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A combinatorial interpretation of the Perrin sequence

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Abstract

This study focuses on exploring a combinatorial interpretation for the Perrin sequence, with the goal of making a meaningful contribution to the field of Mathematics education. In view of studies focused on the History of Mathematics containing the teaching of recurring numerical sequences present in textbooks, there is an interest in carrying out a combinatorial approach for a sequence regarded as akin to the Fibonacci numbers, which is the Perrin sequence. In this way, we have the definition of Perrin's bracelets, investigating Perrin's combinatorial model. Finally, combinatorial interpretations for the Perrin sequence extensions are discussed, generalizing the Perrin combinatorial model.

Subject Classification: *11B37, 11B39, 11B75.*

Keywords: *Combinatorics, Mathematics teaching, Perrin sequence.*

Introduction

Since the 1960s, there has been a notable increase in scientific interest, with researchers from diverse countries actively involved in exploring generalized properties of the Fibonacci sequence. Among the various methods and approaches used to generalize properties for studying not only the Fibonacci sequence but also other recurrent sequences, the Perrin sequence emerges as significant. Thus, it becomes apparent that, within the framework of the combinatorial approach, there is a component that retains a heuristic and intuitive nature, which proves beneficial for the teaching of Mathematics [2, 12].

Given this scenario, this paper discusses and defines a combinatorial interpretation related to the Perrin sequence. Thus, the reader will be able to see discussions related to the Tetrarrin sequences and their generalization, as we add terms in their recurrence, thus enhancing an extension of these Perrin numbers. It should also be noted that the combinatorial approach originates from the notion of the board, defined by Spreafico [7].

Introducing a brief study around the sequence that will be addressed, we have that the Perrin numbers is a third order sequence, numerical recurrent given by: $R_n = R_{n-2} + R_{n-3}$, $R_0 = 3$, $R_1 = 0$, $R_2 = 2$, $R_2 = 2$, These numbers bear the name of Raoul Perrin (1841-1910), a French engineer who, during his leisure time, engaged in the creation of scientific works. He held a pronounced preference and a particular inclination for the field of mathematics. Several of his memoirs, reflecting considerable depth of thought, have served as the basis for highly regarded presentations at the Academy of Sciences and other scholarly societies, earning him commendable distinctions [8].

It is worth mentioning the existence of other works, presenting the advance of Perrin numbers, as is the case of Shtayat and Al-Kateeb [5] which presents a model of cryptography based on matrices.

For that, it is important to portray the Padovan sequence (P_n), in which it has the same recurrence relation as the Perrin sequence, differentiating in its initial values, these being given by: $P_0 = P_1 = P_2 = 1$. These two sequences also present other similarities, which can be deepened in the work of Vieira [4, 6], [9].

With that premise, our exploration into combinatorial analysis began with the definition of a board as provided by Spreafico [7]. In accordance with this definition, a board is comprised of squares, each denoted as a position. These positions are methodically enumerated, with these enumerations functioning to uniquely describe each position. A specific board will be referred to as an *n*-board.

Vieira, Alves and Catarino [11] in their research, they defined the Padovan combinatorial model, with the Padovan board, based on a construction rule given by: In the context of Padovan tiling, a Padovan tile is constructed by assembling various shapes, including an elongated domino defined as a rectangle of size 1 x 3, a 1 x 1 black square, and the 1 x 2 domino. The inclusion of the black square serves to complement vacant tiles, adhering to the stipulation that it is placed solely at the beginning and only once on each tile. These specific regulations are established within the framework of the theorem governing Padovan tiling [11].

Thus, a *n*-board is considered, with the following tile shapes: black square 1 x 1, blue dominoes 1 x 2, gray extended dominoes 1 x 3, all with weight 1. In Figure 1, on the left side, some examples are provided in order to fill in the *n*-board corresponding to the Padovan sequence. On the right side are the terms corresponding to the Padovan numbers. With this, it is possible to perceive the term p_n as being the amount of tile shapes on the *n*-board, following the aforementioned rules, determining the relation: $p_n = P_n, n \geqslant 0$. Next, $p_0 = 1 = P_0$ is fixed, when there is no board of size 1 x *n*, and $p_1 = 1 = P_1$, when there is only one black square.

Indeed, there are propositions associated with Padovan tiles, specifically: the count of tiles commencing with black squares corresponds to a Padovan sequence number, precisely defined by the formula p_{n-3}. Conversely, the count of tiles that do not commence with a black square is expressed as p_{n-2} (see Figure 1).

For any $n \geqslant 0$, the potential tiling arrangements of a 1 x *n* board, initiating with black square tiles, are represented by p_{n-3}. Similarly, for any $n \geqslant 0$, the feasible tilings of a 1 x *n* board that do not commence with black square tiles are expressed as p_{n-2}.

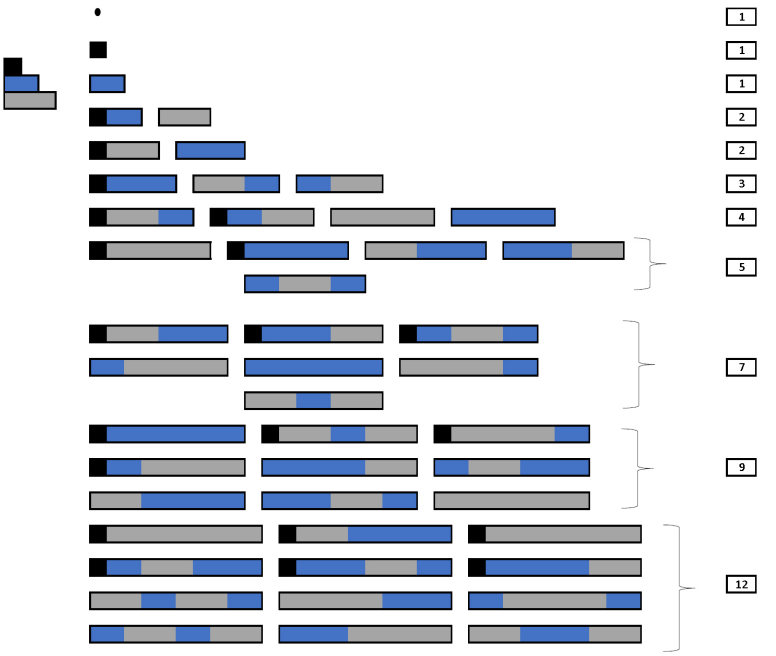


Figure 1

Padovan tiling. Source: [11]

In this way, Perrin's bracelets will be defined in a primordial way in this research, introducing Perrin's combinatorial model before a combinatorial interpretation for the recurrent numerical sequence.

1. Perrin's combinatorial model

With the discussions carried out in the introduction of this research and in its highlighted sources, we will approach Padovan's combinatorial model theorem, since the sequences have similarities.

Theorem 1: *For any non-negative integer n, the potential arrangement of a 1 x n board, involving black square, blue domino, and gray extended domino tiles, is represented by [11]:*

$$p_n = P_n.$$

Proof: The demonstration can be viewed in the Vieira, Alves and Catarino [11].

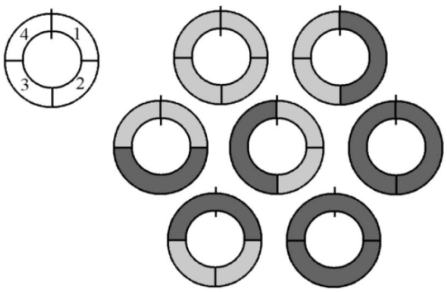


Figure 2

Lucas bracelets size 4. Source: [1]

Initially, one can observe the relationship between the Fibonacci sequence and Lucas [3], having the same recurrence relationship and presenting different initial values between these two sequences. In this way, [1] study the Fibonacci combinatorial model, briefly presented in the previous section, and, in a complementary way, Lucas' combinatorial model. Thus, the Lucas numbers operate in circles, called bracelets, where l_n represents the count of possible arrangements to tile a circular board consisting of *n* cells marked with squares and 1 x 2 dominoes. In the Figure 2 the number of tiles in Lucas' bracelet of size 4 is presented, that is, l_4, obtaining a total of 7 ways to tile the bracelet.

Theorem 2: *For any non-negative integer n, let* l_n *denote the count of ways to tile a circular board of size n using squares and dominoes. Since* l_n *corresponds to the nth Lucas number, we can express that [1]:*

$$l_n = L_n.$$

Proof: Similarly to the demonstration of the theorem 1, we have the validation of this mathematical theorem.

From the studies carried out on Padovan's combinatorial model and the Lucas sequence bracelet, there is the investigation of Perrin's combinatorial model. In view of this, the algebraic relationship between these two sequences is presented, as a way to obtain the combinatorial study of Perrin numbers.

Yilmaz and Taskara [13], established the following relationship between Padovan, $\{P_n\}_{n\in\mathbb{N}}$, and Perrin, $\{R_n\}_{n\in\mathbb{N}}$: $R_n = 3P_{n-5} + 2P_{n-4}$. From which, performing algebraic operations, one can obtain: $R_n = 3P_{n-5} + 2P_{n-4} = 2(P_{n-2} + P_{n-4}) + P_{n-5} = 2P_{n-2} + P_{n-5}$.

Moedas	$n-2$	$n-5$
○	p_{n-2}	
●	p_{n-2}	
◌		p_{n-5}

Figure 3

Cases of Perrin's combinatorial model. Source: Prepared by the authors

From the relation $R_n = 2P_{n-2} + P_{n-5}$, the Perrin bracelet, r_n, of sizes 1 x $n-2$ and 1 x $n-5$ is defined , consisting of five types of components, namely: coins (white, black and gray), tiles of p_{n-2}, tiles of p_{n-5}, tiles of size $n-2$ and tiles of size $n-5$. There are also three types of bracelets, namely:

Case 1: Started with a white coin, juxtaposed with tiles of p_{n-2}, juxtaposed with tiles of size $n-5$.

Case 2: Started with a black coin, juxtaposed with tiles of size p_{n-2}, juxtaposed with tiles of size $n-5$.

Case 3: Started with a gray coin, juxtaposed with tiles of size $n-2$, juxtaposed with tiles of p_{n-5}.

The result is exemplified in Figure 3.

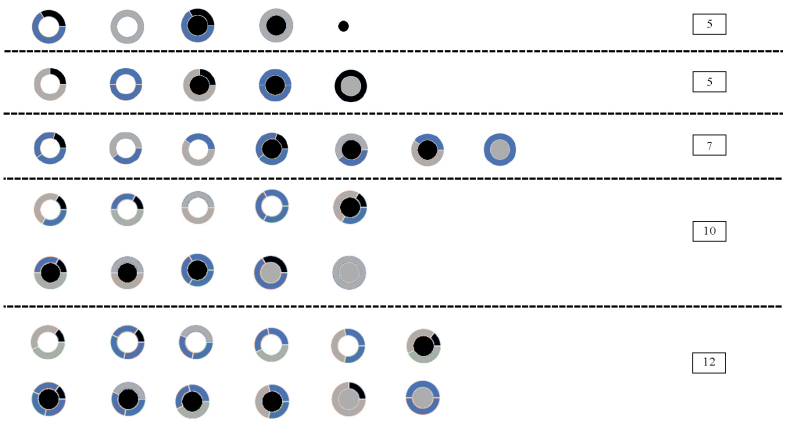


Figure 4

Perrin bracelet. Source: Prepared by the authors.

In order to revisit Padovan's tiling configurations [11], the black square 1 x 1, blue domino 1 x 2 and gray extended domino 1 x 3 are used. if you look at Figure 4.

Theorem 3: *For any* $n \geqslant 5$, *the potential arrangements of bracelets on a circular board with the previously mentioned parts are expressed by:*

$$r_n = R_n,$$

on what r_n *represents the count of ways to fill the 1 x n Perrin bracelet, and* R_n *denotes the nth term of the Perrin sequence.*

Proof: Similarly to the proof of the theorem 1, we have the validation of this mathematical theorem.

Vieira, Alves and Catarino [11] investigated and introduced the Padovan binomial, given by: $P_n = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{O(n,k)}{k}$. With that, based on Padovan's binomial and on the relation $R_n = 2P_{n-2} + P_{n-5}$, we have the investigation of Perrin's binomial, allowing to obtain a definition for the deepening of the study referring to those numbers.

Definition 1: The Perrin combinatorial model, for $n \geqslant 5$, is given by:

$$R_n = 2P_{n-2} = \sum_{k=0}^{\lceil \frac{n-2}{2} \rceil} \binom{O(n-2,k)}{k} + \sum_{k=0}^{\lceil \frac{n-5}{2} \rceil} \binom{O(n-5,k)}{k},$$

where $O(n,k) = \lfloor \frac{n+(k-1)}{3} \rfloor$.

In this way, the first Perrin terms are obtained, through the defined binomial:

$$R_5 = 2\left[\binom{0}{0} + \binom{1}{1}\right] + \binom{0}{0} = 5$$

$$R_6 = 2\left[\binom{1}{0} + \binom{1}{1}\right] + \binom{0}{0} = 5$$

$$R_7 = 2\left[\binom{1}{0} + \binom{1}{1} + \binom{2}{2}\right] + \binom{0}{0} = 7$$

$$R_8 = 2\left[\binom{1}{0} + \binom{2}{1} + \binom{2}{2}\right] + \left[\binom{0}{0} + \binom{1}{1}\right] = 10$$

$$R_9 = 2\left[\binom{2}{0} + \binom{2}{1} + \binom{2}{2} + \binom{3}{3}\right] + \left[\binom{1}{0} + \binom{1}{1}\right] = 12$$

$$R_{10} = 2\left[\binom{2}{0} + \binom{2}{1} + \binom{3}{2} + \binom{3}{3}\right] + \left[\binom{1}{0} + \binom{1}{1} + \binom{2}{2}\right] = 17.$$

1.1 *The combinatorial model of Tetrarrin*

The Tetrarrin sequence is a fourth-order sequence, with recurrence formula $R_{4(n)} = R_{4(n-2)} + R_{4(n-3)} + R_{4(n-4)}$, with the following initial values: $R_{4(0)} = 3$, $R_{4(1)} = 0$, $R_{4(2)} = 2$ and $R_{4(3)} = 3$. It should be noted that this is an extension of the Perrin sequence, with the addition of the term $R_{4(n-4)}$ in the Perrin recurrence.

The Tridovan sequence, discussed by Vieira [9], is a fourth-order sequence, given by the recurrence: $P_{3(n)} = P_{3(n-2)} + P_{3(n-3)} + P_{3(n-4)}$, with the following initial values: $P_{3(0)} = 1$, $P_{3(1)} = 0$, $P_{3(2)} = 1$ and $P_{3(3)} = 1$.

Based on Yilmaz and Taskara [13], in which they presented a relationship between the Padovan sequence and Perrin, we then sought to obtain a linear combination of the terms of the Tetrarrin sequence $(R_{4(n)})$ and Tridovan $(P_{3(n)})$. Taking as a premise that this linear combination is possible, the following system of equations was modeled: $Ax = y$, presenting the following definitions: $A = \begin{bmatrix} P_{3(0)} & P_{3(1)} & P_{3(2)} & P_{3(3)} \\ P_{3(1)} & P_{3(2)} & P_{3(3)} & P_{3(4)} \\ P_{3(2)} & P_{3(3)} & P_{3(4)} & P_{3(5)} \end{bmatrix}$, $y = \begin{bmatrix} R_{4(4)} \\ R_{4(5)} \\ R_{4(6)} \\ R_{4(7)} \end{bmatrix}$ and x is a vector of coefficients satisfying the system. Thus, it was possible to obtain the relation:

$$R_{4(n)} = 3P_{3(n-4)} + 3P_{3(n-3)} + 4P_{3(n-2)}. \quad (1)$$

Since other identities can be obtained, arising from arithmetic operations on the mathematical relation presented in Equation 1, we have: $R_{4(n)} = 2P_{3(n)} + P_{3(n-3)} + P_{3(n-4)}$. From this, the Tetrarrin bracelet, $r_{4(n)}$, of sizes 1 x n, 1 x $n-3$ and 1 x $n-4$, formed by seven types of components, namely: coins (white, black, gray and orange), $p_{3(n)}$ tiles, $p_{3(n-3)}$ tiles, $p_{3(n-3)}$ tiles of $p_{3(n-4)}$, tiles of size n, tiles of size $n-3$ and tiles of size $n-4$. There are also four types of bracelets, namely:

Case 1: Started with a white coin, juxtaposed with tiles of $p_{3(n)}$, juxtaposed with tiles of size $n-3$ and $n-4$.

Case 2: Started with a black coin, juxtaposed with tiles of $p_{3(n)}$, juxtaposed with tiles of size $n-3$ and $n-4$.

Case 3: Started with a gray coin, juxtaposed with tiles of size n and $n-4$, juxtaposed with tiles of $p_{3(n-3)}$.

Case 4: Started with an orange coin, juxtaposed with tiles of size n and $n-3$, juxtaposed with tiles of $p_{3(n-4)}$.

The result is exemplified in Figure 5.

Moedas	n	$n-3$	$n-4$
	$p_{3(n)}$		
	$p_{3(n)}$		
		$p_{3(n-3)}$	
			$p_{3(n-4)}$

Figure 5

Cases of the Tetrarrin combinatorial model. Source: Prepared by the authors.

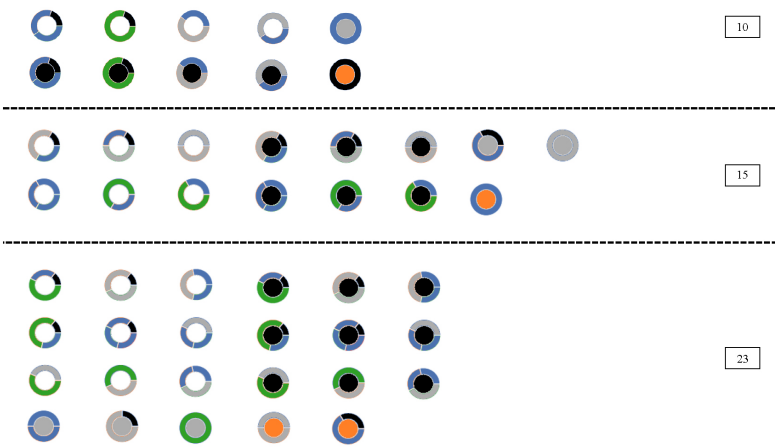


Figure 6

Bracelet of Tetrarrin. Source: Prepared by the authors.

Preserving the configurations of Tridovan's tiling, the black square 1 x 1, blue domino 1 x 2, gray extended domino 1 x 3 and green double domino 1 x 4 are used. observe the Figure 6.

Theorem 4: *For* $n \geqslant 5$, *the bracelets of a circular board of size 1 x n, with the previously mentioned parts is given by:*

$$r_{3(n)} = R_{3(n)} + R_{3(n-1)},$$

where $r_{3(n)}$ *is the count of ways to populate the 1 x n Tetrarrin bracelet and* $R_{3(n)}$ *is the nth term of the Perrin sequence.*

Proof: Similarly to the demonstration of the theorem 1, we have the validation of this mathematical theorem.

1.2 *The Z-nth-rrin combinatorial model*

Based on the Perrin and Tetrarrin sequences, there is the process of generalizing the order of the sequence, calling it a Z-nth-rrin sequence. The sequence Z-nth-rrin, denoted $R_{z(n)}$ with $n \geqslant z$ and $n \in \mathbb{N}$, is defined by: $R_{z(n+z)} = \sum_{i=0}^{z-2} R_{z(n+1)}$, where z represents the sequence order. The Z-dovan sequence is given by $P_{Z(n+z)} = \sum_{i=0}^{z-2} P_{Z(n+i)}$, $n \geqslant z$, $n \in \mathbb{N}$, where $P_{Z(n)}$ represents the nth term of the Z-dovan sequence [9, 10].

For the Z-thirrin combinatorial model, $R_{z(n)}$, it is assumed that there is a linear combination between the Z-dovan sequences $(P_{z(n)})$ and Z-nth-rrin of the same order. Based on this, one can model the following system of equations $Ax = y$ for sequences of order z, presenting the following definitions:

$$A = \begin{bmatrix} P_{z(0)} & P_{z(1)} & \cdots & P_{z(z-2)} & P_{z(z-1)} \\ P_{z(1)} & P_{z(2)} & \cdots & P_{z(z-1)} & P_{z(z)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{z(z-3)} & P_{z(z-2)} & \cdots & P_{z(2z-3)} & P_{z(2z-4)} \\ P_{z(z-2)} & P_{z(z-1)} & \cdots & P_{z(2z-2)} & P_{z(2z-3)} \end{bmatrix},$$

$$y = \begin{bmatrix} R_{z(z)} \\ R_{z(z+1)} \\ \cdots \\ R_{z(2z-2)} \\ R_{z(2z-1)} \end{bmatrix}$$

and x is a vector of coefficients satisfying the system.

The combinatorial representation of Z-nth-rrin is defined as a combination of bracelets of Z-dovan, where the number of units of a given bracelet is determined by the coefficient associated with the Z-dovan term to which it belongs. The multiplicative property of the coefficients is represented through coins of different colors and, therefore, it is verified that the total number of colors of coins is given by the sum of the elements of the vector of coefficients, thus having z coins.

In view of this, there is the immediate corollary, determining the generalization of the Perrin sequence, through the Z-nth-rrin sequence. It should be noted that the generalization of Z-nth-rrin tilings is given from the Tetrarrinn sequence.

Corollary 1: *For* $n \geqslant 5$, *the bracelets of a circular board of size 1 x n, with z coins of different colors and the black square tiles 1 x 1, blue dominoes 1 x 2, 1 x 3 gray extended dominoes, 1 x 4 green double extended dominoes, 1 x 5 yellow extended dominoes … and an orange 1 x z square is given by:*

$$r_{z(n)} = R_{z(n)} + R_{z(n-1)},$$

where z_n *refers to the number of ways to populate the board 1 x n and* $R_{z(n)}$ *is the nth term of the sequence of Z-nth-rrin.*

5. Conclusion

The present research enabled a study of Perrin's combinatorial interpretation, introducing Perrin's combinatorial model of numbers. In this way, it is possible to explore the combinatorial approach of the recurrent numerical sequence, contributing to the teaching of Mathematics in the History of Mathematics.

From the Perrin sequence, it was possible to define its respective combinatorial model and, by inserting elements in the main recurrence, define the Tetrarrin sequence and its generalization. After that, the combinatorial models of the other sequences derived from Perrin were introduced until their Z-nth-rrin generalization.

For future research, we seek to develop didactic teaching situations, allowing application in the classroom in initial training courses for Mathematics teachers.

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