

## Pseudo-resolvent energy of weighted graphs

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### Abstract

Let  $G$  be a finite order graph. Its resolvent matrix and its resolvent energy are respectively given by  $R_i(A(G)) = (A(G) - \xi I)^{-1}$  and  $ER(G) = \sum_{i=1}^n (n - \lambda_i)^{-1}$ . For some classes of weighted graphs, we know that  $\lambda_i = n$  which means that the formula for  $ER(G)$  is not applicable. In this work, the pseudo-resolvent energy for those classes of graphs are obtained where  $\lambda_i = n$ .

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## 1. Introduction

For a finite graph, the resolvent matrix is defined as  $R_\lambda(A) := (A - \lambda I)^{-1}$ , where  $\lambda$  is a complex number and  $I$  is the unit matrix. The matrix  $R_\lambda(A)$  has many applications in different areas such as: operator algebra, spectral analysis, singularities, for more details, we refer to [1]. In [3], the resolvent matrix  $R_\xi(A)$  was investigated in graph theory and some properties of the so-called *resolvent energy of graphs*  $G$  were established.

Let  $|V(G)| = n$  be a simple graph. Let  $v_1, v_2, \dots, v_n$  be the vertices and  $E(G)$  be the edge set of  $G$ .  $e_{i,j}$  denotes a directed edge of  $G$  from  $v_i$  to  $v_j$ . The adjacency matrix of  $G$  is the matrix

$$A(G) = A_{ij}(G) = \begin{cases} 1 & \text{if } e_{ij} \in E(G); \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $A(G)$ . The spectrum  $Sp(G)$  of a graph is the set of the eigenvalues of  $A(G)$ . The resolvent set of  $A(G)$  is the set of all values  $\lambda$  of  $A(G)$ , i.e. each  $\lambda \in \mathbb{C} \setminus Sp(G)$ . The eigenvalues in graph theory help in calculating the energy and resolvent energy of graph  $G$ . Energy is

$$E = E(G) = \sum_{i=1}^n |\lambda_i|$$

and resolvent energy is

$$ER = ER(G) = \sum_{i=1}^n \frac{1}{n - \lambda_i}. \quad (1.1)$$

Since the concept of graph energy was conceived in [4] and resolvent energy in [3], the number of studies on those objects is growing every year, see [5], [6], [8].

Note that the eigenvalues of the resolvent matrix never coincides with the eigenvalues of  $A(G)$ , in particular,  $R_\lambda(A(G))$  exists for all values of  $\lambda$  except when  $\lambda = \lambda_i$ . Besides, (1.1) can be defined only for the classes of graphs considered by the authors in [3], i.e. simple graphs without weights and multi-edges where  $\lambda_i \leq n-1$ . This definition is not applicable for some special classes of graphs, say weighted graphs such that for some weights, we may have  $\lambda_i = n$ .

To estimate how an object is represented by a matrix, we have to study the pseudospectrum in addition to the eigenvalues and spectrum. The concept of pseudospectrum of a matrix has a number of applications in

**Table 1**

**Monad graphs of the residue class cyclic group for  $n \leq 7$ .  $A_n$  is the connected graph on  $2n$  vertices, consisting of the directed cycle of length  $n$  to which  $n$  one-edge branches are attached, each for every vertex of the cycle. On is the directed cycle on  $n$  vertices.  $T_{2^n}$  is the rooted binary tree on  $2n$  vertices and  $n$  leaves.**

|                        |                                                                                                                                                                                                                    |              |
|------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------|
| $\Gamma(\mathbb{Z}_2)$ | $1 \rightarrow 0 \curvearrowright$                                                                                                                                                                                 | $A_1$        |
| $\Gamma(\mathbb{Z}_3)$ | $\begin{array}{c} \curvearrowright \\ 0 \end{array} \quad 1 \curvearrowright 2$                                                                                                                                    | $O_1 + O_2$  |
| $\Gamma(\mathbb{Z}_4)$ | $\begin{array}{c} 1 \\ \searrow \\ 2 \end{array} \quad \begin{array}{c} \curvearrowright \\ 0 \end{array} \\ \nearrow \\ 3$                                                                                        | $T_4$        |
| $\Gamma(\mathbb{Z}_5)$ | $\begin{array}{ccc} 1 & \rightarrow & 2 \\ \uparrow & & \downarrow \\ 4 & \leftarrow & 3 \end{array} \quad \begin{array}{c} \curvearrowright \\ 0 \end{array}$                                                     | $O_1 + O_4$  |
| $\Gamma(\mathbb{Z}_6)$ | $1 \rightarrow 2 \curvearrowright 4 \leftarrow 5 \quad 3 \rightarrow \begin{array}{c} \curvearrowright \\ 0 \end{array}$                                                                                           | $O_1 + A_2$  |
| $\Gamma(\mathbb{Z}_7)$ | $\begin{array}{ccc} & 2 & \\ & \swarrow \searrow & \\ 1 & & 4 \end{array} \quad \begin{array}{ccc} & 6 & \\ & \swarrow \searrow & \\ 3 & & 5 \end{array} \quad \begin{array}{c} \curvearrowright \\ 0 \end{array}$ | $O_1 + 2O_3$ |

different objects such as dynamical systems, hydrodynamic stability, Markov chains and non-Hermitian quantum mechanics, see [2]. In [7], the phenomena of pseudospectrum was applied to graph theory and pseudospectrum energy of graph  $G$  was defined with unlucky calling as follows:

For any  $\varepsilon > 0$ , the pseudospectrum energy of a graph  $G$  is the following subset

$$PE = PE(G) = \{\lambda \in \mathbb{C} : \sum_{i=1}^n \left| \frac{1}{n - \lambda_i} \right| > \varepsilon_j^{-1} \ j = 1, 2, \dots\}. \quad (1.2)$$

In particular, the  $PE(G)$  is a countable open subset of complex plane which is bounded by the  $\varepsilon_j^{-1}$  levels for each  $\lambda_i$ . One may ask what the pseudospectrum energy of a graph can serve? Perhaps it is not obvious at first if  $PE(G)$  is not large, this means that  $R_\lambda(A(G))$  is close to an eigenvalue of  $A(G)$ .

The difficulty is to determine  $\varepsilon_j^{-1}$  which contains all eigenvalues except that eigenvalue  $\lambda_1 = n$ . In this paper, we exploit  $PE(G)$  to calculate the resolvent energy of some classes of graphs when  $\lambda_1 = n$ . Moreover, we provide some computational studies. In Table 1, we gave a list of monad

graphs of the residue class cyclic group for  $n \leq 7$  to be used in further calculations in this paper.

## 2. Main results

A weighted graph  $W(G)$  is an ordered pair  $(G, f)$  where  $G$  is the underlying graph having finite number of vertices and

$$f: E \rightarrow \mathbb{R} \quad (2.1)$$

is called the weight function which connects every  $u_i$  in the set of edges to a non-zero weight  $f(u_i)$ . The function (2.1) corresponds to the distance between  $v_i$  and  $v_j$  where  $i \neq j$  such as  $f(u) = \max |v_j - v_i|$ . Sometimes this is called a metric edge map. It is very convenient and well-known that each graph can be considered as a weighted graph with constant weight 1.

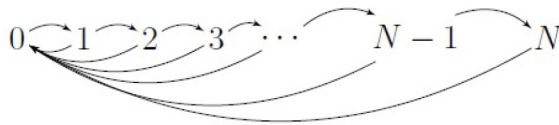
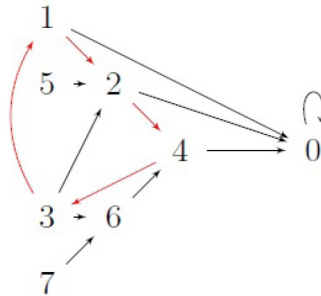
Graph energy forms a subfield of graph theory called spectral graph theory which has been extensively studied. Moreover, in chemical graph theory, if the underlying molecule is a hydrocarbon, then  $G$  is a simple, unweighted graph, but if the conjugated molecule contains atoms different from carbon and hydrogen (in chemistry referred to as "heteroatoms"), then  $G$  must possess pertinently weighted edges (see [11]). Such weights are mostly positive but they also maybe negative. For more details, we refer to [12], [13].

**Lemma 2.1:** *Let  $W(G)$  be a weighted graph and  $G$  be isomorphic to a complete graph  $K_n$ . If  $w_j = \frac{n}{2}$  for  $j = 1, 2, \dots, n$ , then  $\lambda_1 = n$ .*

**Lemma 2.2:** *Let  $W(G)$  be a weighted graph and  $G$  be isomorphic to a path graph  $P_n$ . If  $w_1 = n$  and  $w_j = 0$  for  $j = 2, \dots, n$ , then  $\lambda_1 = n$ .*

**Lemma 2.3:** *Let  $W(G)$  be a weighted graph and  $G$  be isomorphic to a star graph  $S_n$ . If  $w_1 = n$  and  $w_j = 0$  for  $j = 2, \dots, n$ , then  $\lambda_1 = n$ .*

From Lemmas 2.1, 2.2 and 2.3, we see that the resolvent energy of  $ER(W(K_n))$ ,  $ER(W(P_n))$  and  $ER(W(S_n))$  with respect to the weights mentioned in the above lemmas can not be calculated by (1.1). By (1.2), one can check that  $\varepsilon^{-1}$  must not exceed  $n$  and calculate the resolvent energy for the  $\lambda_i \in PE(G)$ . By  $ER_p(G)$ , we denote the pseudo-resolvent energy of graphs where  $\lambda_1 = n$ .

**Figure 1****Multi-Monad graph generated by the shift function****Figure 2****Multi-Monad of type  $T_{2^3, 1, 2, O_4}$** 

**Corollary 2.4:** Let  $W(G)$  be a weighted graph and  $G$  be isomorphic to  $K_n$ ,  $P_n$  or  $S_n$ . Then

$$ER(G) \leq \frac{2(n-1)}{n},$$

where  $G$  isomorphic to  $W(K_n)$ ,  $W(P_n)$  or  $W(S_n)$ .

**Proof:** Assume that  $\varepsilon^{-1} < n$ . Then all eigenvalues belong to  $PE(G)$  with values less than  $\frac{n}{2}$ . Thus we have

$$ER_p(G) = \sum_{i=2}^n \frac{1}{n - \frac{n}{2}} \leq \frac{2(n-1)}{n},$$

where  $G$  isomorphic with  $W(K_n)$ , or  $W(P_n)$ , or  $W(S_n)$ .

The above corollary fix a problem for a very limited class of graphs, but one may ask about the generalisation to another classes with more than one eigenvalue equal to  $n$ . The difficulty here is that how to determine  $\varepsilon^{-1}$  and this complexity appear for graphs with non-normal adjacency matrix. In the next section, we provide a computational study due to this issue.

### 3. Computational studies

In order to determinate  $\varepsilon^{-1}$  for some classes of graphs with more complicated properties such as the adjacency matrix is non-normal or the eigenvalues are complex, we will focus on weighted digraphs, especially on multi monad graphs defined in [9].

Arnold defined a monad graph as a directed graph such that every vertex belonging to a finite group  $G$  is adjacent under the action of  $f$  to its image by a directed edge. For more details, see [10].

Let's compute monad graphs of the first few residue classes of cyclic groups under the action of square function:

In [9], the multi-monad graph was defined as

$$\Gamma(G_1, f) \cup \Gamma(G_2, f) \cup \dots \cup \Gamma(G_N, f) = \cup_{i=1}^N \Gamma(G_i, f) := \Gamma(\Psi_{G_N, f})$$

and denoted by  $\Gamma(\Psi_{G_N, f})$ . For instance, if  $f$  is a forward shift function, i.e.  $f(g) = g + 1$  for all  $g \in \mathbb{Z}$ , then one can easily paint  $\Gamma(\Psi_{\mathbb{Z}_N, f})$  for any  $N > 0$ , as given in Figure 1.

Moreover, in [9] a troubled binary rooted tree with  $2^r$  vertices was denoted by  $T_{2^r, \tau, \zeta, O_l}$  where  $\tau$  is the number of edges between the separated leaves and  $\zeta$  is the number of edges entering the root (except one from itself and the other from the unique vertex of the first level) which includes a dicycle part  $O_l$  of order  $l$ . See Figure 2.

**Observation.** Let us consider a weighted multi-monad graph generated by a square function of type  $W(\Gamma(\Psi_{2^3, f})) = W([\cup_{m=1}^3 \Gamma(\mathbb{Z}_{2^m})] \cup \Gamma(\mathbb{Z}_5))$  where the loop vertex weighted with 8 (i.e. equal to  $n$ ) and the other weighted-edges belong to  $\mathbb{Z}$  which is pertaining in Figure 2. The eigenvalues of  $A(T_{2^3, 1, 2, O_4})$  are  $\lambda_1 = 8$ ,  $\lambda_i = 0$  for  $i = 2, 3, 4$ ,  $\lambda_5 = 1.395$ ,  $\lambda_6 = -0.4746$  and  $\lambda_{7,8} = -0.4604 \pm 1.1393i$ . From those values of  $\lambda$  it follows that if  $\varepsilon^{-1} < \frac{n}{2}$ , then  $ER_p(W(T_{2^3, 1, 2, O_4})) \approx 0.877$ .

### Conflict of Interest

The authors hereby declare that there is no potential conflict of interest.

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