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The k-Bronze Fibonacci sequence

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Abstract

In the present work, we consider the generalized k-Bronze Fibonacci sequence and we give the generalized Binet formula and its consequences, the limit of the quotient of two consecutive terms, the sum of the first terms and the generating function.

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1. Introduction

Numerical sequences provide an important and very challenging area of research, thus attracting the interest of many researchers. We refer, for example, to the important Fibonacci and Lucas sequences, which have attracted much attention among researchers. But several other sequences have also been attracting much interest in research. The reasons for that rely both on the mathematical beauty intrinsically encoded in such sequences and also by their applications.

Lately, many works have appeared in this field of research. We mention, for example, to the works of Catarino [2], where the sequence of k-Pell hybrid numbers was introduced; of Kizilates [9], with the study of another sequence of hybrid numbers (Fibonacci and Lucas); of Koshy [10], with the study of some applications of the Fibonacci and Lucas sequences;

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of Catarino and Borges [4], with the introduction of the numerical sequence constituted by Leonardo's numbers; of Catarino and Vasco [5], with the study of the k-Pell generalized numbers of order m, where m is a non negative integer; of Kartal [8], studying Gaussian Bronze Fibonacci sequences, among others. Also, the paper of Nanhongkai and Leerawat [12], yielding closed formulas for the generating function and the exponential generating function of the D_k-sequence, where D_k stands for the k-th diagonal row of Pascal's triangle.

We also point out the recent works of Akbiyik and Alo [1], dedicated to third-order Bronze Fibonacci numbers; of Catarino and Almeida [6], studying some properties of Vietoris' numbers sequence; Catarino and Campos [3], offering a tour through k-Fibonacci numbers, k-Jacobsthal numbers, k -Pell numbers, balancing numbers, k-telephone numbers, hyper k-pell numbers and incomplete numbers; and the work of Shablya, Kruchinin, and Shelupanov [13], with developments concerning properties for composition of ordinary generating functions, allowing to obtain primality criteria for several known sequences.

In this paper, we introduce a new generalization of the Bronze Fibonacci sequence, which is defined by a recurrence formula that depends on one parameter. For this generalization, we prove some interesting properties: the Binet formula, the generating function, the Tagiuri-Vajda's identity, the Convolution identity, the Gelin-Cesáro identity, the Melham identity and, also, other resulting identities.

This article is organized as follows. In Section 2, we define the k-Bronze Fibonacci sequence and establish some interesting properties of this sequence. Section 3 is devoted to find the sum of the first terms and, in Section 4, we obtain the generating function. The last section of the paper brings some prospects for future work.

2. The k-Bronze Fibonacci sequence and some properties

We introduce a new sequence defined recursively through the equality

$$BF_{k,n+1} = 3BF_{k,n} + kBF_{k,n-1}, \quad k \in \mathbb{Z}^+, \tag{2.1}$$

with the initial conditions $BF_{k,0} = 0$ and $BF_{k,1} = 1$. We call it the k-Bronze Fibonacci sequence. Table 1 shows some terms of this sequence.

Table 1

The k-Bronze Fibonacci numbers $BF_{k,n}$, for $n \in \{0,1,2,3,4,5,6\}$.

$BF_{k,0}$	=	0
$BF_{k,1}$	=	1
$BF_{k,2}$	=	3
$BF_{k,3}$	=	3^2+k
$BF_{k,4}$	=	3^3+6k
$BF_{k,5}$	=	$3^4+3^3k+k^2$
$BF_{k,6}$	=	$3^5+(3^4+3^3)k+3^2k^2$

If we take $k=1$, then the k-Bronze Fibonacci sequence is reduced to the Bronze Fibonacci sequence, or 3-Fibonacci sequence, $\{B_n\}$, listed in the Online Encyclopedia of integers sequences [14] as the sequence A006190:

$$B_{n+1} = 3B_n + B_{n-1}, \tag{2.2}$$

with $B_0 = 0$ and $B_1 = 1$.

The characteristic equation associated with the recurrence relation (2.2) is $x^2-3x-1=0$, which has two roots $x_1 = \frac{3+\sqrt{13}}{2}$, called the Bronze number, and $x_2 = \frac{3-\sqrt{13}}{2}$. Note that $x_1+x_2=3$, $x_1-x_2 = \sqrt{13}$, $x_1x_2 = -1$, $\frac{x_1}{x_2} = -x_1^2$ and $\frac{x_2}{x_1} = -x_2^2$.

From the recurrence relation (2.1), defining the k-Bronze Fibonacci sequence, we get the characteristic equation $x^2-3x-k=0$, yielding the two roots

$$r_1 = \frac{3+\sqrt{9+4k}}{2} \quad \text{and} \quad r_2 = \frac{3-\sqrt{9+4k}}{2}. \tag{2.3}$$

We are going to call $r_1 = \frac{3+\sqrt{9+4k}}{2}$ the k-Bronze number. Note that

$$r_1+r_2=3, \qquad r_1-r_2=\sqrt{9+4k},$$
$$r_1r_2=-k, \qquad \frac{r_1}{r_2}=-\frac{1}{k}r_1^2, \qquad \frac{r_2}{r_1}=-\frac{1}{k}r_2^2.$$

We also observe that $r_2 < 0 < r_1$ and $|r_2| < r_1$.

2.1 *The Binet formula*

In this section, we start finding an explicit formula (the Binet Formula) that allows us to express the general term of the sequence $\{BF_{k,n}\}_{n\geq 0}$ using just the roots r_1 and r_2 obtained in (2.3). Based on the Binet formula, many other interesting properties are derived.

Theorem 2.1 (Binet's formula): *The general term of the sequence* $\{BF_{k,n}\}_{n\geq 0}$ *is obtained as*

$$BF_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{2.4}$$

where r_1 *and* r_2 *are the roots presented in (2.3).*

Proof: We clearly have $BF_{k,n} = c_1 r_1^n + c_2 r_2^n$, for some constants c_1 and c_2. By taking $n=0$ we obtain $c_1 + c_2 = 0$, and, if $n=1$ we get $c_1 r_1 + c_2 r_2 = 1$. Thus $c_1 = \frac{1}{r_1 - r_2}$ and $c_2 = -\frac{1}{r_1 - r_2}$, from which the result follows.

Corollary 2.2: *For negative integers,*

$$BF_{k,-n} = (-1)^{n+1} k^{-n} BF_{k,n}, \qquad n \in \mathbb{Z}^+. \tag{2.5}$$

Proof: The Binet formula (2.4), together with the relations $\frac{1}{r_1} = -\frac{1}{k} r_2$ and $\frac{1}{r_2} = -\frac{1}{k} r_1$, leads to

$$\begin{aligned} BF_{k,-n} &= \frac{r_1^{-n} - r_2^{-n}}{r_1 - r_2} = \frac{\left(\frac{1}{r_1}\right)^n - \left(\frac{1}{r_2}\right)^n}{r_1 - r_2} = \frac{\left(-\frac{1}{k} r_2\right)^n - \left(-\frac{1}{k} r_1\right)^n}{r_1 - r_2} \\ &= -\left(-\frac{1}{k}\right)^n \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) = (-1)^{n+1} k^{-n} BF_{k,n}, \end{aligned}$$

that is the desired equality.

As illustration, in Table 2 is depicted some k-Bronze Fibonacci numbers $BF_{k,n}$ and $BF_{k,-n}$.

Table 2

$\boldsymbol{BF_{k,n}}$ **and** $\boldsymbol{BF_{k,-n}}$ **for** $\boldsymbol{n \in \{0,1,2,3,4,5,6\}}$.

$BF_{k,0}$	$=$	0			
$BF_{k,1}$	$=$	1	$BF_{k,-1}$	$=$	$\frac{1}{k}$
$BF_{k,2}$	$=$	3	$BF_{k,-2}$	$=$	$-\frac{3}{k^2}$
$BF_{k,3}$	$=$	3^2+k	$BF_{k,-3}$	$=$	$\frac{3^2}{k^3}+\frac{1}{k^2}$
$BF_{k,4}$	$=$	3^3+6k	$BF_{k,-4}$	$=$	$-\frac{3^3}{k^4}-\frac{6}{k^3}$
$BF_{k,5}$	$=$	$3^4+3^3k+k^2$	$BF_{k,-5}$	$=$	$\frac{3^4}{k^5}+\frac{3^3}{k^4}+\frac{1}{k^3}$
$BF_{k,6}$	$=$	$3^5+(3^4+3^3)k+3^2k^2$	$BF_{k,-6}$	$=$	$-\frac{3^5}{k^6}-\frac{3^4+3^3}{k^5}-\frac{3^2}{k^4}$

2.2 *Limit of the quotient of two consecutive terms*

It is well-known that the quotient of consecutive terms of the classical Fibonacci sequence gives the golden number $\Phi = \frac{1+\sqrt{5}}{2}$, just as the quotient of consecutive terms of the Bronze Fibonacci sequence gives the bronze number $\Psi = \frac{3+\sqrt{13}}{2}$. It is natural to wonder whether a similar result holds for the generalized k-Bronze Fibonacci sequence. The next proposition shows that the quotient of two consecutive elements of the latter sequence approaches the k-Bronze number.

Proposition 2.3: $\lim_{n\to\infty} \frac{BF_{k,n}}{BF_{k,n-1}} = r_1$ and $\lim_{n\to\infty} \frac{BF_{k,n-1}}{BF_{k,n}} = \frac{1}{r_1}$.

Proof: The Binet formula (2.4) yields

$$\lim_{n\to\infty} \frac{BF_{k,n}}{BF_{k,n-1}} = \lim_{n\to\infty} \frac{\frac{r_1^n - r_2^n}{r_1 - r_2}}{\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}} = \lim_{n\to\infty} \frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}}$$

$$= \lim_{n\to\infty} \frac{1 - \left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1} - \left(\frac{r_2}{r_1}\right)^{n-1}} = r_1,$$

since $\left|\frac{r_2}{r_1}\right| < 1$, implying that $\lim_{n\to\infty} \left(\frac{r_2}{r_1}\right)^n = 0$. Basic properties of the calculus of limits allows to get the second equality.

2.3 *Tagiuri-Vajda's identity and other resulting identities*

Using the Binet formula we find next the Tagiuri-Vajda identity for the *k*-Bronze Fibonacci sequence. Then, as a consequence of this identity, we obtain other important identities. In particular, Catalan's identity, D'Ocagne's identity and Cassini's identity are also derived below.

Theorem 2.4 (Tagiuri-Vajda's identity): *For n, s, r integers we have*

$$BF_{k,n+s}BF_{k,n+r}-BF_{k,n}BF_{k,n+s+r}=(-k)^{n}BF_{k,r}BF_{k,s}. \tag{2.6}$$

Proof: The Binet formula (2.4) allows to write

$$\begin{aligned} BF_{k,n+s}BF_{k,n+r}-BF_{k,n}BF_{k,n+s+r} &= \\ &=\frac{(r_1^{n+s}-r_2^{n+s})(r_1^{n+r}-r_2^{n+r})-(r_1^{n}-r_2^{n})(r_1^{n+s+r}-r_2^{n+s+r})}{(r_1-r_2)^2} \\ &=\frac{(r_1r_2)^n(-r_1^sr_2^r-r_1^rr_2^s+r_2^{r+s}+r_1^{r+s})}{(r_1-r_2)^2} \\ &=\frac{(-k)^n[r_1^s(r_1^r-r_2^r)-r_2^s(r_1^r-r_2^r)]}{(r_1-r_2)^2} \\ &=(-k)^n\left(\frac{r_1^r-r_2^r}{r_1-r_2}\right)\left(\frac{r_1^s-r_2^s}{r_1-r_2}\right) \\ &=(-k)^n BF_{k,r}BF_{k,s}. \end{aligned}$$

Corollary 2.5 (Catalan's identity). *For n, r integers,*

$$BF_{k,n-r}BF_{k,n+r}-BF_{k,n}^2=(-1)^{n+r+1}k^{n-r}BF_{k,r}^2. \tag{2.7}$$

Proof: For $s=-r$, the identity (2.6) becomes

$$BF_{k,n-r}BF_{k,n+r}-BF_{k,n}^2=(-k)^{n}BF_{k,r}BF_{k,-r}.$$

Using (2.5), we obtain

$$\begin{aligned} BF_{k,n-r}BF_{k,n+r}-BF_{k,n}^2 &=(-k)^{n}BF_{k,r}(-1)^{r+1}k^{-r}BF_{k,r} \\ &=(-1)^{n+r+1}k^{n-r}BF_{k,r}^2. \end{aligned}$$

Corollary 2.6 (D'Ocagne's identity): *For n, m integers,*

$$BF_{k,n+1}BF_{k,m}-BF_{k,n}BF_{k,m+1}=(-k)^{n}BF_{k,m-n}. \tag{2.8}$$

Proof: Identity (2.6) with $s=1$ and $m=n+r$ becomes

$$BF_{k,n+1}BF_{k,m}-BF_{k,n}BF_{k,m+1}=(-k)^n BF_{k,r}BF_{k,1}=(-k)^n BF_{k,m-n}.$$

Corollary 2.7 (Cassini's identity): *For n integer,*

$$BF_{k,n-1}BF_{k,n+1}-BF_{k,n}^2=(-1)^n k^{n-1}. \tag{2.9}$$

Proof: It follows immediately from equality (2.6) taking $r=1$ and $s=-1$.

Corollary 2.8: *For n and r integers,*

$$BF_{k,n+1}BF_{k,n+r}-BF_{k,n}BF_{k,n+r+1}=(-1)^n k^n BF_{k,r} \tag{2.10}$$

$$BF_{k,n-1}BF_{k,n+r}-BF_{k,n}BF_{k,n+r-1}=(-1)^n k^{n-1} BF_{k,r}. \tag{2.11}$$

Proof: It follows immediately from equality (2.6), with $s=1$ and $s=-1$, respectively, since $BF_{k,1}=1$ and $BF_{k,-1}=\frac{1}{k}$.

In particular, taking $r=2$ in equality (2.10) and $r=-2$ in equality (2.11), we obtain, respectively, the two interesting equalities,

$$BF_{k,n+1}BF_{k,n+2}-BF_{k,n}BF_{k,n+3}=3(-1)^n k^n,$$

$$BF_{k,n-2}BF_{k,n-1}-BF_{k,n-3}BF_{k,n}=3(-1)^{n+1} k^{n-3}.$$

2.4 *Other interesting identities*

In this subsection, some additional properties of the sequence $\{BF_{k,n}\}_{n\geq 0}$ are stated and proved.

Proposition 2.9 (Convolution identity): For *m, n* positive integers

$$BF_{k,m+n}=kBF_{k,m-1}BF_{k,n}+BF_{k,m}BF_{k,n+1}. \tag{2.12}$$

Proof: Without loss of generality, let us fix *m* and go ahead by induction on *n*. For $n=1$, since $BF_{k,1}=1$ and $BF_{k,2}=3$, we get

$$BF_{k,m+1}=kBF_{k,m-1}BF_{k,1}+BF_{k,m}BF_{k,2}=kBF_{k,m-1}+3BF_{k,m}, \tag{2.13}$$

which is trivially true (see (2.1) applied to *m*). Now, let us assume that

$$BF_{k,m+l}=kBF_{k,m-1}BF_{k,l}+BF_{k,m}BF_{k,l+1}, l\geq 1.$$

By the definition of *k*-Bronze Fibonacci numbers, we have

$$BF_{k,m+l+1} = 3BF_{k,m+l} + kBF_{k,m+l-1}.$$

The induction hypothesis and (2.1) lead to

$$\begin{aligned} BF_{k,m+l+1} &= 3kBF_{k,m-l}BF_{k,l} + 3BF_{k,m}BF_{k,l+1} + k^2BF_{k,m-l}BF_{k,l-1} + kBF_{k,m}BF_{k,l} \\ &= kBF_{k,m-l}(3BF_{k,l} + kBF_{k,l-1}) + BF_{k,m}(3BF_{k,l+1} + kBF_{k,l}) \\ &= kBF_{k,m-l}BF_{k,l+1} + BF_{k,m}BF_{k,l+2}. \end{aligned}$$

Horadam and Shannon in [7] and Melham in [11], proposed, respectively, Gelin-Cesáro's identity and Melham's identity for a general sequence $\{W_n\}$, defined by

$$W_n = aW_{n-1} - bW_{n-2} \quad (n \geq 2), \quad W_0 = w_0, \quad W_1 = w_1,$$

where a, b, w_0 and w_1 are integers. For the sake of completeness, we shall include also Gelin-Cesáro's identity and Melham's identity, and their proofs, obtained specifically for the *k*-Bronze Fibonacci sequence.

Proposition 2.10 (Gelin-Cesáro's identity): For n integer,

$$BF_{k,n+2}BF_{k,n+1}BF_{k,n-1}BF_{k,n-2} - BF_{k,n}^4 = (-1)^n k^{n-2}(k-9)BF_{k,n}^2 - 9k^{2n-3}.$$

Proof: The Cassini identity and the Catalan identity with $r = 2$ allow to write, respectively,

$$\begin{aligned} BF_{k,n-1}BF_{k,n+1} &= BF_{k,n}^2 + (-1)^n k^{n-1}, \quad \text{and} \\ BF_{k,n-2}BF_{k,n+2} &= BF_{k,n}^2 + (-1)^{n+3} k^{n-2} BF_{k,2}^2 \\ &= BF_{k,n}^2 + (-1)^{n+3} 9k^{n-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} & BF_{k,n+2}BF_{k,n+1}BF_{k,n-1}BF_{k,n-2} - BF_{k,n}^4 = \\ = \ & [BF_{k,n}^2 + (-1)^n k^{n-1}][BF_{k,n}^2 + (-1)^{n+3} 9k^{n-2}] - BF_{k,n}^4 \\ = \ & (-1)^{n+3} 9k^{n-2} BF_{k,n}^2 + (-1)^n k^{n-1} BF_{k,n}^2 + (-1)^{2n+3} 9k^{2n-3} \\ = \ & (-1)^n (-9k^{n-2} BF_{k,n}^2 + k^{n-1} BF_{k,n}^2) - 9k^{2n-3} \\ = \ & (-1)^n k^{n-2}(k-9) BF_{k,n}^2 - 9k^{2n-3}. \end{aligned}$$

Proposition 2.11 (Melham's identity): For n integer,

$$BF_{k,n+1}BF_{k,n+2}BF_{k,n+6} - BF_{k,n+3}^3 = (-1)^n k^{n+1}(3^3 BF_{k,n+2} - k^2 BF_{k,n+1}). \tag{2.14}$$

Proof: Accordingly with (2.1), we consider

$$BF_{k,n+3} = 3BF_{k,n+2} + kBF_{k,n+1}. \tag{2.15}$$

We obtain

$$BF_{k,n+6} = (3^4 + 3^3 k + k^2)BF_{k,n+2} + (3^3 k + 6k^2)BF_{k,n+1}. \tag{2.16}$$

Using (2.16), we find

$$BF_{k,n+1}BF_{k,n+2}BF_{k,n+6} = (3^4 + 3^3 k + k^2)BF_{k,n+1}BF_{k,n+2}^2 + (3^3 k + 6k^2)BF_{k,n+1}^2 BF_{k,n+2},$$

and, from (2.15), we get

$$BF_{k,n+3}^3 = 9k^2 BF_{k,n+1}^2 BF_{k,n+2} + 3^3 kBF_{k,n+1}BF_{k,n+2}^2 + k^3 BF_{k,n+1}^3 + 3^3 BF_{k,n+2}^3.$$

Therefore, the left-hand side of (2.14) becomes

$$\begin{aligned} & BF_{k,n+1}BF_{k,n+2}BF_{k,n+6} - BF_{k,n+3}^3 = \\ = \ & (3^4 + k^2)BF_{k,n+1}BF_{k,n+2}^2 + 3k(3^2 - k)BF_{k,n+1}^2 BF_{k,n+2} - \\ & -k^3 BF_{k,n+1}^3 - 3^3 BF_{k,n+2}^3. \end{aligned}$$

By the other hand, from the Cassini's identity (see (2.9)), we have

$$BF_{k,n+1}BF_{k,n+3} - BF_{k,n+2}^2 = (-1)^n k^{n+1}.$$

Hence, the right-hand side of (2.14) turns into

$$\begin{aligned} & (-1)^n k^{n+1}(3^3 BF_{k,n+2} - k^2 BF_{k,n+1}) = \\ = \ & (BF_{k,n+1}BF_{k,n+3} - BF_{k,n+2}^2)(3^3 BF_{k,n+2} - k^2 BF_{k,n+1}) \\ = \ & (3^4 + k^2)BF_{k,n+1}BF_{k,n+2}^2 + 3k(3^2 - k)BF_{k,n+1}^2 BF_{k,n+2} - \\ & -k^3 BF_{k,n+1}^3 - 3^3 BF_{k,n+2}^3, \end{aligned}$$

thus proving the desired equality.

3. A particular sum

We denote by $S_{k,n}$ the sum of the first $n+1$ terms of the k-Bronze Fibonacci sequence, that is, $S_{k,n} = \sum_{i=0}^{n} BF_{k,i}$.

Proposition 3.1 (Sum of first terms): The following sum is valid

$$S_{k,n} = \sum_{i=0}^{n} BF_{k,i} = \frac{1}{k+2}(BF_{k,n+1} + kBF_{k,n} - 1). \tag{3.1}$$

Proof: From (2.4) we get

$$\begin{aligned} S_{k,n} &= \sum_{i=0}^{n} BF_{k,i} = \sum_{i=0}^{n} \frac{r_1^i - r_2^i}{r_1 - r_2} = \frac{1}{r_1 - r_2} \sum_{i=0}^{n} (r_1^i - r_2^i) \\ &= \frac{1}{r_1 - r_2} \left(\frac{1 - r_1^{n+1}}{1 - r_1} - \frac{1 - r_2^{n+1}}{1 - r_2} \right) \\ &= \frac{1}{r_1 - r_2} \frac{(r_1 - r_2) - (r_1^{n+1} - r_2^{n+1}) - k(r_1^n - r_2^n)}{-2 - k} \\ &= \frac{1}{k+2} \left(\frac{r_1^{n+1} + r_2^{n+1}}{r_1 - r_2} + k \frac{r_1^n - r_2^n}{r_1 - r_2} - 1 \right), \end{aligned}$$

thus proving the result.

4. The generating function for the *k*-Bronze Fibonacci sequence

Let $f_k(x)$ be the generating function for the sequence (2.1). Hence

$$f_k(x) = BF_{k,0} + BF_{k,1}x + BF_{k,2}x^2 + \ldots + BF_{k,n}x^n + \ldots = \sum_{n=0}^{+\infty} BF_{k,n}x^n.$$

Proposition 4.1 (The generating function): We have

$$f_k(x) = \frac{x}{1 - 3x - kx^2}. \tag{4.1}$$

Proof: By definition

$$f_k(x) = BF_{k,0} + BF_{k,1}x + BF_{k,2}x^2 + \ldots + BF_{k,n}x^n + \ldots.$$

It follows that

$$\begin{aligned} xf_k(x) &= BF_{k,0}x + BF_{k,1}x^2 + BF_{k,2}x^3 + \ldots + BF_{k,n-1}x^n + \ldots \\ x^2 f_k(x) &= BF_{k,0}x^2 + BF_{k,1}x^3 + BF_{k,2}x^4 + \ldots + BF_{k,n-2}x^n + \ldots \end{aligned}$$

Since $BF_{k,n+2} = 3BF_{k,n+1} + kBF_{k,n}$ and $BF_{k,0} = 0$ and $BF_{k,1} = 1$ we conclude that

$$f_k(x)(1 - 3x - kx^2) = x,$$

and the result follows.

5. Future Work

In this article, a generalization of the Bronze Fibonacci sequence was introduced, having presented the respective Binet formula and its use, not only for the determination of the general term of the sequence, as well as for the study of some identities. It is our intention for future work to dedicate ourselves to the study of sequences of polynomials whose coefficients are k-Bronze Fibonacci numbers, as well as the dual version of this sequence.

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