Phi^*(v_h v_h^j)=[\Phi(v_h)+\Phi(v_h^j)] \bmod 2\beta$$

$$\Phi^*(v_h v_h^j)=\begin{cases} 8t(h-1)+4j-3 & \text{if } h=1,2,3,\ldots, g, \text{ and } g \text{ is even;} \\ 8t(h-1)+4j-3 & \text{if } h=1,2,3,\ldots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi^*(v_g v_g^j)=8t(g-1)+4j-1, \; g \text{ is odd,}$$

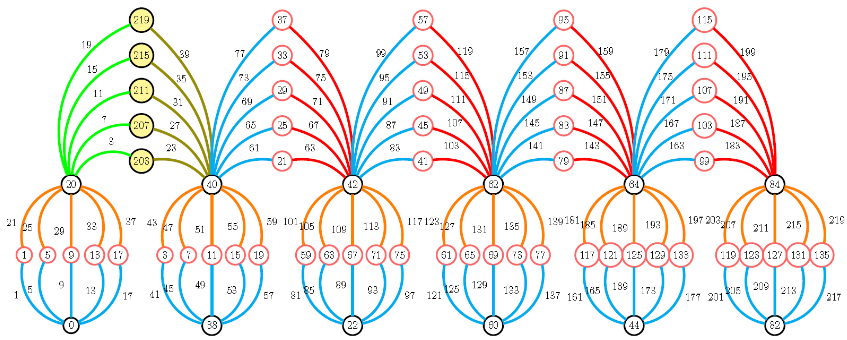

Figure 5

$SSD_{(2,t)}[P_6 \odot K_1]$ **of** $P_6 \odot K_1$ **and its semi-odd harmonious labeling.**

$$\Phi^*(u_h v_h^j) = \begin{cases} 4t(2h-1)+4j-3 & h=1,3,5,\cdots,g-1, \text{ and } g \text{ is even;} \\ & \text{or } h=1,3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 8t(h-1)+4j-1 & h=2,4,6,\cdots,g, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi^*(u_g v_g^j) = 8t(g-1)+4j-3, \quad \text{if } g \text{ is odd,}$$

$$\Phi^*(u_1 u_1^j) = [\Phi(u_1)+\Phi(u_1^j)] \bmod (2\beta) = [8gt-4t+4j-1] \bmod (8tg-4t) = 4j-1,$$

$$\Phi^*(u_h u_h^j) = \begin{cases} 8t(h-1)+4j-1 & h=3,5,\cdots,g-1, \text{ and } g \text{ is even;} \\ & \textit{or } h=3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 4t(2h-1)+4j-3 & h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \textit{or } h=2,4,6,\cdots,g-1, \text{ and } g \text{ is odd,} \end{cases}$$

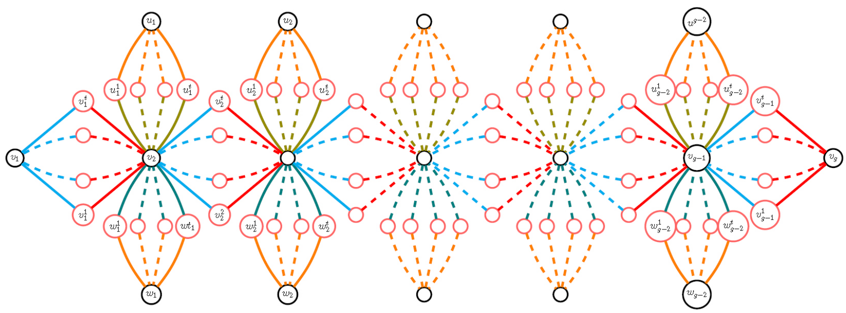

Figure 6

$SSD_{(2,t)}$ $[TW_g]$ **of the graph** TW_9**.**

$$\Phi^*(u_2u_1^j) = [\Phi(u_2) + \Phi(u_1^j)] \bmod (2\beta) = [8gt + 4j - 1] \bmod (8tg - 4t) = 4t + 4j - 1,$$

$$\Phi^*(u_{h+1}u_h^j) = 4t(2h-1) + 4j - 1, \quad h \in [1, g-1].$$

As a consequence, the edges' labels are all odd, distinct, and located in the codomain $\{1, 3, \cdots, 8tg - 4t - 1\}$ that complete the proof.

Illustration: The super subdivision $SSD_{(2,t)}[P_6 \odot K_1]$ of $P_6 \odot K_1$ and its semi-odd harmonious labeling is sketched in Figure 5, where $t = 5$.

The twig graph $TW_g, n \geq 4$ is the graph derived from a path $P_g = \{v_1, v_2, \cdots, v_g\}$ by linking precisely two pendant edges to every inside vertex of P_g, then $V(TW_g) = \{v_h,\ h \in [1,\ g]\} \cup \{u_h,\ w_h,\ h \in [1,\ g-2]\}$.

Theorem 2.5: *For integer $g \geq 4$ and a twig graph TW_g, the super subdivision $SSD_{(2,t)}\ [TW_g]$ is semi-odd harmonious graph.*

Proof: The graph $SSD_{(2,t)}\ [TW_g]$ consists of $V[SSD_{(2,t)}\ (TW_g)] = \{v_h,\ h \in [1, n]\} \cup \{v_h^j,\ h \in [1,\ g-1]\} \cup \{u_h,\ w_h,\ u_h^j,\ v_h^j,\ h \in [1,\ g-2]\}$. So, $SSD_{(2,t)}\ (TW_g)$ has $\alpha = t\ (3g-5) + (3g-4)$ and $\beta = 2t\ (3g-5)$, see Figure 6.

We indicate the vertex labeling function $\Phi : V[SSD_{(2,t)}\ (TW_g)] \to \{0, 1, 2, \cdots, 12tg - 20t - 1\}$ as follows:

$$\Phi(v_h) = \begin{cases} 4t(3g-5) - 4 & \text{for } h = 1; \\ 3h(t+1) - (6t+4) & \text{for } h = 2, 4, 6, \cdots, g-2 \text{ and } g \text{ is even;} \\ & \text{or } h = 2, 4, 6, \cdots, g-1, \text{ and } g \text{ is odd;} \\ 3h(t+1) - (3t+7) & \text{for } h = 3, 5, 7, \cdots, g-1, \text{ and } g \text{ is even;} \\ & \text{or } h = 3, 5, 7, \cdots, g-2, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(v_g) = \begin{cases} 3g(t+1) - (4t+10) & \text{for } g \text{ is even;} \\ (t+1)(3g-7) & \text{for } g \text{ is odd,} \end{cases}$$

$$\Phi(u_h) = \begin{cases} (3h-3)(t+1) & \text{for } h = 1, 3, 5, \cdots, g-3, \text{ and } g \text{ is even;} \\ & \text{or } h = 1, 3, \cdots, g-2, \text{ and } g \text{ is odd;} \\ 3h(t+1) - 6t & \text{for } h = 2, 4, 6, \cdots, n-2, \text{ and } g \text{ is even;} \\ & \text{or } h = 2, 4, 6, \cdots, g-3, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(w_1) = 4t(3g-5) - 8 \text{ and } \Phi(w_2) = 4t(3g-5) - 2,$$

$$\Phi(w_h)=\begin{cases} 3h(t+1)-(3t+11) & \text{for } h=3,5,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 3h(t+1)-(6t+8) & \text{for } h=4,6,8,\cdots,n-2, \text{ and } n \text{ is even;} \\ & \text{or } h=4,6,\cdots,n-3, \text{ and } n \text{ is odd,} \end{cases}$$

$$\Phi(v_h^j)=\begin{cases} 3h(3t-1)-9t+12j-2 & \text{for } h=1,3,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 3h(3t-1)+6(j-t)+1 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(v_{n-1}^j)=\begin{cases} 3g(3t-1)-18t+2j+9 & \text{for } g \text{ is even;} \\ 3g(3t-1)-15t+2j+6 & \text{for } g \text{ is odd,} \end{cases}$$

$$\Phi(u_h^j)=\begin{cases} 3h(3t-1)-9t+12j-4 & \text{for } h=1,3,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 3h(3t-1)+6(j-t)-1 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi(w_h^j)=\begin{cases} 3h(3t-1)-9t+12j & \text{for } h=1,3,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 3h(3t-1)+6(j-t)+3 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd.} \end{cases}$$

Using the labelling pattern outlined above, the induced edge labels Φ^* are

$$\Phi^*(v_1v_1^j)=[\Phi(v_1)+\Phi(v_1^j)] \bmod 2\beta=[4t(3g-5)+12j-9] \bmod (12tg-20t)=12j-9,1$$

$$\Phi^*(v_hv_h^j)=\begin{cases} 12t(h-1)+12j-9 & \text{for } h=3,5,\cdots,g-3, \text{ and } g \text{ is even;} \\ & \text{or } h=3,5,\cdots,g-2, \text{ and } g \text{ is odd;} \\ 12t(h-1)+6j-3 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even;} \\ & \text{or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd,} \end{cases}$$

$$\Phi^*(v_{n-1}v_{g-1}^j)=12t(g-2)+2j-1,$$

$$\Phi^*(v_{h+1}v_h^j)=\begin{cases}12t(h-1)+12j-3 & \text{for } h=1,3,\cdots,g-3, \text{ and } g \text{ is even; or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;}\\ 6t(2h-1)+6j-3 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd,}\end{cases}$$

$$\Phi^*(v_n v_{g-1}^j)=2t(6g-11)+2j-1,$$

$$\Phi^*(u_h u_h^j)=\begin{cases}12t(h-1)+12j-7 & \text{for } h=1,3,\cdots,g-3, \text{ and } g \text{ is even; or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;}\\ 12t(h-1)+6j-1 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd,}\end{cases}$$

$$\Phi^*(v_{h+1}u_h^j)=\begin{cases}12t(h-1)+12j-5 & h=1,3,\cdots,g-3, \text{ and } g \text{ is even; or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;}\\ 6t(2h-1)+6j-5 & h=2,4,\cdots,g-2, \textit{ and } g \text{ is even; or } h=2,4,\cdots,g-3, \text{ and } g \text{ is odd,}\end{cases}$$

$$\Phi^*(w_1w_1^j)=[\Phi(w_1)+\Phi(w_1^j)] \bmod 2\beta=[12tg-20t+12j-11] \bmod (12tg-20t)=12j-11,$$

$$\Phi^*(w_2w_2^j)=[\Phi(w_2)+\Phi(w_2^j)] \bmod 2\beta=[12tg-8t+6j-5] \bmod (12tg-20t)=6j+12t-5,$$

$$\Phi^*(w_h w_h^j)=\begin{cases}12t(h-1)+12j-11 & \text{for } h=3,5,\cdots,g-3, \text{ and } g \text{ is even; or } h=3,5,\cdots,g-2, \text{ and } g \text{ is odd;}\\ 12t(h-1)+6j-5 & \text{for } h=4,6,\cdots,g-2, \text{ and } g \text{ is even; or } h=4,6,\cdots,g-3, \text{ and } g \text{ is odd,}\end{cases}$$

$$\Phi^*(v_{h+1}w_h^j)=\begin{cases}12t(h-1)+12j-1 & \text{for } h=1,3,\cdots,g-3, \text{ and } g \text{ is even; or } h=1,3,\cdots,g-2, \text{ and } g \text{ is odd;}\\ 6t(2h-1)+6j-1 & \text{for } h=2,4,6,\cdots,g-2, \text{ and } g \text{ is even; or } h=2,4,6,\cdots,g-3, \text{ and } g \text{ is odd.}\end{cases}$$

As a result, the proof is complete because all of the edge labels are odd, distinctive, and located in the codomain $\{1,3,\cdots,4t(3g-5)-1\}$.

Illustration: In Figure 7 we depicted $SSD_{(2,t)}[TW_7]$ of the twig graph TW_7 with its semi-odd harmonious labeling.

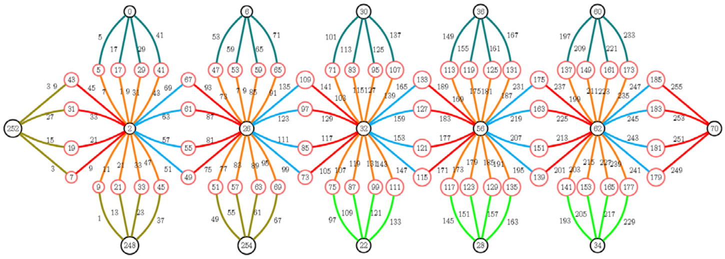

Figure 7

$SSD_{(2,t)}[TW_7]$ **of** TW_7 **with its semi-odd harmonious labeling.**

Theorem 2.6: *For* $g \equiv 2,4 \bmod 6$ *and a crown graph* Cr_g, *the super subdivision* $SSD_{(2,t)}[Cr_g]$ *is semi-odd harmonious graph.*

Proof: A crown $Cr_g = C_g \odot K_1$ [11] is formed by adding g extra pendent vertices $\{v_1, v_2, \cdots, v_g\}$ and g more edges $(u_i v_i)$, $i \in [1,g]$ for $g \geq 3$ to the cycle $C_g = \{u_1, u_2, \cdots, u_g\}$, then $\alpha[Cr_g] = \beta[Cr_g] = 2g$. The graph $SSD_{(2,t)}[Cr_g]$ consists of $V[SSD_{(2,t)}(Cr_g)] = \{u_h, u_h^j, v_h, v_h^j, h \in [1, g]\}$ and $E[SSD_{(2,t)}(Cr_g)] = \{u_h u_h^j, v_h v_h^j, h \in [1,g]\} \cup \{u_{h+1} u_h^j, h \in [1, g-1]\}. \cup \{u_1 u_n^j\} \cup \{u_k v_h^j, h \in [1,g]\}$. The graph $SSD_{(2,t)}(Cr_g)$ includes $\alpha = 2g(t+1)$ and $\beta = 4tg$.

We define the vertex labeling function $\Phi : V[SSD_{(2,t)}(Cr_g)] \to \{0,1,2, \cdots, 8tg-1\}$ as follows:

$\Phi(u_1) = 8t$, $\Phi(u_2) = 0$,

$\Phi(u_h) 8t(g+2-h)$, $\quad h \in [3,g]$,

$$\Phi(v_h) = \begin{cases} 16t-4 & \text{if } h=1 \\ 8t(3-h)+4 & \text{if } h=2,\ 3 \\ 8t(g-h+3)+4 & \text{if } h \in \left[4, \frac{g}{2}+1\right]; \\ 4t(g+2)-4 & \text{if } h = \frac{g}{2}+2; \\ 8t(g-h+3)-4 & \text{if } h \in \left[\frac{g}{2}+3, g\right] \end{cases}$$

$$\Phi(u_h^j) = \begin{cases} 8(3t\,h+j-3t)-5 & \text{if } h \in \left[1, \frac{g}{2}-1\right]; \\ 4(t\,g+2j-6)t-5 & \text{if } h = \frac{g}{2}; \\ 8(t\,g+j-h-1)-1 & \text{if } h \in \left[\frac{g}{2}+1, g-1\right]; \\ 8(t\,g+j-t)-1 & \text{if } h = g, \end{cases}$$

$$\Phi(v_h^j)=\begin{cases} 8t(g+j-1)-3 & \text{if } h=1; \\ 8t(3t\,h-6+j)-7 & \text{if } h\in\left[2,\frac{g}{2}\right]; \\ 4t(g-6)+8j-7 & \text{if } h=\frac{g}{2}+1; \\ 8t(g-h)+8j-3 & \text{if } h\in\left[\frac{g}{2}+2,g\right]. \end{cases}$$

Applying the labelling pattern presented above, then the induced edge labels Φ^* are

$$\Phi^*(u_h u_h^j)=\begin{cases} 8(t+j)-5 & \text{if } h=1; \\ 8(3t+j)-5 & \text{if } h=2; \\ 8t(2h-1)+8j-5 & \text{if } h\in\left[3,\frac{g}{2}-1\right]; \\ 8t(g-1)+8j-5 & \text{if } h=\frac{g}{2}; \\ 8t(2g-2h+1)+8j-1 & \text{if } h\in\left[\frac{g}{2}+1,g-1\right]; \\ 8t+8j-1 & \text{if } h=g \end{cases}$$

$$\Phi^*(u_{h+1} u_h^j)=\begin{cases} 8j-5 & \text{if } h=1; \\ 16t(h-1)+8j-5 & \text{if } h\in\left[2,\frac{g}{2}-1\right]; \\ 8t(g-2)+8j-5 & \text{if } h=\frac{g}{2}; \\ 8t(2g-2h)+8j-1 & \text{if } g\in\left[\frac{g}{2}+1,g-1\right], \end{cases}$$

$$\Phi^*(u_1 u_g^j)=8j-1,$$

$$\Phi^*(v_h v_h^j)=\begin{cases} 8(t+j)-7 & \text{if } h=1; \\ 8t(2h-3)+8j-3 & \text{if } h\in\left[2,\frac{g}{2}\right]; \\ 8t(g-1)+8j-3 & \text{if } h=\frac{g}{2}+1; \\ 8t(g-1)+8j-7 & \text{if } h=\frac{g}{2}+2; \\ 8t(2g-2h+3)+8j-7 & \text{if } h\in\left[\frac{g}{2}+3,g\right], \end{cases}$$

$$\Phi^*(u_h v_h^j)=\begin{cases} 8j-3 & \text{if } h=1; \\ 8j-7 & \text{if } h=2; \\ 8t(2h-4)+8j-7 & \text{if } h\in\left[3,\frac{g}{2}\right]; \\ 8t(g-2)+8j-7 & \text{if } h=\frac{g}{2}+1; \\ 16t(g-h+1)+8j-3 & \text{if } h\in\left[\frac{g}{2}+2,g\right]. \end{cases}$$

Because all of the labels on the edges are odd, distinct, and located in the codomain $\{1,3,\cdots,\ 8tg-1\}$, the proof is complete.

Illustration: Figure 8 displays a semi-odd harmonic labelling of $SSD_{(2,t)}[Cr_8]$ of the crown graph Cr_8 when $t=4$.

3. Odd harmonious labeling

Definition 3.1: [19] An injective function $\Phi : V(\Gamma) \to [0,2\beta-1]$, is said to be *odd harmonious labeling* of $\Gamma(V,E)$ if the induced mapping $\Phi^* : E(\Gamma) \to \{1,3,5,\cdots,\ 2\beta-1\}$, given by: $\Phi^*(uv) = [\Phi(u)+\Phi(v)]$, is a bijective function. So, in this case of labeling of a graph with β edges modular arithmetic is not done except for the case when the sum of two vertex labels is 2β. A graph that accepts an odd harmonious labelling is referred to an odd harmonious graph.

Theorem 3.2: *For an integer $g \geq 2$ and a path P_g, the super subdivision $SSD_{(2,t)}[P_g]$ is an odd harmonious graph for any integer $t \geq 1$.*

Proof: Consider $V(P_g) = \{u_h,\ h \in [1,\ g]\}$, the super subdivision $SSD_{(2,t)}[P_g]$ of P_g includes $V[SSD_{(2,t)}(P_g)] = \{u_h,\ h \in [1,\ n]\} \cup \{v_h^j,\ h \in [1,\ g-1]\}$. Consequently, we obtain $\alpha = t\ (g-1) + g$ and $\beta = 2t\ (g-1)$.

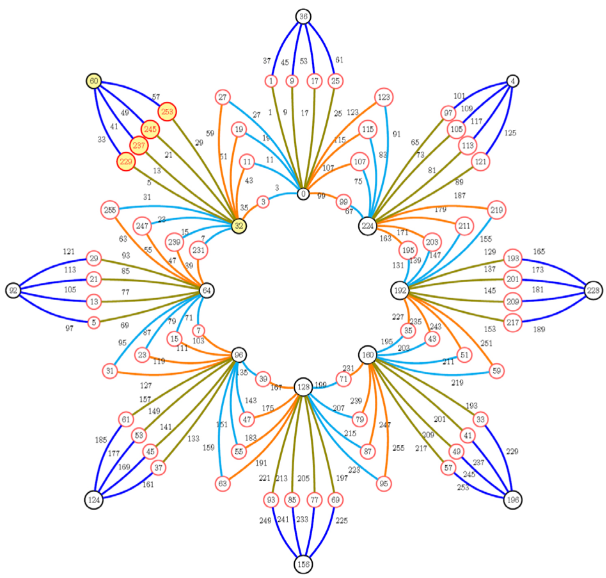

Figure 8

A semi-odd harmonious labeling $SSD_{(2,4)}[Cr_8]$ of the crown Cr_8.

The vertex labeling function $\Phi : V[SSD_{(2,t)}(P_g)] \to \{0,1,2,\cdots, 4tg - 4t - 1\}$ is proposed as follows:

$$\Phi(u_h) = 2th - 2t, \quad h \in [1, g],$$

$$\Phi(v_h^j) = 2(th - t + j) - 1 \qquad h \in [1, g-1],$$

Given the previous labelling pattern, the induced edge labels Φ^* are

$$\Phi^*(u_h v_h^j) = \Phi(u_h) + \Phi(v_h^j) = 4th + 2j - 4t - 1, \qquad h \in [1, g-1],$$

$$\Phi^*(u_{h+1} v_h^j) = \Phi(u_{h+1}) + \Phi(v_h^j) = 2th + 2j - 2t - 1, \qquad h \in [1, g-1],$$

The verification is thus finished by the labels of the edges, which are all odd, distinct, and lie in the codomain $\{1,3,\cdots, 4tg - 4t - 1\}$.

Illustration: The super subdivision $SSD_{(2,7)}[P_7]$ of P_7 and its odd harmonious labeling is depicted in Figure 9, where $t = 7$.

Theorem 3.2: *For even positive integer g and a cycle C_g, the super subdivision $SSD_{(2,t)}[C_g]$ is an odd harmonious graph for any integer $t \geq 1$.*

Proof: For a cycle C_g, the super subdivision $SSD_{(2,t)}[C_g]$ contains $\alpha = gt + g$ and $\beta = 2tg$ The vertex labelling function is defined Φ : $V[SSD_{(2,t)}(C_g)] \to \{0,1,2,\cdots, 4tg - 1\}$ as follows:

$$\Phi(u_h) = 2t(h-1), \quad h \in [1, g],$$

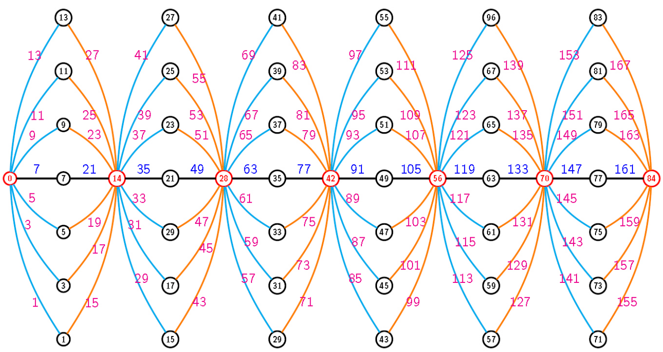

Figure 9

The graph $SSD_{(2,7)}[P_7]$ and its odd harmonious labeling.

$$\Phi(v_h^j)=\begin{cases}2ht-t+2j-2t-1 & \text{if} \quad h\in\left[1,\frac{g}{2}\right];\\ 2ht+2j-1 & \text{if} \quad h\in\left[\frac{g}{2}+1,g\right],\end{cases}$$

The induced edge labels $\Phi^*:E[SSD_{(2,t)}(C_g)]\to\{1,3,\cdots,4tg-1\}$ are

$$\Phi^*(u_hv_h^j)=\begin{cases}4th+2j-4t-1 & \text{if} \quad h\in\left[1,\frac{g}{2}\right];\\ 4th+2j-2t-1 & \text{if} \quad h\in\left[\frac{g}{2}+1,n\right],\end{cases}$$

$$\Phi^*(u_{h+1}v_h^j)=\begin{cases}4th+2j-2t-1 & \text{if} \quad h\in\left[2,\frac{g}{2}\right];\\ 4th+2j-1 & \text{if} \quad h\in\left[\frac{g}{2}+1,\ g-1\right],\end{cases}$$

$$\Phi^*(u_0v_n^j)=2tg+2j-1.$$

As a result, the modular arithmetic is not done, the labels of the edges are all odd, distinct, and exist in the codomain $\{1,3,\cdots,4tg-1\}$, which brings the proof to an end.

Illustration: $SSD_{(2,5)}[C_6]$ and its odd harmonious labeling is depicted in Figure 10.

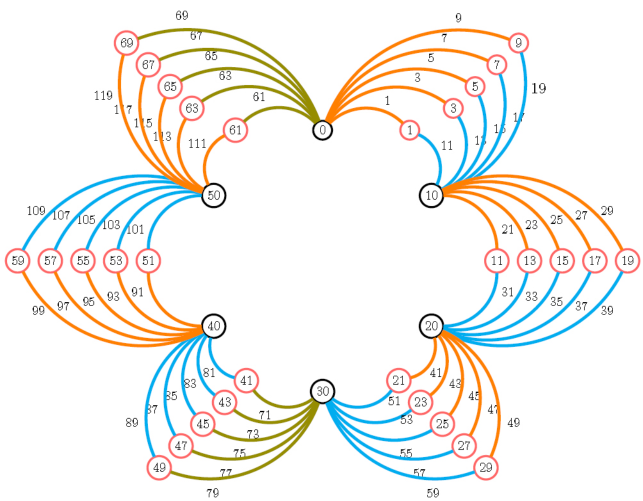

Figure 10

$SSD_{(2,5)}[C_6]$ and its odd harmonious labeling.

Theorem 3.4: *For an integers $t, m \geq 1$, any even integer $g \geq 4$ and a dragon graph $P_m(C_g)$ [28], the super subdivision $SSD_{(2,t)}[P_m(C_g)]$ is an odd harmonious graph.*

Proof: Dragon graph $P_m(C_g)$ has vertex set $\{v_h,\ h \in [1,\ m-1]\} \cup \{u_h,\ h \in [1, g]\}$. the super subdivision graph $SSD_{(2,t)}\ [P_m(C_g)]$ contains $V[SSD_{(2,t)}\ (P_m(C_g))] = \{v_h\ ,\ v_h^j,\ h \in [1,\ m-1]\} \cup \{u_h,\ u_h^j,\ h \in [1, g\,]\}$. Because of this, $SSD_{(2,t)}\ [P_m(C_g)]$ comprises $\alpha\ =\ (t+1)\ (m+g-1)$ and $\beta = 2t\ (m+g-1)$.

The vertex labelling function $\Phi : V(SSD_{(2,t)}\ [P_m(C_g)]) \to \{0,1,2, \cdots, 4t(m\ +\ g-1)\ -1\}$ is :

$$\Phi(v_h) = 2t\,(h-1)\,, \qquad h \in [1, m-1\,],$$

$$\Phi(u_h) = 2t\,m + 2th - 4t\,, \qquad h \in [1, g\,],$$

$$\Phi(v_h^j) = 2t\,h + 2j - 2t - 1\,, \qquad h \in [1, m-1\,],$$

$$\Phi(u_h^j) = \begin{cases} 2\,t\,m + 2t\,h + 2j - 4t - 1 & \text{if } \ h \in \left[1, \frac{g}{2}\right]; \\ 2t\,m + 2t\,h + 2j - 2t - 1 & \text{if } \ h \in \left[\frac{g}{2}+1, g\right], \end{cases}$$

Using the labelling pattern mentioned above, the induced edge labels Φ^* are

$$\Phi^*(v_h v_h^j) = 4t\,h + 2j - 4t - 1, \qquad h \in [1, m-1\,],$$

$$\Phi^*(v_{h+1} v_h^j) = 2t\,(2h-1) + 2j - 1\,, \qquad h \in [1, m-2\,],$$

$$\Phi^*(v_{m-1}^j u_1) = 2t\,(2m-3) + 2j - 1,$$

$$\Phi^*(u_h u_h^j) = \begin{cases} 4t\,(m+h-2) + 2j - 1 & \text{if } \ h \in \left[1, \frac{g}{2}\right]; \\ 2t\,(2m+2h-3) + 2j - 1 & \text{if } \ h \in \left[\frac{g}{2}+1, n\right], \end{cases}$$

$$\Phi^*(u_{h+1} u_h^j) = \begin{cases} 2t\,(2m+2h-3) + 2j - 1 & \text{if } \ h \in \left[1, \frac{g}{2}\right]; \\ 4t\,(m+h-1) + 2j - 1 & \text{if } \ h \in \left[\frac{g}{2}+1, g-1\right], \end{cases}$$

$$\Phi^*(u_1 u_g^j) = 2t\,(2m+g-2) + 2j - 1,$$

As a result, the proof is complete because all of the edges' labels are odd, distinct, and located in the codomain $\{1, 3, \cdots, 2\beta - 1 = 4t(m+g-1) - 1\}$.

Illustration: An odd harmonious labeling of $SSD_{(2,5)}[P_5(C_6)]$ of the dragon $P_5(C_6)$ is depicted in Figure 11, where $t=5$. .

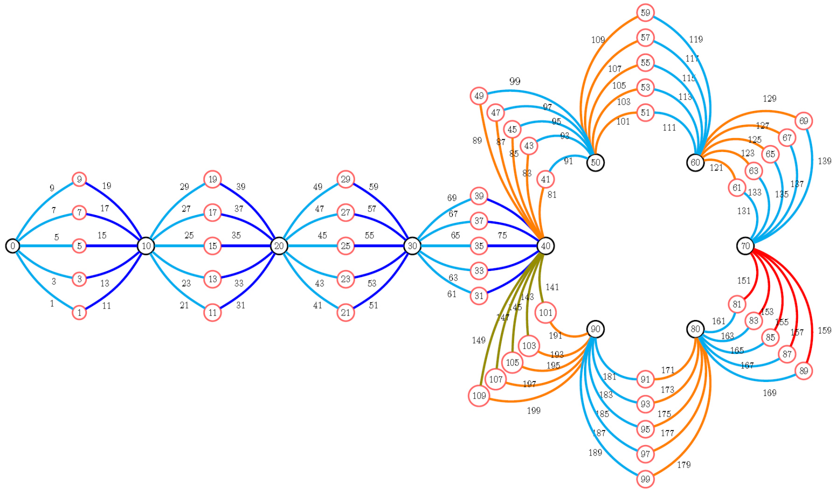

Figure 11

An odd harmonious labeling of super subdivision SSD$_{(2,5)}$[P5(C$_6$)].

A planar graph known as the friendship graph, Fr_g, is created for even numbers g by combining $\frac{g}{2}$ copies of the cycle graph C_3 at a hub vertex. The graph Fr_g, encompasses $\alpha(Fr_n)=(g+1)$ and $\beta(Fr_n)=\frac{3g}{2}$.

Theorem 3.5: *For an integer* $t\geq 1$, $g\equiv 0 \bmod 4$ *and a friendship graph* Fr_g, *the super subdivision* $SSD_{(2,t)}(Fr_g)$ *is an odd harmonious graph.*

Proof: The graph $SSD_{(2,t)}(Fr_g)$ consists of vertex set $V[SSD_{(2,t)}(Fr_g))]$ $=\{u_0, v_h, u_h^j, h\in[1,g], j\in[1,t]\} \cup \{v_h^j, h\in[1, g-1]\}$ and edge set $E[SSD_{(2,t)}(Fr_g))]=\{u_0u_h^j, v_hu_h^j, h\in[1,g]\}\cup\{v_hv_h^j, v_{h+1}v_h^j, h=1,3,5,\cdots, g-1\}$. Thus, $SSD_{(2,t)}(Fr_g)$ includes $\alpha=g\ (\frac{3}{2}t+1)+1$ and $\beta=3tg$.

Here is a description of vertex labelling function $\Phi : V[SSD_{(2,t)}(Fr_n)] \to \{0,1,2, 6tg-1\}$:

$$\Phi(u_0)=0, \quad \Phi(v_h)=2t(2g-2h+1), \quad h\in[1,g],$$

$$\Phi(v_h^j)=\begin{cases}2tg+2j+2ht-2t-1 & \text{if} \quad h=1,5,9,\cdots,\\ 2tg+2j+2ht-1 & \text{if} \quad h=3,7,11,\ldots g-1,\end{cases}$$

$\Phi(u_h^j) = 2t\,h + 2j - 2t - 1, \qquad h \in [1, g].$

Considering the labelling pattern described above, the induced edge labels Φ^* are

$\Phi^*(u_0 u_h^j) = 2t\,h + 2j - 2t - 1, \qquad h \in [1, g],$

$\Phi^*(v_h u_h^j) = 4t\,g + 2j - 2h\,t - 1, \qquad h \in [1, g],$

$$\Phi^*(v_h v_h^j) = \begin{cases} 6t\,g + 2j - 2h\,t - 1 & \text{if} \quad h = 1,5,9,\ldots; \\ 6t\,g + 2t + 2j - 2h\,t - 1 & \text{if} \quad h = 3,7,11,\ldots g-1, \end{cases}$$

$$\Phi^*(v_{h+1} v_h^j) = \begin{cases} 6t\,g + 2j - 2h\,t - 4t - 1 & \text{if} \quad h = 1,5,9,\ldots; \\ 6t\,g + 2j - 2h\,t - 2t - 1 & \text{if} \quad h = 3,7,11,\ldots\ g-1, \end{cases}$$

Because all of the labels of the edges are odd, distinguishable, and found in the codomain $\{1,3,\cdots,6tg-1\}$, the proof is complete.

Illustration: The super subdivision $SSD_{(2,3)}[Fr_8]$ of the friendship graph Fr_8 and its odd harmonious labeling is shown in Figure 12, where $t = 3$.

When disjoint cycles $C_g = \{v_1, v_2, \cdots, v_g\}$ and $C_m = \{u_1, u_2, \cdots, u_m\}$ are combined by a cutting edge $r = u_1 v_g$, the resulting graph is the dumbbell graph, represented as, $Db_{m,g}$.

Theorem 3.6: *For even positive integers g, m, and a dumbbell graph $Db_{m,g}$, the super subdivision $SSD_{(2,t)}[Db_{m,g}]$ is an odd harmonious graph for integer $t \geq 1$.*

Proof: Let the vertex set of the graph $SSD_{(2,t)}\,[Db_{m,g}]$ be $V[SSD_{(2,t)}\,(Db_{m,g})]$ $= \{v_h,\ v_h^j,\ 1 \leq h \in [1,g]\} \cup \{u_h,\ u_h^j,\ 1 \leq h \in [1,m]\} \cup \{r^j\}$ and the edge set be

$E[SSD_{(2,t)}\ (Db_{m,g})] = \{v_h v_h^j,\ h \in [1,g]\} \cup \{v_{h+1} v_h^j,\ h \in [1,\ g-1]\} \cup \{u_h u_h^j,\ h \in [1,m]\} \cup \{u_{h+1} u_h^j,\ h \in [1, m-1]\} \cup \{v_1 v_n^j,\ v_g r^j,\ u_1 r^j,\ u_1 u_m^j\}$. Consequently, $SSD_{(2,t)}\ (Db_{m,g})$ includes $\alpha = (g+m)\ (t+1) + t$ and $\beta = 2t\ (g+m+1)$.

The following is the vertex labelling function $\Phi : V[SSD_{(2,t)}\ (Db_{m,n})] \to \{0,1,2,\ \cdots,\ 4t(g+m+1)\ -1\}$:

$\Phi(v_h) = 2t\,h - 2t, \qquad h \in [1, g],$

$\Phi(r^j) = 2t\,g + 2j + 2t - 1,$

$\Phi(u_h) = 2t\,g + 2t\,h - 2t, \qquad h \in [1, m],$

$$\Phi(v_h^j) = \begin{cases} 2t\,h + 2j - 2t - 1 & \text{if} \quad h \in \left[1, \frac{g}{2}\right]; \\ 2\,t\,h + 2j - 1 & \text{if} \quad h \in \left[\frac{g}{2} + 1, g\right], \end{cases}$$

$$\Phi(u_h^j) = \begin{cases} 2t\,g + 2h\,t + +2j + 2t - 1 & \text{if} \quad h \in \left[1, \frac{m}{2}\right]; \\ 2t\,g + 2t\,h + +2j + 4t - 1 & \text{if} \quad h \in \left[\frac{m}{2} + 1, m\right], \end{cases}$$

The induced edge labels Φ^* are then identified using the following method.

$\Phi^*(v_1 v_g^j) = 2t\,g + 2j - 1,$

$\Phi^*(u_1 u_m^j) = 2t\,(2g + m + 2) + 2j - 1,$

$\Phi^*(v_g r^j) = 4t\,g + 2j - 1,$

$\Phi^*(u_1 r^j) = 2t\,(2g + 1) + 2j - 1,$

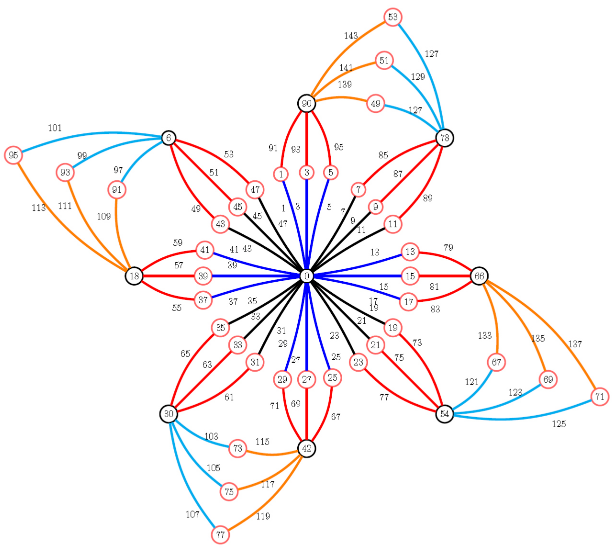

Figure 12

$SSD_{(2,3)}[Fr_8]$ with an odd harmonious labeling.

$$\Phi^*(v_h v_h^j) = \begin{cases} 4t\,h + 2j - 4t - 1 & \text{if} \quad h \in \left[1, \frac{g}{2}\right]; \\ 4t\,h + 2j - 2t - 1 & \text{if} \quad h \in \left[\frac{g}{2} + 1, g\right], \end{cases}$$

$$\Phi^*(u_h u_h^j) = \begin{cases} 4t\,n + 4t\,h + 2j - 1 & \text{if} \quad h \in \left[1, \frac{m}{2}\right]; \\ 2t\,(2g + 2h) + 2j + 2t - 1 & \text{if} \quad h \in \left[\frac{m}{2} + 1, g\right], \end{cases}$$

$$\Phi^*(v_{h+1} v_h^j) = \begin{cases} 4h\,t + 2j - 2t - 1 & \text{if} \quad h \in \left[1, \frac{g}{2}\right]; \\ 4h\,t + 2j - 1 & \text{if} \quad h \in \left[\frac{g}{2} + 1, g - 1\right], \end{cases}$$

$$\Phi^*(u_{h+1} u_h^j) = \begin{cases} 4t\,g + 4h\,t + 2t + 2j - 1 & \text{if} \quad h \in \left[1, \frac{m}{2}\right]; \\ 4t\,g + 4t\,h + 2j + 4t - 1 & \text{if} \quad h \in \left[\frac{m}{2} + 1, m - 1\right]. \end{cases}$$

The proof is therefore finished since all of the edges' labels are odd, distinctive, and located in the codomain $\{1, 3, \cdots, 4t(g + m + 1) - 1\}$ Additionally, the modular arithmetic is not done.

Illustration: The graph $SSD_{(2,3)}[Db_{4,6}]$ of the dumbbell graph $Db_{4,6}$ and its odd harmonious labeling is represented in Figure 13.

Theorem 3.7: *For an odd positive integers g and a fan graph F_g, the super subdivision $SSD_{(2,t)}[F_g]$ is an odd harmonious graph.*

Proof: The join of P_g with K_1 produces the fan graph $F_g = P_g + K_1$ [1]. A super subdivision $SSD_{(2,t)}[F_g]$ consists of $V[SSD_{(2,t)}(F_g)] = \{u_0, v_h, u_h^j, h \in [1, g]\} \cup \{v_h^j, h \in [1, g-1]\}$ and $E[SSD_{(2,t)}(F_g)] = \{u_0 u_h^j, v_h u_h^j, h \in [1, g]\} \cup \{v_h v_h^j, v_{h+1} v_h^j, h \in [1, g-1]\}$. Thereby, $SSD_{(2,t)}(F_g)$ includes $\alpha = g\,(2t+1) + (1-t)$ and $\beta = 2t(2g-1)$. There are two cases

[1] If $n \equiv 1 \bmod 4$. Consider $\Phi : V[SSD_{(2,t)}(F_g)] \to \{0, 1, 2, \cdots, 8tg - 4t - 1\}$ as follows:

$\Phi(u_0) = 0$, $\quad \Phi(v_h) = 4t\,g + 2t - 4t\,h, \quad h \in [1, n],$

$\Phi(u_h^j) = 2\,t\,h + 2j - 2t - 1, \quad h \in [1, g],$

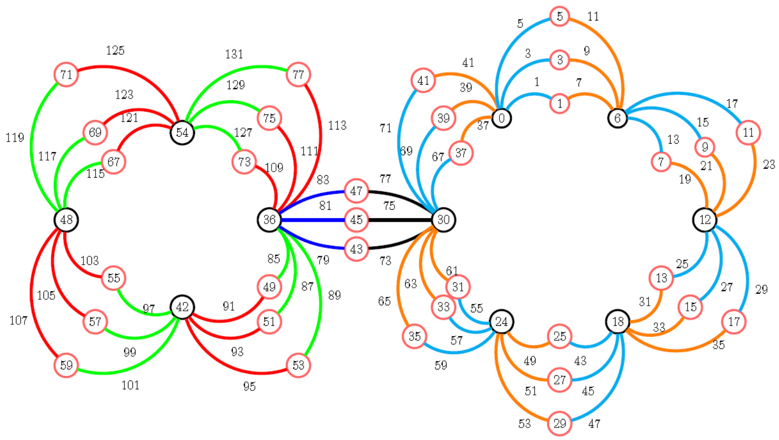

Figure 13

The graph $SSD_{(2,3)}[Db_{4,6}]$ with its odd harmonious labeling.

$$\Phi(v_h^j)=\begin{cases}2t\,g+2\,t\,h+2j-1 & \text{if} \quad h\equiv 0 \bmod 4;\\ 2t\,g+2\,t\,h+2j-2t-1 & \text{if} \quad h\equiv 2 \bmod 4,\\ 4t\,g+2\,t\,h+2j-6t-1 & \text{if} \quad h\equiv 1 \bmod 4;\\ 4t\,g+2t\,h+2j-4t-1 & \text{if} \quad h\equiv 3 \bmod 4,\end{cases}$$

Using the subsequent labelling approach, the induced edge labels Φ^* are then determined.

$$\Phi^*(u_0u_h^j)=2t\,h+2j-2t-1, \qquad h\in[1,n],$$

$$\Phi^*(v_hu_h^j)=4t\,g-2th+2j-1, \qquad h\in[1,n],$$

$$\Phi^*(v_hv_h^j)=\begin{cases}8t\,g+2j-2th-4t-1 & \text{if} \quad h\equiv 1 \bmod 4;\\ 6t\,g+2j-2ht-1 & \text{if} \quad h\equiv 2 \bmod 4;\\ 2t\,g+2j-2ht-2t-1 & \text{if} \quad h\equiv 3 \bmod 4;\\ 6t\,g+2t+2j-2ht-1 & \text{if} \quad h\equiv 0 \bmod 4,\end{cases}$$

$$\Phi^*(v_{h+1}v_h^j)=\begin{cases}6t\,g+2j-2t\,h-2t-1 & \text{if} \quad h\equiv 0 \bmod 4;\\ 6t\,g+2j-2\,th-4t-1 & \text{if} \quad h\equiv 2 \bmod 4;\\ 8t\,g+2j-2h\,t-8t-1 & \text{if} \quad h\equiv 1 \bmod 4;\\ 8g\,t+2j-2h\,t-6t-1 & \text{if} \quad h\equiv 3 \bmod 4.\end{cases}$$

[2] If $n \equiv 3 \bmod 4$. The vertex labelling function Φ can be defined similarly to the previous case, but with the alterations listed below:

$$\Phi(v_h^j) = \begin{cases} 2tg + 2th + 2j - 1 & \text{if} \quad h \equiv 2 \bmod 4; \\ 2tg + 2th - 2t + 2j - 1 & \text{if} \quad h \equiv 0 \bmod 4, \end{cases}$$

In this case, the induced edge labels Φ^* is defined as in the previous one, with the following modifications

$$\Phi^*(v_h v_h^j) = \begin{cases} 6tg + 2t + 2j - 2ht - 1 & \text{for} \quad h \equiv 2 \bmod 4; \\ 6gt + 2j - 2ht - 1 & \text{for} \quad h \equiv 0 \bmod 4; \end{cases}$$

$$\Phi^*(v_{h+1} v_h^j) = \begin{cases} 6tg + 2j - 2ht - 2t - 1 & \text{for} \quad h \equiv 2 \bmod 4; \\ 6tg + 2j - 2ht - 4t - 1 & \text{for} \quad h \equiv 0 \bmod 4; \end{cases}$$

As a consequence, the proof is complete in every case because the labels of the edges are all odd, distinct, and located in the codomain $\{1, 3, \cdots, 4t(2g-1)-1\}$.

Illustration: An odd harmonious labeling of $SSD_{(2,3)}[F_5]$ for the fan F_5 is depicted in Figure 14.

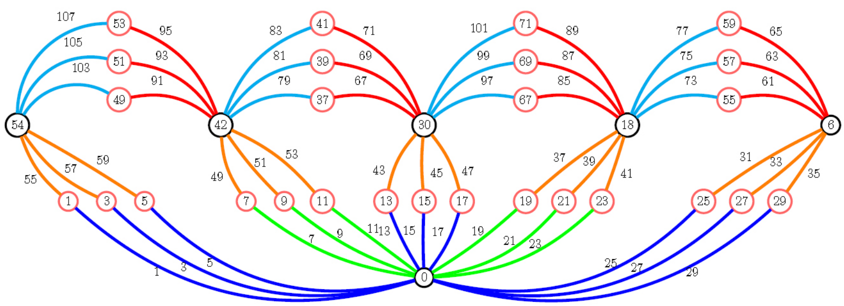

Figure 14

$SSD_{(2,3)}[F_5]$ of the fan F5 with an odd harmonious labeling.

Theorem 3.8: *For an even positive integers g, and a wheel graph* W_g, *the super subdivision* $SSD_{(2,t)}[W_g]$ *is an odd harmonious for integer* $t \geq 1$.

Proof: A super subdivision $SSD_{(2,t)}[W_g]$ of W_g with central vertex u_0, consists of a vertex set $V[SSD_{(2,t)}(W_g)] = \{u_0, v_h, v_h^j, u_h^j, h \in [1, g]\}$ and edge set $E[SSD_{(2,t)}\ (W_g)] = \{u_0 u_h^j,\ v_h v_h^j,\ v_h u_h^j,\ h \in [1, g]\} \cup \{v_{h+1} v_h^j,\ h \in [1, g-1]\} \cup \{v_1 v_g^j\}$. So, $SSD_{(2,t)}\ (W_g)$ contains $\alpha = g(2t+1) + 1$ and $\beta = 4t\,g$.

[1] If $n \equiv 0\ mod\ 4$. The labelling $\Phi : V[SSD_{(2,t)}\ (W_g)] \to \{0, 1, 2, \cdots, 8t\,g-1\}$ is specified as:

$\Phi(u_0) = 0$,

$\Phi(v_h) = 2t\,g + 2t - 4h\,t, \qquad h \in [1, g],$

$\Phi(u_h^j) = 2t\,h + 2j - 2t - 1, \qquad h \in [1, g],$

$$\Phi(v_h^j) = \begin{cases} 2t\,g + 2h\,t + 2j - 1 & \text{for} \quad h \equiv 1 \bmod 4; \\ 4t\,g + 2j + 2t\,h - 1 & \text{for} \quad h \equiv 2 \bmod 4; \\ 2t\,g + 2h\,t + 2j + 2t - 1 & \text{for} \quad h \equiv 3 \bmod 4; \\ 4g\,t + 2h\,t + 2j - 2t - 1 & \text{for} \quad h \equiv 0 \bmod 4; \\ 4g\,t + 2j - 2t - 1 & \text{for} \quad h = g, \end{cases}$$

The induced edge labels Φ^* are then identified using the subsequent labelling technique.

$\Phi^*(u_0 u_h^j) = 2t\,h + 2j - 2t - 1, \qquad h \in [1, g],$

$\Phi^*(v_h u_h^j) = 4t\,g + 2j - 2h\,t - 1, \qquad h \in [1, g],$

$$\Phi^*(v_h v_h^j) = \begin{cases} 6t\,g + 2j - 2h\,t + 2t - 1 & \text{for} \quad h \equiv 1 \bmod 4; \\ 8t\,g + 2j - 2t\,h + 2t - 1 & \text{for} \quad h \equiv 2 \bmod 4; \\ 6t\,g - 2h\,t + 2j + 2t - 1 & \text{for} \quad h \equiv 3 \bmod 4; \\ 8t\,g + 2j - 2t\,h - 1 & \text{for} \quad h \equiv 0 \bmod 4; \\ 4t\,g + 2j - 1 & \text{for} \quad h = g, \end{cases}$$

$$\Phi^*(v_{h+1} v_k^j) = \begin{cases} 6t\,g + 2j - 2h\,t - 2t - 1 & \text{for} \quad h \equiv 1, \bmod 4; \\ 8t\,g + 2j - 2t - 2t\,h - 1 & \text{for} \quad h \equiv 2, \bmod 4; \\ 6t\,g + 2j - 2h\,t - 1 & \text{for} \quad h \equiv 3, \bmod 4; \\ 8t\,g + 2j - 2t - 2t\,h - 1 & \text{for} \quad h \equiv 0, \bmod 4, \end{cases}$$

$\Phi^*(v_1 u_g^j) = 4t\,(2g-1) + 2j - 1.$

[2] If $n \equiv 2 \bmod 4$. The labelling Φ can be created as in the case before, but with the following modifications,

$$\Phi(v_h^j) = \begin{cases} 2tg + 2th + 2j + 2t - 1 & \text{for} \quad h \equiv 1 \bmod 4; \\ 2tg + 2j + 2ht - 1 & \text{for} \quad h \equiv 3 \bmod 4, \end{cases}$$

As in aforementioned situation, the edge labels Φ^* have the following modifications

$$\Phi^*(v_h v_h^j) = \begin{cases} 6tg + 2j + 2t - 2th - 1 & \text{for} \quad h \equiv 1 \bmod 4; \\ 6tg + 2j + 2t - 2ht - 1 & \text{for} \quad h \equiv 3 \bmod 4; \end{cases}$$

$$\Phi^*(v_{h+1} v_h^j) = \begin{cases} 6tg + 2j - 2th - 1 & \text{for} \quad h \equiv 1 \bmod 4; \\ 6tg + 2j - 2t - 2ht - 1 & \text{for} \quad h \equiv 3 \bmod 4; \end{cases}$$

In all cases, the edges' labels are all odd, distinct, and found in the codomain $\{1, 3, \cdots, 2\beta - 1 = 8tg - 1\}$. The proof is therefore complete.

Illustration: A super subdivision $SSD_{(2,3)}[W_6]$ of the wheel W_6 with an odd harmonious labeling is illustrated in Figure 15.

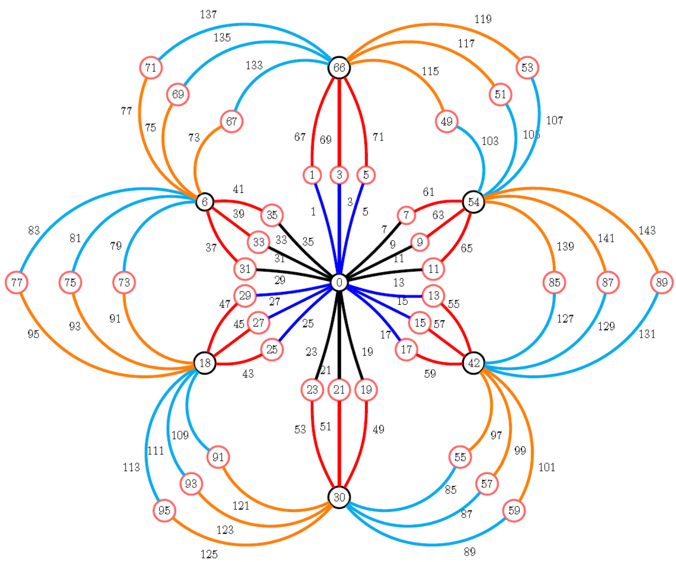

Figure 15

An odd harmonious labeling of SSD$_{(2,3)}[W_6]$.

Helm graph H_g is contracted by adding g more pendent vertices $\{w_1, w_2, \cdots, w_g\}$ and g more edges $(w_i v_i)$, $i \in [1, g]$ for $g \ge 3$ to the wheel $W_g = \{u_0, v_1, \cdots, v_g\}$, then $\alpha = |V(H_g)| = 2g+1$ and $\beta = |E(H_g)| = 3g$.

Theorem 3.9: *For an even positive integer* g, *and a helm graph* H_g, *the super subdivision* $SSD_{(2,t)}$ $[H_g]$ *is an odd harmonious graph for integer* $t \ge 1$.

Proof: A super subdivision $SSD_{(2,t)}$ $[H_g]$ of the helm H_g consists of $V[SSD_{(2,t)}$ $(H_g)]$ = $\{u_0,\ v_h,\ w_h\ ,\ v_h^j,,\ u_h^j,\ w_h^j,\ h \in [1, g]\}$ and $E[SSD_{(2,t)}$ $(H_g)]$ = $\{u_0 u_h^j,\ v_h v_h^j,\ w_h w_h^j,\ v_h w_h^j,\ h \in [1, g]\}$ $\cup$ $\{v_{h+1} v_h^j,\ h \in [1,\ g-1]\}$ $\cup$ $\{v_1 v_g^j\}$. The graph $SSD_{(2,t)}$ (H_g) includes $\alpha = g\ (3t+2)$ +1 and $\beta = 6t\,g$.

[1] If $n \equiv 0 \bmod 4$. A labeling Φ : $V[SSD_{(2,t)}$ $(H_g)] \to \{0, 1, 2, \cdots, 12t\,g-1\}$ is specified as follows:

$\Phi(u_0) = 0,$

$\Phi(v_h) = 4t\,g - 4t\,h + 2t, \qquad h \in [1, g],$

$\Phi(w_h) = 4t\,g + 4t - 4h\,t, \qquad h \in [1, g],$

$\Phi(u_h^j) = 2t\,h + 2j - 2t - 1, \qquad h \in [1, g],$

$\Phi(w_h^j) = 4t\,g + 8h\,t + 2j - 6t - 1, \qquad h \in [1, g],$

$$\Phi(v_h^j) = \begin{cases} 2t\,g + +2j2h\,t - 1 & \text{for} \quad h \equiv 1 \bmod 4; \\ 4t\,g + +2j + 2h\,t - 1 & \text{for} \quad h \equiv 2 \bmod 4; \\ 2t\,g + 2t\,h + 2j + 2t - 1 & \text{for} \quad h \equiv 3 \bmod 4; \\ 4t\,g + 2h\,t + 2j - 2t - 1 & \text{for} \quad h \equiv 0 \bmod 4; \\ 4t\,g + 2j - 2t - 1 & \text{for} \quad h \equiv g, \end{cases}$$

The following labelling procedure is then used to determine the induced edge labels Φ^*.

$\Phi^*(u_0 u_h^j) = 2t\,g + 2j + 2h\,t - 1, \qquad h \in [1, g],$

$\Phi^*(v_h u_k^j) = 4t\,g + 2j - 2h\,t - 1, \qquad h \in [1, g],$

$\Phi^*(w_h w_h^j) = 8t\,g + 4h\,t + 2j - 2t - 1, \qquad h \in [1, g],$

$$\Phi^*(v_h w_h^j) = 8tg + 4th + 2j - 4t - 1, \qquad h \in [1, g],$$

$$\Phi^*(v_h v_h^j) = \begin{cases} 6tg + 2t + 2j - 2ht - 1 & \text{for } h \equiv 1 \bmod 4; \\ 8tg + 2t + 2j - 2th - 1 & \text{for } h \equiv 2 \bmod 4; \\ 6tg + 4t + 2j - 2ht - 1 & \text{for } h \equiv 3 \bmod 4; \\ 8tg + 2j - 2th - 1 & \text{for } h \equiv 0 \bmod 4; \\ 4tg + 2j - 1 & \text{for } h \equiv g, \end{cases}$$

$$\Phi^*(v_{h+1} v_k^j) = \begin{cases} 6tg + 2j - 2th - 2t - 1 & \text{for } h \equiv 1 \bmod 4; \\ 8tg + 2j - 2th - 2t - 1 & \text{for } h \equiv 2 \bmod 4; \\ 2tg + 2j - 2ht - 1 & \text{for } h \equiv 3 \bmod 4; \\ 8tg + 2j - 2th - 4t - 1 & \text{for } h \equiv 1 \bmod 4, \end{cases}$$

$$\Phi^*(v_1 u_g^j) = 8tg + 2j - 4t - 1.$$

[2] If $n \equiv 2 \bmod 4$. A labeling Φ can be described as above with the following modification.

$$\Phi(v_h^j) = \begin{cases} 2tg + 2ht + 2j + 2t - 1 & \text{if } h \equiv 1 \bmod 4; \\ 2tg + 2j + 2ht - 1 & \text{if } h \equiv 3 \bmod 4; \end{cases}$$

The induced labelling map Φ^* can therefore be defined as in the previous case, with the following modifications

$$\Phi^*(v_h v_h^j) = \begin{cases} 6tg + 4t + 2j - 2th - 1 & \text{for } h \equiv 1 \bmod 4; \\ 6tg + 2t + 2j - 2ht - 1 & \text{for } h \equiv 3 \bmod 4; \end{cases}$$

$$\Phi^*(v_{h+1} v_h^j) = \begin{cases} 6tg + 2j - 2ht - 1 & \text{if } h \equiv 1 \bmod 4; \\ 6tg + 2j - 2ht - 2t - 1 & \text{if } h \equiv 3 \bmod 4; \end{cases}$$

In every case the labels of the edges are all odd, located in the codomain $\{1, 3, \cdots, 12tg - 1\}$ and distinct.

Illustration: In Figure 16, an odd harmonious labeling of $SSD_{(2,t)}[H_6]$ for the helm graph H_6, where $t = 3$ is illustrated .

4. Strong odd harmonious graphs

Definition 4.1: An injective function $\Phi : V(\Gamma) \to \{0,1,2,\cdots,\beta\}$ is called *strong odd harmonious labeling* of $\Gamma(\alpha,\beta)$ if the induced mapping $\Phi^* : E(\Gamma) \to \{1,3,5,\cdots,2\beta-1\}$, given by: $\Phi^*(uv) = [\Phi(u) + \Phi(v)]$, is a bijective function. Strong harmonious graph is defined as the graph that accepts strong odd harmonious labelling.

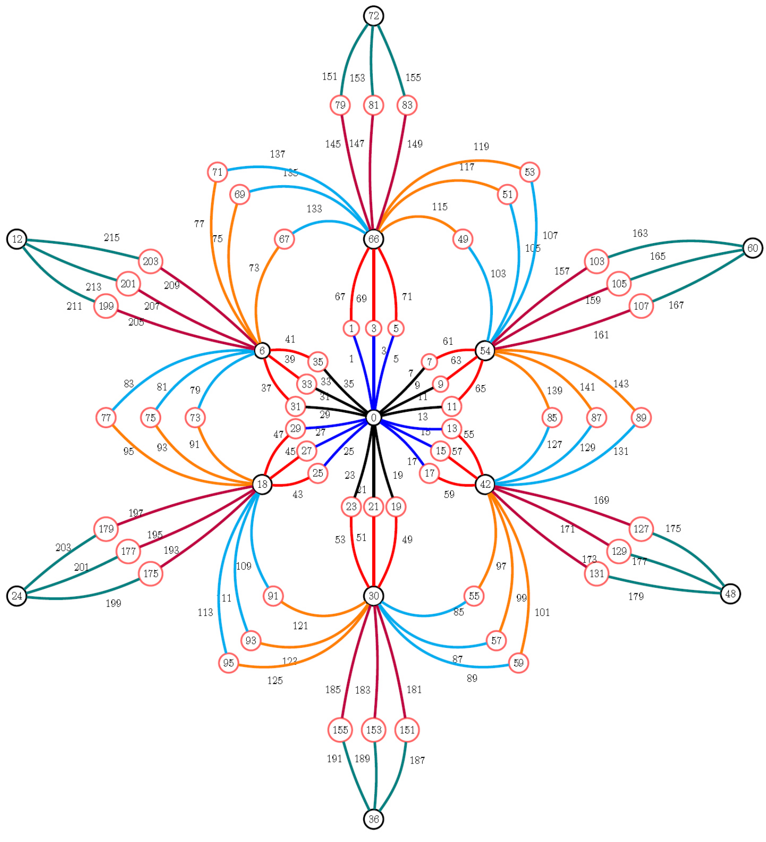

Figure 16

$SSD_{(2,3)}(H_6)$ of helm graph H6 with an odd harmonious labeling.

Definiton 4.2: A cycle $C_g = \{u_1,u_2,\cdots,u_g\}$ with a $P_{\frac{g}{2}} = \{v_0,v_1,\cdots,v_{\frac{g}{2}}\}$ -chord matching the vertices v_0 with u_1 and $v_{\frac{g}{2}}$ with $u_{\frac{g}{2}+1}$ construct the graph $C_g \divideontimes P_{\frac{g}{2}}$, [28]. So, $V[C_g \divideontimes P_{\frac{g}{2}}] = \{u_h,\ h\in[1,g]\}.\ \bigcup\{v_h^{\frac{g}{2}},\ h\in[1,\frac{g}{2}-1]\}$.

Theorem 4.3: *For even positive integer g and positive integer t. The super subdivision* $SSD_{(2,t)}\,[C_g \circledast P_{\frac{g}{2}}]$ *of the cycle* C_g *with a* $P_{\frac{g}{2}}$ *chord is strong odd harmonious graph.*

Proof: The vertex set of the graph $SSD_{(2,t)}\,[C_g \circledast P_{\frac{g}{2}}]$ is $V[SSD_{(2,t)}\,(C_g \circledast P_{\frac{g}{2}})]$ $= \{v_h\,,\ h \in [1,\ \frac{g}{2}-1]\}\ \cup \{u_h\,,\ u_h^j,\ 1 \le h \in [1, g]\}\ \cup \{v_h^j,\ h \in [1, \frac{g}{2}]\}$. The graph $SSD_{(2,t)}\,(C_g \circledast P_{\frac{g}{2}})$ includes $\alpha = [\frac{3g(t+1)}{2} - 1]$ and $\beta = 3gt$, see Figure 17.

We present the labeling function $\Phi : V[SSD_{(2,t)}\,(C_g \circledast P_{\frac{n}{2}})] \to \{0, 1, 2, \cdots, 6tg-1\}$ as follows:

$$\Phi(u_h) = \begin{cases} 0 & \text{if} \quad h = 1; \\ 6t(h-1)+4 & \text{if} \quad h \in \left[2, \frac{g}{2}\right], \\ 3tg & \text{if} \quad h = \frac{g}{2}+1; \\ 6t(g-h+1)-4 & \text{if} \quad h \in \left[\frac{g}{2}+2, g\right], \end{cases}$$

$$\Phi(u_h^j) = \begin{cases} 6t(h-1)+6j-5 & \text{if} \quad h \in \left[1, \frac{g}{2}\right]; \\ 6t(g-h)+6j-1 & \text{if} \quad h \in \left[\frac{g}{2}+1, g\right], \end{cases}$$

$$\Phi(v_h) = 6th\,, \qquad h \in [1, \tfrac{g}{2}-1],$$

$$\Phi(v_h^j) = 6t(h-1)+6j-3\,, \qquad h \in [1, \tfrac{g}{2}].$$

Taking into account the labelling pattern mentioned above, the induced edge labels Φ^* are obtained.

$$\Phi^*(u_h u_h^j) = \begin{cases} 6j-5 & \text{if} \quad h = 1; \\ 12t(h-1)+6j-1 & \text{if} \quad h \in \left[2, \frac{g}{2}\right], \\ 6t(g-1)+6j-1 & \text{if} \quad h = \frac{g}{2}+1; \\ 6t(2g-2h+1)+6j-5 & \text{if} \quad h \in \left[\frac{g}{2}+2, g\right], \end{cases}$$

$$\Phi^*(u_{h+1} u_h^j) = \begin{cases} 6t(2h-1)+6j-1 & \text{if} \quad h \in \left[1, \frac{g}{2}-1\right]; \\ 6t(g-1)+6j-5 & \text{if} \quad h = \frac{g}{2}, \\ 12t(g-h)+6j-5 & \text{if} \quad h \in \left[\frac{g}{2}+1, g-1\right], \end{cases}$$

$$\Phi^*(u_1 u_n^j) = 6j-1,$$

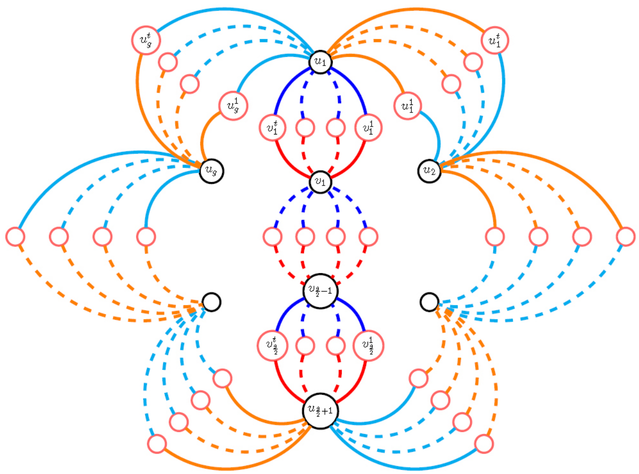

Figure 17

A cycle C_g with a $P_{\frac{g}{2}}$ chord and its $SSD_{(2,t)}\ [C_g \circledast P_{\frac{g}{2}}]$

$\Phi^*(v_h v_h^j) = 6t(2h-1) + 6j - 3, \qquad h \in [1, \frac{g}{2} - 1],$

$\Phi^*(u_{\frac{g}{2}+1} v_{\frac{g}{2}}^j) = 6t(g-1) + 6j - 3,$

$\Phi^*(v_h v_{h+1}^j) = 12th + 6j - 3, \qquad h \in [1, \frac{g}{2} - 1],$

$\Phi^*(u_1 v_1^j) = [\Phi(u_1) + \Phi(v_1^j)] = 6j - 3.$

Thus, the modular arithmetic is not done, the labels of the edges are all odd, distinct, and reside in the codomain $\{1, 3, \cdots, 6tg - 1\}$, which completes the proof.

Illustration: The super subdivision graph $SSD_{(2,t)}[C_6 \circledast P_3]$ with a strong odd harmonious labeling is illustrated in Figure 18, where $t = 4$.

5. Conclusion

Graph labeling have received way too much attention lately, although these objects still hold interest in the fields of discrete mathematics and graph theory. There are a significant number of outcomes and published publications. Numerous graphs have not yet revealed whether they are oddly harmonic or not.

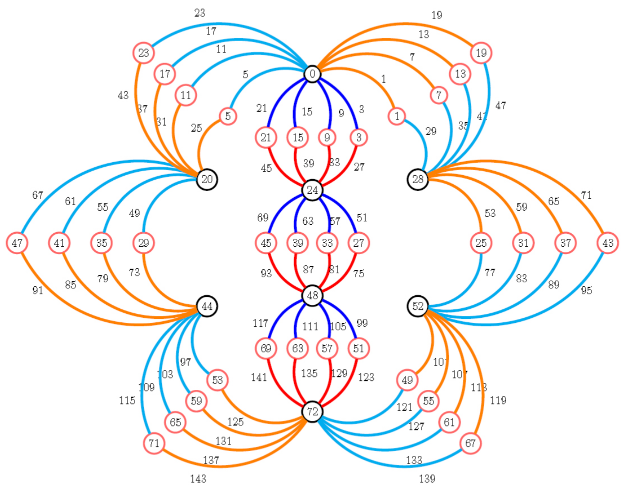

Figure 18

***$SSD_{(2,4)}[C_6 \divideontimes P_3]$* with a strong odd harmonious labeling**

In this paper, we extend the odd harmonious labelling by introducing the semi-odd harmonious labelling. For the super subdivision of the diverse graphs, including the $Y-$tree, the graph $T_{n,n-2}$, the graph $P_n \odot K_1$, as well as the twig and crown graph. We demonstrate the existence of odd harmonious labelling for the super subdivision of the various graphs such as : cycle C_n, dragon graph $P_m(C_n)$, friendship graph Fr_n, dumbbell graph $Db_{m,n}$, fan graph F_n, the wheel W_n and the helm H_n. The last part illustrates the existence of strong odd harmonic labelling for a super subdivision graph.

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