

Even vertex odd mean labeling of some graphs

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Abstract

This work introduces the principle of “an even point (vertex) odd ratio (mean) labeling, which is specifically applied to a graph ‘G’ consisting of ‘p’ vertices and ‘q’ edges. Even point (vertex) odd ratio (mean) labeling is exhibited by a graph G in the presence of an injection-based function $f: V \text{ of } G \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ ensuring that the function derived from it (induced map) $g^*: E \text{ of } G \rightarrow \{1, 3, 5, \dots, 2q - 1\}$ specified by $g^*(uv) = g(u) + g(v)/2$ is a bijection. Graphs that meet these criteria are termed an even point (vertex) odd ratio (mean) graphs.

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This paper explores the properties of an even point (vertex) odd ratio (mean) labeling in various graph structures.

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1. Introduction

Graph theory is becoming a popular subject of study in mathematics and some sciences. Under certain circumstances, graph labeling assigns an integer to the vertices or edges, or vice versa. Numerous scientists and mathematicians have contributed to and developed various labeling techniques. The tree pattern matching, graph packing problem [4], and graph coloring problem are the three most commonly encountered graph theory problems. All three of these challenges have labeled graphs as a common element. An integer is assigned to either or both of the graph's vertices and edges, based on the circumstances. Rosa was the first to pioneer the graceful labeling of graphs in the late 1960s [3]. Gnanajothi R. B. introduced the odd graceful graph in [2]. Mean labeling was established, and Somasundaram S and Ponraj R explored the meanness of various conventional graphs [5][6][7][8]. Odd mean labeling of a graph was initially introduced by K. Manickam and M. Marudai [9]. R. Vasuki, A. Nagarajan, and S. Arockiaraj introduced the concept of even vertex odd mean labeling of graphs to examine this labeling behavior in various standard graphs [10]. M. Basher investigates some graphs' even vertex odd meanness in [11]. Selvaraj R et al. analysed radial radio mean graph of Grotzsch graph and some graphs[12]. A finite, undirected, simple graph is what we refer to as a graph. For terminology and notations follows[1]. This paper demonstrates that certain graph operations exhibit even vertex odd meanness in graphs labeled with even vertex odd mean labeling.

2. Preliminaries:

Definition 1 A graph G with order p and size q is said to possess even vertex odd mean labeling if there exists a bijective function $f: V \text{ of } G \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ such that the induced function $g: E \text{ of } G \rightarrow \{1, 3, 5, \dots, 2q - 1\}$ defined by $g^* \text{ of } uv = \frac{g(u) + g(v)}{2}$ is a bijection.

Definition 2 For every vertex $v \in V(G)$, the open neighborhood set N of v is the set of all vertices adjacent to v in G .

Theorem 3 If G is an even vertex odd mean graph, then G is a bipartite graph.

Theorem 4 Cycle C_n is an even vertex odd mean graph if and only if $n \equiv 0 \pmod{4}$.

Theorem 5 If $n \geq 3$, $K_{1,n}$ is not an even vertex odd mean graph.

Theorem 6 $K_{2,n}$ is an even vertex odd mean graph for all n .

Theorem 7 The Bistar $B_{m,n}$ is an even vertex odd mean graph for $n = m, m + 1$.

3. Main Results

Theorem 1: *A product of alternate double quadrilateral snake and K_1 admits an Even point (vertex) odd ratio (mean) graph.*

Proof: Let G_n denote the product of alternate quadrilateral snake and K_1 .

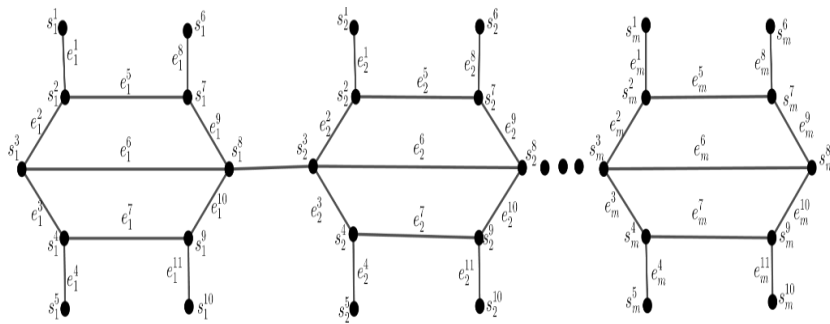


Figure 1
Alternate quadrilateral snake Graph

Let $\{s_1^1, s_1^2, \dots, s_1^{10}\} \cup \{s_2^1, s_2^2, \dots, s_2^{10}\} \cup \dots \cup \{s_m^1, s_m^2, \dots, s_m^{10}\}$ be the vertices and $e_1^1, e_1^2, \dots, e_1^{11}, e_2^1, e_2^2, \dots, e_2^{11}, \dots, e_m^1, e_m^2, \dots, e_m^{11}$ be the edges as illustrated in Figure 1. Define the unique mapping function $f: V$ of $G \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows.

$$f(s_j^i) = \begin{cases} 2(i-1) + j(12), 1 \leq i \leq 5 \\ 2(i-1) + 4 + j(12), 6 \leq i \leq 10 \end{cases} \text{ and } j = 2(m-1), m = 1, 2, 3, \dots, n$$

and define the edge function $g^*: E$ of $G \rightarrow \{1, 3, \dots, 2q - 1\}$ such as g^* of $uv = \frac{g(u) + g(v)}{2}$, u and v are adjacent is a bijection as follows $g^*(e_i) = 2$ of $i - 1 + 1$ if $i = 1, 2, \dots, n$. As a result g^* of $E = \{1, 3, \dots, 2q - 1\}$. This establishes f as a Uniform (even) Vertex odd Mean Labeling, which shows that the graph produced from the product of an alternate quadrilateral, a double snake, and K_1 is an example of a graph with Uniform (even) Vertex odd Mean Labeling.

Theorem 2: *The triangular graph G admits Uniform (even) Vertex odd Mean Labeling.*

Proof: Let G be a graph with p vertices and q edges.

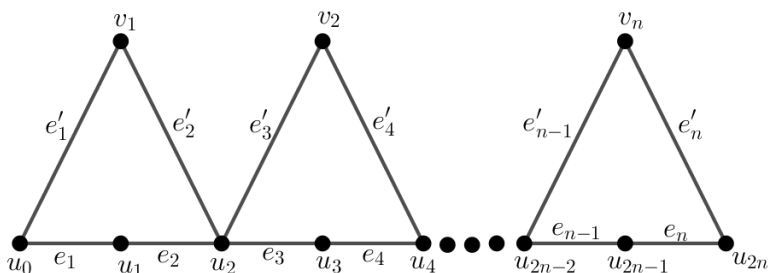


Figure 2
Triangular Graph

Let $u_0, u_1, \dots, u_{2n}$ and $v_0, v_1, \dots, v_n$ be the vertices and $e_1, e_2, \dots, e_n$ and $e'_1, e'_2, \dots, e'_n$ are edges of the graph G , as illustrated in Figure 2. Define the unique mapping function $f: V \text{ of } G \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows $u_i = 0, i = 0$ $u_i = 4i, i = 2, 4, \dots, 2n$ $u_i = 2, i = 1$ $u_i = u_{i-2} + 8, i = 3, 5, \dots, 2n - 1$ and $v_i = 8(i - 1) + 6, i = 1, 2, \dots, n$. Define a edge function $g^*: E \text{ of } G \rightarrow \{1, 3, \dots, 2q - 1\}$ such as $g^*(uv) = \frac{g(u) + g(v)}{2}$ is bijection. The edge values are defined by g^* of $e_i = 4$ of $i - 1$ and g^* of $e'_i = 4i - 1$. Therefore $g^* \text{ of } E = 1, 3, \dots, 2q - 1$. So f is Uniform(even) Vertex odd Mean Labeling. A graph that admits Uniform(even) Vertex odd Mean Labeling is a Uniform(even) Vertex odd Mean Labeling graph.

Theorem 3: *An alternate triangular snake graph satisfies Uniform(even) Vertex odd Mean Labeling.*

Proof: Let G be a graph with $s, u_1, \dots, u_n, v_1, \dots, v_n, x_1, \dots, x_n$ are the vertices and $e_1, e_2, \dots, e_n$ are the edges of the graph G as illustrated in Figure 3

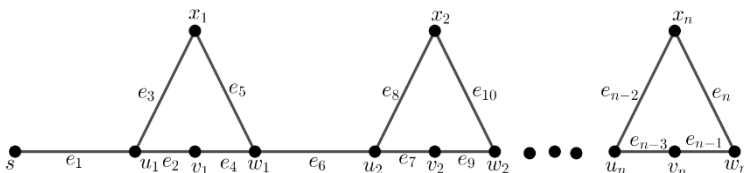


Figure 3
Alternate triangular snake Graph

Define the the unique mapping function $f: V(G) \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows $s = 0, x_i = 10i - 6, 1 \leq i \leq n$, $u_i = 10i - 8, 1 \leq i \leq n$, $v_i = 10i - 8, 1 \leq i \leq n$, $w_i = 10i, 1 \leq i \leq n$ and the edges function defined by $g^*: G(E) \rightarrow \{1, 3, \dots, 2q - 1\}$ as follows $g^*(E) = \frac{g(u) + g(v)}{2}$ is a bijection function. The edge values are $1, 3, 5, \dots, 2q - 1$. Hence $g^*(E) = 1, 3, \dots, 2q - 1$.

So G is Uniform(even) Vertex odd Mean Labeling. A graph that admits Uniform (even) Vertex odd Mean Labeling is an even vertex odd mean graph.

Theorem 4: *The caterpillar graph admits Uniform(even) Vertex odd Mean Labeling.*

Proof: Let G be a graph with p vertices and q edges and $v_1, v_2, \dots, v_n$ and $e_1, e_2, \dots, e_n$ are edges of the graph of G and n vertices and $n - 1$ edges. Shown in Figure 4.

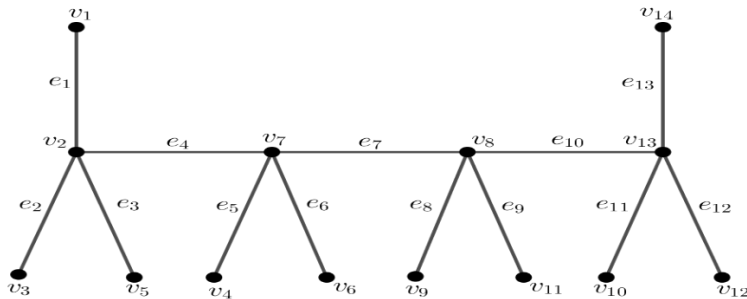


Figure 4
Caterpillar Graph

Define the unique mapping function $f: V(G) \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows

$$f(v_i) = \begin{cases} i - 1, i = 1 \\ i, i = 2 \\ i + 1, i = 3, 5 \\ i + 4, i = 4, 6 \\ i + 3, i = 8 \\ i + 7, i = 9 \\ i + 10, i = 10, 12 \\ i + 11, i = 13 \\ i + 12, i = 14 \end{cases}$$

The edge function $g^*: E(G) \rightarrow 1, 3, \dots, 2q - 1$ then the induced edge labels follow $1, 3, 5, \dots, 2q - 1$ by taking the mean value of two adjacent vertices. Therefore $g^*(E) = 1, 3, \dots, 2q - 1$. So f is Uniform(even) Vertex odd Mean Labeling. A graph that admits Uniform(even) Vertex odd Mean Labeling is an even vertex odd mean graph.

Theorem 5: *The graph $P_2(QS_2)$ admits an even point (vertex) odd ratio (mean) labeling.*

Proof: Let G be a graph with 14 vertices and 17 edges. Let $v_1, v_2, \dots, v_n$, $n = 14$ $e_1, e_2, \dots, e_n$, $n = 17$ are edges of the graph of G . As illustrated in Figure 5. Now define the unique mapping function

$f: V(G) \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows.

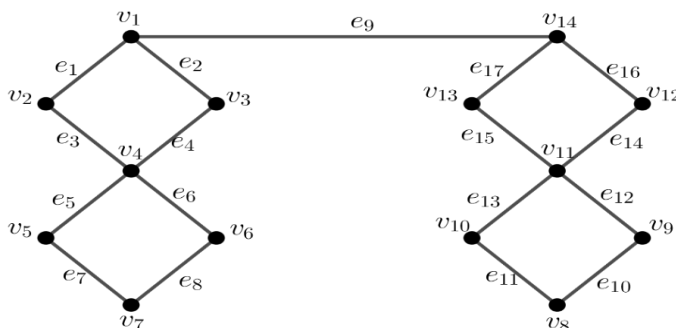


Figure 5
 $P_2(QS_2)$ Graph

$$f(v_i) = \begin{cases} 2i - 2, 1 \leq i \leq 2 \\ 2i, 3 \leq i \leq 5 \\ 2i + 2, 6 \leq i \leq 9 \\ 2i + 4, 10 \leq i \leq 12 \\ 2i + 6, 13 \leq i \leq 14 \end{cases}$$

The edge function $g^*: E(G) \rightarrow 1, 3, \dots, 2q - 1$ then the induced edge labels are obtained by $1, 3, 5, \dots, 2q - 1$ by taking the mean value of two adjacent vertices. Therefore $g^*(E) = 1, 3, \dots, 2q - 1$. So f is an even vertex odd mean labelling. A graph that admits an even point (vertex) odd ratio (mean) labelling is an even vertex odd mean graph.

Theorem 6: *The double alternate triangular graph admits an even point (vertex) odd ratio (mean) labelling.*

Proof: A graph G with p vertices and q edges. Let $v_1, v_2, \dots, v_n$ vertices $e_1, e_2, \dots, e_n$ are edges of the graph of G . As illustrated in Figure 6. Here $V(G) = 5i + 1$ and $E(G) = 7i, i = 2$. Now define a unique mapping function $f: V(G) \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows.

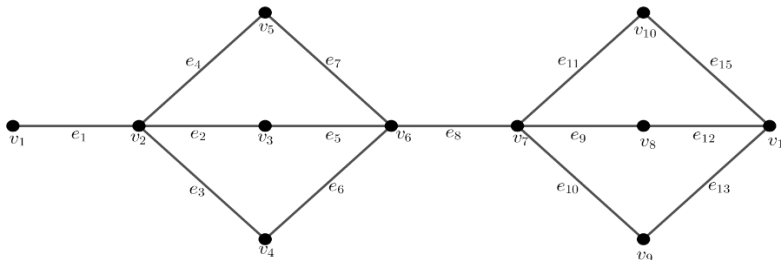


Figure 6
Double alternate triangular graph

$$f(v_i) = \begin{cases} 2i - 2, 1 \leq i \leq 3 \\ 2i, i = 4 \\ 2i + 2, 5 \leq i \leq 8 \\ 2i + 4, i = 9 \\ 2i + 6, 10 \leq i \leq 11 \end{cases}$$

The edge function $g^*: E(G) \rightarrow 1, 3, \dots, 2q - 1$ then the induced edge labels are obtained by $1, 3, 5, \dots, 2q - 1$ by taking the mean value of two adjacent vertices. The edge values are obtained by $e_i = g^*(uv) = 2i - 1$. Therefore $g^*(E) = 1, 3, \dots, 2q - 1$. So f is an even vertex odd mean labeling. A graph that admits an even point (vertex) odd ratio (mean) labeling is an even vertex odd mean graph.

Theorem 7: *The double alternate triangular with K_1 admits an even point (vertex) odd ratio (mean) labelling.*

Proof: Let $u_1, u_2, \dots, u_n$ be the vertices and $e_1, e_2, \dots, e_n$ are edges of the alternate triangular graph G with K_1 . Here $V(G) = 7i + 1, E(G) = 9i, i = 2$. Shown in Figure 7. Now define the unique mapping function $f: V(G) \rightarrow \{0, 2, 4, \dots, 2q - 2, 2q\}$ as follows.

$$f(v_i) = \begin{cases} 2i - 2, 1 \leq i \leq 5 \\ 2i + 2, 6 \leq i \leq 8 \end{cases}$$

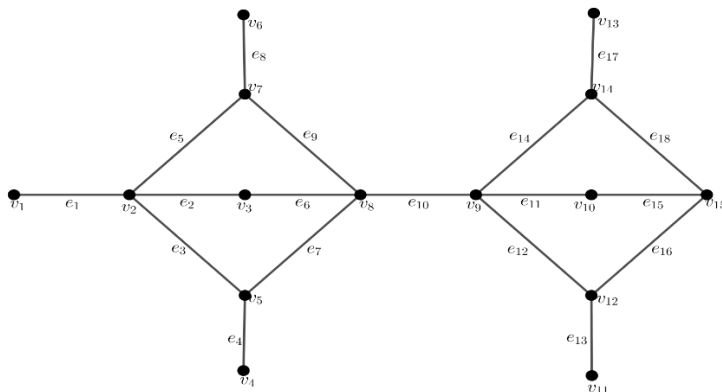


Figure 7
Double alternate triangular with K_1

The edge function $g^*: E(G) \rightarrow 1, 3, \dots, 2q - 1$ then the induced edge labels are obtained by $1, 3, 5, \dots, 2q - 1$ by taking the mean value of two adjacent vertices of the graph G . The edge values are obtained by $e_i = g^*(uv) = 2i - 1$. Therefore $g^*(E) = 1, 3, \dots, 2q - 1$. So f is an even point (vertex) odd ratio (mean) labelling." A graph that allows for even vertex odd mean labeling is referred to as an even vertex odd mean graph.

Future Research Directions

Exploring the impact of even vertex odd mean labeling on topological indices in chemical graph theory or molecular networks could uncover valuable cross-disciplinary applications.

Conclusion

This work introduced (demonstrates) an even point (vertex) Odd ratio (mean) Labeling for certain graph families and suggests that this method can be extended to other types of graph families.

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