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δ and prime-labelling of a tree from caterpillars

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Abstract

This paper explores δ-labelling and prime labelling for trees derived from caterpillar structures. In δ-labelling, vertex labels induce injective edge labels, while prime labelling ensures adjacent vertices have a greatest common divisor of one. An algorithm for efficiently labelling caterpillar trees using both methods is introduced. These labelling techniques have practical applications in network security and cryptography, providing new strategies to strengthen the security and robustness of communication networks and cryptographic systems.

Subject Classification: *Primary 05C78, Secondary 05C69.*

Keywords: *δ - Labelling, Prime labelling, Tree and caterpillars.*

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1. Introduction

In graph theory, assigning non-negative integers to the vertices or edges of a graph is a fundamental concept[2]. A labelling of a graph, denoted as G with vertex set V and edge set E, is formally defined as a one-to-one function from the set of vertices V to a finite set of non-negative integers X. This vertex labelling naturally induces an edge labelling, where each edge, connecting vertices u and v, is assigned the label corresponding to the absolute difference between the labels of u and v. For graphs with a specific number of edges, q, particular attention is given to labellings where X is either the set containing integers from zero to q, or from zero to two q. Rosa referred to these types of labelling as "valuations" and examined four particular kinds. When the function f is a valuation for a graph with q edges, it is called an alpha-valuation if X is a subset of the set from zero to q, the resulting edge labels are the integers from one to q, and there exists a specific value in X such that for every edge, the label of one vertex is less than or equal to this value while the other is greater. If X is a subset of the set from zero to q and the edge labels still range from one to q, the labelling is termed a beta-valuation. When X is a subset of the set from zero to two q and the edge labels are one to q, it is known as a sigma valuation. A rho-valuation occurs when X is a subset of the set from zero to two q, and the edge labels are a set of integers such that for each i between one and q, the label of the edge is either i or two q plus one minus i. These definitions form a hierarchy where every alpha-valuation is also a beta-valuation, every beta-valuation is a sigma-valuation, and every sigma-valuation is a rho-valuation. The term "graceful labelling," introduced by Golomb, corresponds to what is now referred to as a beta-valuation. A graph possessing a graceful labelling is referred to as a graph with graceful properties. Rosa's foundational research [1] has sparked extensive studies in this domain, resulting in numerous publications. Much of the focus has been on identifying new categories of graphs that exhibit graceful properties. Additionally, various extensions and modifications to the graceful labelling concept have been explored. For a comprehensive overview of this research, Gallian's ongoing survey [7] and other relevant works [3, 4, 5, 9, 13, 14] provide valuable insights. A significant unresolved challenge in this area is the "Ringel-Kotzig" conjecture, which proposes that all trees can be labelled gracefully. According to Gallian [8], "Aldred and McKay", using computational methods, demonstrated that all trees with up to "27" vertices can be gracefully labelled, a result later extended to trees with up to 35 vertices [6]. Furthermore, paths, certain subclasses of trees, and other tree structures are known to have graceful properties. In 2014, N. Parvathi and S. Vidyanandini proved the graceful labelling of trees derived from caterpillars [12]. δ-labelling is a novel type of labelling introduced by G. Sethuraman and V. Murugan in 2021 [10]. They demonstrated that the decomposition of complete graphs into arbitrary trees is possible and that all trees can be δ-labeled. This paper introduces the concepts of δ-labelling and prime labelling specifically for trees derived from caterpillars. This paper introduces the concepts of δ-labelling and prime labelling specifically for trees structured from caterpillars

2. Preliminary

Definition: [11] "Let $G = (V(G), E(G))$ be a graph with m edges. An injection $f: V(G) \rightarrow \{1, 2, 3, \ldots, 4m\}$ is said to be a δ - labelling of G if the induced edge labelling function $f': E(G) \rightarrow \{1, 2, 3, \ldots, 2m\}$ is such that for each $uv \in E(G)$.

$$f'(uv) = \begin{cases} |f(u) - f(v)| & \text{if } |f(u) - f(v) \leq 2m \\ 4m + 1 - |f(u) - f(v)| & \text{if } |f(u) - f(v) > 2m \end{cases}$$

is injective. The graph that admits a δ – labelling is called δ – labelling of the graph.

Definition: [8] Let $G = (V(G), E(G))$ be a graph with P vertices. A bijection f : $V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\}$ is called a prime labelling if for each edge $e = uv$, $gcd(f(u), f(v)) = 1$. A graph which admits prime labelling is called a prime graph."

Definition : [15] "A caterpillar is a tree such that if one removes all of its leaves, the remaining graph is a path this path can be termed as backbone of the caterpillar."

3. δ-labelling of tree from caterpillars:

In this section, we construct a tree with δ-labelling derived from a collection of δ-caterpillars. Let G_1 represent a caterpillar containing m_1 edges and having a δ-labelling δ_1. Define the bipartition of the node set of $V(T_1)$ as "(V_{11}, V_{12})", where the labels assigned to the vertices in V_{11} as specified in δ_1 are less than or equal to β_1, the width of the δ-labelling for G_1. Similarly, let G_2 denote another caterpillar with m_2 edges and a δ-labelling δ_2. Let (V_{21}, V_{22}) represent the bipartition of the vertex set of $V(T_2)$, where the vertex labels in V_{21} , as specified by δ_2, are less than or equal to β_2, which denotes the width of the δ-labelling for G_2. We construct a new tree G^* by connecting an edge between the vertex labeled 0 in G_1 and the vertex labeled $2(m_2 - 1)$ in G_2. Note that G^* contains. The sum of the cardinalities of V_1 and V_2 vertices and The sum of m_1, m_2 and one edge.

The δ-labelling δ^* for the tree G^* is given by the function

$$\delta^* = \begin{cases} \delta_1(v) & if \quad v \in V_{11} \\ \delta_2(v) + \beta_1 + 1 & \text{if} \quad v \in V_{21} \\ \delta_2(v) + \beta_1 + 1 & \text{if} \quad v \in V_{22} \\ \delta_1(v) + m_2 + 1 & \text{if} \quad v \in V_{12} \end{cases}$$

Lemma 3.1: *The labels assigned to the vertices of G^* by the labelling function δ^* are all distinct.*

Proof: Consider a caterpillar graph G_1 with m_1 edges and another caterpillar graph G_2 with m_2 edges. Let δ_1 denote the δ-labelling for G_1 and δ_2 denote the δ-labelling for G_2. Since G_1 is a bipartite graph, let V_{11} and V_{12} be the two partitions of its vertex set. Similarly, let V_{21} and V_{22} be the partitions of the vertex set of G_2. Consider two arbitrarily chosen vertices p and q of G^*. To demonstrate that $\delta(p)$ and $\delta(q)$ are distinct, suppose for contradiction that $\delta(p) = \delta(q)$.

Case 1: Assume $p \in V_{11}$ and $q \in V_{12}$. According to the labelling function δ^*, it follows that $\delta(p) = \delta_1(p)$ and $\delta(q) = \delta_1(q) + m_2 + 1$. This results in the equation $m_2 = -1$, which is contradictory, as a tree cannot have a negative number of edges.

Case 2: Suppose $p \in V_{11}$ and $q \in V_{21}$. According to the labelling function, we have $\delta(p) = \delta_1(p)$ and $\delta(q) = \delta_2(q) + \beta_1 + 1$.

- **Sub Case 2.1**: If $\delta_1(p) < \delta_2(q)$, then let $\delta_2(q) = \delta_1(p) + c$, where $c \geq 1$. Therefore, $\delta_1(p) = \delta_1(p) + c + \beta_1 + 1$. This results in $c + \beta_1 = -1$, which is a contradiction.
- **Sub Case 2.2:** $\delta_2(q) < \delta_1(p)$, then $\delta_1(p) = \delta_2(q) + d$, $d \geq 1$. Therefore, $\delta_2(q) + d = \delta_2(q) + \beta_1 + 1$, a which leads to an inconsistency

Case 3: Let $u \in V_{11}$ and $v \in V_{22}$. According to the labeling rule, it follows that $\delta(p) = \delta_1(p)$ and $\delta(q) = \delta_2(q) + \beta_1 + 1$. Given that $\delta(p) = \delta(q)$, we can deduce that $\delta_1(p) = \delta_2(q) + \beta_1 + 1$.

- **Sub Case 3.1:** If $\delta_1(p) < \delta_2(q)$, then let $\delta_2(q) = \delta_1(p) + e$, where $e \geq 1$. Therefore, $\delta_1(p) = \delta_1(p) + e + \beta_1 + 1$. This implies $e + \beta_1 = -1$, which is a contradiction.
- **Sub Case 3.2:** $\delta_2(q) < \delta_1(p)$, then $\delta_1(p) = \delta_2(q) + f$, $f \geq 1$. Therefore, $\delta_2(q) + f = \delta_2(q)\beta_1 + 1$, a contradiction.

Case 4: Assume $q \in V_{11}$ and $p \in V_{12}$. According to the labelling function δ^*, we have $\delta(q) = \delta_1(q)$ and $\delta(p) = \delta_1(p) + m_2 + 1$. However, based on our assumption, $\delta(q) = \delta(p)$. This results in the equation $\delta_1(q) = \delta_1(p) + m_2 + 1$. Consequently, we find that $m_2 = -1$, which presents a contradiction, as a tree cannot have a negative number of edges.

Case 5: $q \in V_{11}$ and $p \in V_{21}$ by the labelling function, we have $\delta(q) = \delta_1(q)$ and $\delta(p) = \delta_2(p) + \beta_1 + 1$. since $\delta_1(q) = \delta_2(p) + \beta_1 + 1$.

- **Sub Case 5.1:** If $\delta_1(q) < \delta_2(p)$, then it follows that $\delta_2(p) = \delta_1(q) + r$, $r \geq 1$. Consequently, we have $\delta_1(q) = \delta_1(q) + r + \beta_1 + 1$. This leads to the equation $r + \beta_1 = -1$, which is a contradiction.
- **Sub Case 5.2:** If $\delta_2(p) < \delta_1(q)$, then we can express this as $\delta_1(q) = \delta_2(p) + s$ where , $s \geq 1$. Therefore, we have $\delta_2(p) + s = \delta_2(p) + \beta_1 + 1$, resulting in a contradiction.

Case 6: Let $q \in V_{11}$ and $p \in V_{22}$. According to the labelling function, we find that $\delta(q) = \delta_1(q)$ and $\delta(p) = \delta_2(p) + \beta_1 + 1$. Since $\delta(q) = \delta(p)$, it follows that $\delta_1(q) = \delta_2(p) + \beta_1 + 1$.

- **Sub Case 6.1:** If $\delta_1(q) < \delta_2(p)$, then let $\delta_2(p) = \delta_1(q) + t$, where $t \geq 1$. Therefore, $\delta_1(q) = \delta_1(q) + t + \beta_1 + 1$. This results in $t + \beta_1 = -1$, which is a contradiction.
- **Sub Case 6.2:** If $\delta_2(p) < \delta_1(q)$, then we can write $\delta_1(q) = \delta_2(p) + w$, where $w \geq 1$. Consequently, we have $\delta_2(p) + w = \delta_2(p) + \beta_1 + 1$, leading to a contradiction. Thus, $\delta(p)$ cannot equal $\delta(q)$ in any of the scenarios considered. Thus, the vertex labelling determined by the function δ^* remains unique. Figure [1-3] shows the δ-labelling of trees from caterpillars.

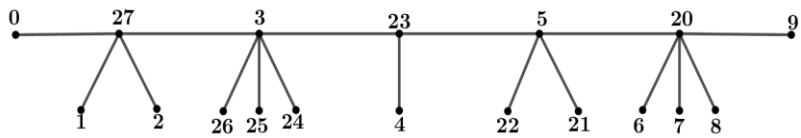

Figure 1
Graph G_1 featuring 17 edges

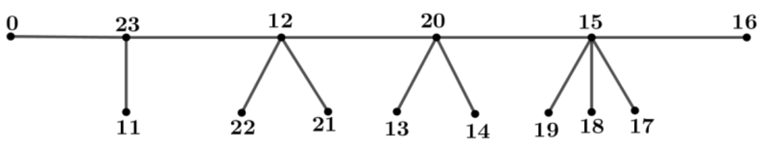

Figure 2
Graph G_2 featuring 13 edges

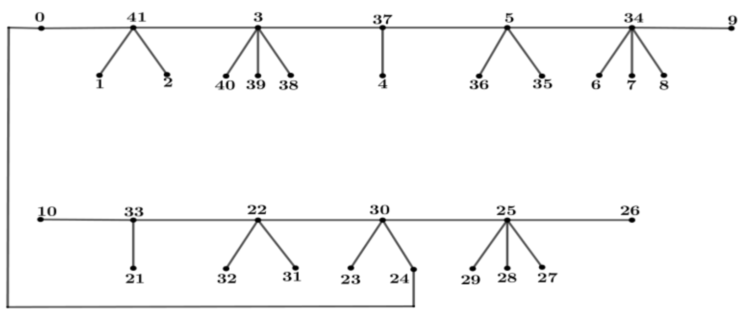

Figure 3
δ^* featuring 31 edges

4. Prime-labelling of tree from caterpillars:

In this section, we aim to create a tree derived from a set of prime-caterpillars. Let G_1 represent a caterpillar with m_1 edges and a prime-labelling denoted as λ_1. Let (V_{11}, V_{12}) represent the bipartition of the vertex collection $V(T_1)$, where the labels of the vertices collection in V_{11}, defined by λ_1, are less than or equal to β_1, which is the width of the prime-labelling of G_1. Additionally, let G_2 be a caterpillar containing m_2 edges with a prime-labelling denoted as λ_2. Let (V_{21}, V_{22}) denote the bipartition of the vertex collection $V(T_2)$, where the labels of the vertices collection in V_{21}, as defined by λ_2, are less than or equal to β_2, the width of the prime-labelling of G_2. A new tree $G^* = (G_1 \cup G_2) + e$ is formed by combining G_1 and G_2 through the addition of a new edge connecting the vertex labeled 6 in G_1 with the vertex labeled $(m_2 + 3)$ in G_2. Note that G^* contains The sum of the cardinalities of V_1 and V_2 vertices and The sum of m_1, m_2 and one edge.

The prime-labelling λ^*, for the tree G^* is defined by the following function:

$$\lambda^* = \begin{cases} p_1(v) & if \quad v \in v_{11} \\ p_2(v) + (\beta 1 - 1) + 1 & if \quad v \in v_{21} \\ p_2(v) + (\beta 1 - 1) + 1 & if \quad v \in v_{22} \\ p_1(v) + m_2 + 1 & if \quad v \in v_{12} \end{cases}$$

Lemma 4.1: *The vertex labels of vertices in $G*$, defined by the labelling function λ^*, are distinct.*

Proof: Let G_1 be a caterpillar with m_1 edges, and G_2 another caterpillar with m_2 edges. Denote λ_1 as the prime-labelling function for G_1 and λ_2 as the prime-labelling function for G_2. Since G_1 is bipartite, let W_{11} and W_{12} represent the bipartition of its vertex set. Similarly, let W_{21} and W_{22} represent the bipartition of the vertex set of G_2. Consider two vertices x and y in $G*$. We will show that $\lambda(x) \neq \lambda(y)$ by assuming the opposite, that $\lambda(x)=\lambda(y)$.

Case 1: Assume $x \in W_{11}$ and $y \in W_{12}$. From the labelling function λ^*, we have $\lambda(x)= \lambda_1(x)$ and $\lambda(y)= \lambda_1(y)+ m_2 +1$. This leads to $m_2 = -1$, which is a contradiction since the number of edges in a tree cannot be negative.

Case 2: Assume $x \in W_{11}$ and $y \in W_{21}$. By the labelling function, $\lambda(x)= \lambda_1(x)$ and $\lambda(y)= \lambda_2(y)+ \gamma_1+1$, where $\gamma 1$ is the width of the prime-labelling for G_1.

- **Sub-case 2.1:** Suppose $\lambda_1(x) < \lambda_2(y)$. Let $\lambda_2(y)= \lambda_1(x)+c$, where $c \geq 1$. This results in $\lambda_1(x)= \lambda_1(x)+c+(\gamma_1-1)+1$, leading to $c+(\gamma_1-1)=-1$, a contradiction.
- **Sub-case 2.2:** Suppose $\lambda_2(y)<\lambda_1(x)$. Let $\lambda_1(x)= \lambda_2(y)+d$, where $d \geq 1$. This gives $\lambda_2(y)+d=\lambda_2(y)+(\gamma_1-1)+1$, another contradiction.

Case 3: Let $x \in W_{11}$ and $y \in W_{22}$. From the labelling function, $\lambda(x)= \lambda_1(x)$ and $\lambda(y)= \lambda_2(y)+(\gamma_1-1)+1$. Since $\lambda(x)=\lambda(y)$, this implies $\lambda_1(x)= \lambda_2(y)+(\gamma_1-1)+1$.

- **Sub-case 3.1:** Suppose $\lambda_1(x) < \lambda_2(y)$. Define $\lambda_2(y) = \lambda_1(x)+e$, where $e \geq 1$. This leads to $\lambda_1(x) = \lambda_1(x)+e+(\gamma_1-1)+1$, which implies $e+(\gamma_1-1)=-1$, a contradiction.
- **Sub-case 3.2:** If $\lambda_1(y) < \lambda_1(x)$, let $\lambda_1(x) = \lambda_2(y)+f$, where $f \geq 1$. This gives $\lambda_2(y)+f=\lambda_2(y)+(\gamma_1-1)+1$, another contradiction.

Case 4: Let $x \in W_{11}$ and $y \in W_{12}$. By the labelling function, $\lambda(x) = \lambda_1(x)$ and $\lambda(y) = \lambda_1(y)+m_2+1$. Since $\lambda(x)=\lambda(y)$, it follows that $\lambda_1(y) = \lambda_1(y)+m_2+1$, implying $m_2=-1$, which contradicts the requirement that the number of edges must be non-negative.

Case 5: Let $x \in W_{11}$ and $y \in W_{21}$. From the labelling function, $\lambda(x) = \lambda_1(x)$ and $\lambda(y) = \lambda_2(y)+(\gamma_1-1)+1$.

- **Sub-case 5.1:** Suppose $\lambda_1(x) < \lambda_2(y)$. Define $\lambda_2(y) = \lambda_1(x)+r$, where $r \geq 1$. This leads to $\lambda_1(x) = \lambda_1(x)+r+(\gamma_1-1)+1$, which implies $r+(\gamma_1-1)=-1$, a contradiction.
- **Sub-case 5.2:** If $\lambda_2(y) < \lambda_1(x)$, let $\lambda_1(x) = \lambda_2(y)+s$, where $s \geq 1$. This gives $\lambda_2(y)+s=\lambda_2(y)+(\gamma_1-1)+1$, another contradiction.

Case 6: Let $x \in W_{11}$ and $y \in W_{22}$. By the labelling function, $\lambda(x) = \lambda_1(x)$ and $\lambda(y) = \lambda_2(y)+(\gamma_1-1)+1$. Since $\lambda(x)=\lambda(y)$, we conclude that $\lambda_1(x) = \lambda_2(y)+(\gamma_1-1)+1$.

- **Sub-case 6.1:** Suppose $\lambda_1(x) < \lambda_2(y)$. Define $\lambda_2(y) = \lambda_1(x)+t$, where $t \geq 1$. This results in $\lambda_1(x) = \lambda_1(x)+t+(\gamma_1-1)+1$, leading to $t+(\gamma_1-1)=-1$, a contradiction.
- **Sub-case 6.2:** If $\lambda_2(y) < \lambda_1(x)$, let $\lambda_1(x) = \lambda_2(y)+w$, where $w \geq 1$. This results in $\lambda_2(y)+w=\lambda_2(y)+(\gamma_1-1)+1$, another contradiction.

Thus, in all possible cases, $\lambda(x) \neq \lambda(y)$, proving that the vertex labels defined by the function λ^* are distinct. Figure [4-6] shows the prime-labelling of trees from caterpillars.

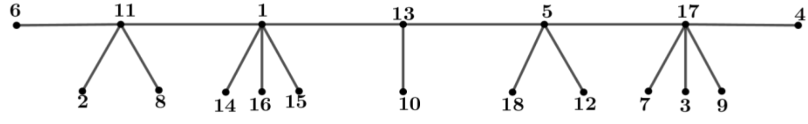

Figure 4
Graph G_1 featuring 17 edges

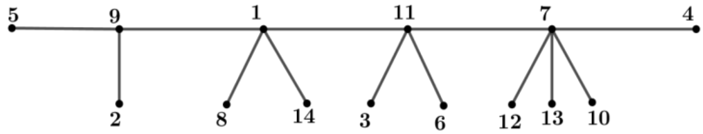

Figure 5
Graph G_2 featuring 13 edges

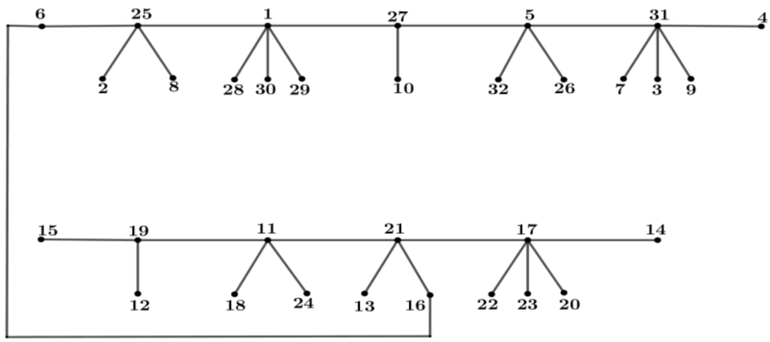

Figure 6
P^* featuring 31 edges

Algorithm 1 Labelling a Caterpillar Tree with Del and Prime Labelling

Step 1: Start
Step 2: Input
Given n vertices and m edges count, where $m = n - 1$.
Step 3: Construct Caterpillar Tree
Construct a caterpillar tree with the given vertices and edges.
Step 4: Label Vertices and Edges
Label vertices as $v_1, v_2, \ldots, v_n$.
Label edges as $e_1, e_2, \ldots, e_n$.
Step 5: Apply Del and Prime Labelling
Apply δ and prime labelling values to the graph.
Step 6: Condition for Del Labelling
The condition for δ labelling is:
"An injection $f : V(G) \to \{1, 2, 3, \ldots, 4m\}$ is said to be a δ-labelling of G if the induced edge labelling function $f' : E(G) \to \{1, 2, 3, \ldots, 2m\}$ is such that for each edge $uv \in E(G)$":
The function f' must be injective.
Step 7: Condition for Prime Labelling

"Let $G = (V(G), E(G))$ be a graph with P vertices. A bijection $f : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ is called a prime labelling if for each edge $e = uv$, $\gcd(f(u), f(v)) = 1$".
Step 8: Verify Distinct Values
Check that all the vertices have distinct values.
Step 9: Stop
End the algorithm.
Stop

5. Application: Network Security and Cryptography

The δ-labelling and prime labelling techniques presented in this paper have significant applications in the field of network security and cryptography. These methods offer innovative ways to enhance the security and resilience of communication networks and cryptographic systems.

5.1. *Cryptographic Key Generation*

The δ-labelling of a graph provides a systematic approach to generate unique labels for vertices and edges. These labels can serve as cryptographic keys, ensuring that each key is distinct and difficult to predict. The injective nature of δ-labelling guarantees that no two edges share the same label, which is crucial for maintaining the integrity of cryptographic algorithms. The large range of potential labels increases the complexity and security of the keys, making it significantly more challenging for unauthorized users to break the encryption.

5.2. *Secure Communication Protocols*

Prime labelling ensures that the labels assigned to vertices of a graph are relatively prime. This property can be utilized in secure communication protocols, where the relatively prime labels help in constructing robust encryption schemes. In these schemes, the greatest common divisor (gcd) condition ensures minimal commonality between keys, reducing the risk of common factor attacks and enhancing the overall security of the communication channels. By ensuring that each pair of connected nodes uses co-prime labels, the system can maintain high security standards, even in the presence of sophisticated attacks. 5.3. Resilient Network Architectures. The combination of δ-labelling and prime labelling can be employed to design network architectures that are resilient to attacks and failures. The unique labelling strategies can be used to create distinct pathways and connections within the network, making it harder for an attacker to predict or intercept communication flows. In the event of a compromised node, the network can reroute communication through alternative paths with distinct labels, maintaining secure and uninterrupted service. This resilience is crucial for critical infrastructure and sensitive data transmissions, ensuring that the network remains functional and secure under various conditions.

5.4. *Enhancing Encryption Algorithms*

The methodologies of δ-labelling and prime labelling can be integrated into existing encryption algorithms to strengthen their security. For example, the labelling can add an additional layer of complexity to symmetric and asymmetric encryption schemes, making the decryption process more challenging for unauthorized users. The injective and prime properties ensure that each part of the encrypted message has a unique and secure representation, thus improving the overall robustness of the encryption. This enhancement is vital for protecting sensitive information in fields such as finance, healthcare, and government.

5.5. *Graph-Based Cryptanalysis Prevention*

The injective nature of δ-labelling and the relative primeness in prime labelling provide a robust framework for preventing graph-based cryptanalysis attacks. These labelling techniques ensure that the structure of the graph, and thus the underlying cryptographic system, is complex and non-trivial to analyze. This complexity makes it significantly more difficult for an attacker to deduce patterns or exploit vulnerabilities within the system. By employing these advanced labelling techniques, the security of cryptographic systems can be substantially increased, providing greater protection for digital communications and data storage.

5.6. *Societal Benefits*

The application of δ-labelling and prime labelling in network security and cryptography has profound societal benefits. Enhanced cryptographic systems safeguard sensitive personal and financial information, protecting individuals from fraud and identity theft. Secure communication protocols ensure the confidentiality and integrity of data exchanged over the internet, which is vital for both personal privacy and national security. Furthermore, resilient network architectures contribute to the robustness of critical infrastructure, such as power grids and communication networks, against cyber-attacks and natural disasters. By improving the security and reliability of these systems, the techniques developed in this research support the creation of a safer and more secure digital society.

6. Conclusion

This paper introduces δ-labelling and Prime labelling trees constructed from pairs of caterpillars, allowing for flexibility even when the caterpillars have unequal numbers of edges. By replacing the initial edge between caterpillars G_1 and G_2 with another edge that induces labels of $2m_2 - 2$ and $m_2 + 3$, various δ and Prime trees can be generated from the same caterpillar inputs. These methods provide powerful tools for enhancing the security and resilience of cryptographic systems and communication networks, applicable in cryptographic key generation, secure communication protocols, network resilience, and graph-based cryptanalysis prevention. Integrating these advanced labelling techniques promises to strengthen system security, reliability, and efficiency, thereby protecting sensitive information and critical infrastructure.

References

[1] A. Rosa. "On certain valuations of the vertices of a graph", in Theory of Graphs. Internat. Symposium, Rome, Gordon and Breach, N. Y. and Dunod Paris, pp. 349 – 355 (1967).

[2] D.B. West, "Introduction to Graph Theory", second ed., Prentice-Hall, Englewood cliffs, NJ (2001).

[3] K.M. Koh, D. G. Rogers and T. Tan. "On graceful trees", Nanta Math. Vol. 10(2), pp. 27 – 31 (1997).

[4] P. Hrnciar and Alfonz Haviar. "All trees of diameter five are graceful", Disc. Math, Vol. 233 (1 – 3), pp. 133-150 (2001).

[5] R.E.L. Aldred and B.D. McKay. "Graceful and harmonious labelings of trees", Bull. Inst. Combin. Appl, Vol. 23, pp. 69-72, (1998).

[6] M. M. Al Aziz, M. F. Hossain, T. Faequa and M. Kaykobad, "Graceful labeling of trees: Methods and applications", 2014 17th International Conference on Computer and Information Technology (ICCIT), Dhaka, Bangladesh, pp. 92-95, (2014).

[7] W. Fang. "A computational approach to the graceful tree conjecture" Accessed July 31(2011). http://arxiv.org/abs/1003.3045.

[8] J. A Gallian, "A Dynamic Survey of Graph Labeling", *The Electronic Journal of Combinatorics*, Vol. 15 (2008).

[9] S. W Golomb, "How to number a graph", in Graph Theory and Computing, R. C. Read,ed., Academic Press, New York, pp. 23-37 (1972).

[10] M Horton, "Graceful Trees: Statistics and Algorithms", Ph. D. Thesis, University of Tasmania (2003).

[11] G. Sethuraman, V. Murugan, "Decomposition of Complete Graphs into Arbitrary Trees" Graphs and Combinatorics 37, 1191-1203 (2021).

[12] N. Parvathi and S. Vidyanandini. "Graceful Labeling of a Tree from Caterpillars", *Journal of Information and Optimization Sciences*. 35:4, pp. 387-393(2014).

[13] A. Sethukkarasi , S Vidyanandini & Nayak, Soumya Ranjan (2024) “Local antimagic vertex coloring of a Myceilski of graphs”. *Journal of Discrete Mathematical Sciences and Cryptography*, 27:4, 1389–1401(2024).

[14] A. Sethukkarasi and S. Vidyanandini, “Graph Composite Labeling techniques and Practical Applications”, 2024 International Conference on Emerging Systems and Intelligent Computing (ESIC), Bhubaneswar, India, pp. 526-532 (2024).

[15] J Maowa. “Study on graceful labeling of trees”. M. Sc. Thesis, Bangladesh University of Engineering and Technology, Dhaka, Bangladesh (2016).

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