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On picture fuzzy almost ideals of semigroups

Pannawit Khamrot †
Department of Mathematics
Faculty of Science and Agricultural Technology
Rajamangala University of Technology Lanna
Phitsanulok
Thailand

Thiti Gaketem *
Department of Mathematics
School of Science
University of Phayao
Phayao 56000
Thailand

Abstract

In this paper, we give the concept of picture fuzzy almost ideals, minimal picture fuzzy almost ideals, and prime (semiprime, strongly prime) picture fuzzy almost ideals in semigroups. We prove the basic properties of picture fuzzy almost ideals in semigroups. Moreover, we investigate the connection between almost ideals and picture fuzzy almost ideals in semigroups.

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1. Introduction

The concept of fuzzy sets was established by L. A. Zadeh in 1965 [20]. This tool is highly effective in managing uncertainties. Since its introduction, numerous authors have utilized it across various fields, including mathematics, computer science, physics, and chemistry. In 1981

† *E-mail:* `pk_g@rmutl.ac.th`
* *E-mail:* `thiti.ga@up.ac.th` **(Corresponding Author)**

Kuroki [7], discussed the concept of fuzzy subsemigroups and fuzzy generalized bi-ideals in semigroups. In 1981, Bogdanvic, [1] studied definitions of almost bi-ideals in semigroups and studied properties of almost bi-ideals in semigroups. Later, Chinram gives definition definitions of the types of almost ideals in semigroups such that almost quasi-ideal [15]. In 2020, Chinram et. al [5] sutdied almost interior and weakly almost interior ideals in semigroups and proved the properties of its. In 2022, R. Chinrm et. al [2] developed the almost bi-quasi-interior ideal and fuzzy almost bi-quasi-interior ideal in semigroups. Moreover, related research to various types of almost ideals has been studied in other algebraic structures (see, e.g., [6, 8, 10, 13, 12, 15, 14]). In 1986 K. T. Atanassov [16], established the concept of intuitionistic fuzzy sets such that it both a degree of membership and a degree of non-membership are real number between 0 and 1. Yager [17, 18] studied Pythagorean membership grades in multicriteria decision making. Picture fuzzy set and Pythagorean fuzzy set is a generalization of the Zadeh 's fuzzy set and the Antanassov intuitionistic fuzzy set. Cuong et al. [3] were studied the picture fuzzy logic. Later in 2020 Yiarayong [19] used the concept of picture fuzzy sets to semigroup theory and picture fuzzy left (right) ideals and picture fuzzy bi-ideals of semigroups. The theory of (m,n)-ideals in semigroups was studied by Lajos in 1963 [11]. The notion of (m,n)-ideals of semigroups generalized the notion of one sided ideals of semigroups. Recently, W. Nakkhasen [9] discussed concept picture fuzzy (m,n)-ideals of semigroups and investigated some basic properties of picture fuzzy (m,n)-ideals of semigroups. In the same year, T. Gaketem [4] studied concept of interval valued fuzzy almost (m,n)-ideals in semigroups.

In this paper, we study concept of picture fuzzy almost ideals, minimal picture fuzzy almost ideals, and prime (semiprime, strongly prime) almost picture fuzzy ideals in semigroups. We proved the basic properties of picture fuzzy almost ideals in semigroups. Finally, we study the connection between almost ideals and picture fuzzy almost ideals in semigroups.

2. Preliminaries

In this section, we introduce certain concepts and findings that will be beneficial in subsequent sections.

Definition 2.1: Let $\mathcal{T}$ be a semigroup (SG).

- A *subsemigroup* (SSG) of $\mathcal{T}$ is a nonempty set L of $\mathcal{T}$ such that $LL \subseteq L$.
- A left ideal (l Id) [right ideal (r Id)] of $\mathcal{T}$ is a nonempty set L of $\mathcal{T}$ such that $\mathcal{T}L \subseteq L$. $[L\mathcal{T} \subseteq L]$. By an ideal (Id) of T, we mean a nonempty set of $\mathcal{T}$ which is both a l Id and a r Id of $\mathcal{T}$.
- An almost ideal (A-Id) L of $\mathcal{T}$ if $uL \cap L \neq \varnothing$ and $Ld \cap L \neq \varnothing$ for all $u, d \in \mathcal{T}$.

Theorem 2.2: [15] *Every Id L of a SG $\mathcal{T}$ is an A-Id of $\mathcal{T}$.*

Let Int be the closed interval zero to one. For any $z_i \in Int$, $i \in \mathcal{J}$, define

$$\underset{i\in\mathcal{J}}{\vee} z_i := \sup_{i\in\mathcal{J}}\{z_i\} \quad \text{and} \quad \underset{i\in\mathcal{J}}{\wedge} z_i := \inf_{i\in\mathcal{J}}\{z_i\}.$$

For any elements $z, r \in Int$, we have

$$z \wedge r = \min\{z, r\} \quad \text{and} \quad z \vee r = \max\{z, r\}.$$

A **fuzzy set (fuzzy subset)** of a nonempty set $\mathcal{T}$ is a function

$$\rho : \mathcal{T} \to Int.$$

Let ρ and υ fuzzy sets of a $\mathcal{T}$, define the symbol as follows:

(1) $\rho \geq \upsilon \Leftrightarrow \rho(\lozenge) \geq \upsilon(\lozenge)$ for all $\lozenge \in \mathcal{T}$,

(2) $\rho = \upsilon \Leftrightarrow \rho \geq \upsilon$ and $\upsilon \geq \rho$,

(3) $(\rho \wedge \upsilon)(\lozenge) = \rho(\lozenge) \wedge \upsilon(\lozenge) = \min\{\rho(\lozenge), \upsilon(\lozenge)\}$ for all $\lozenge \in \mathcal{T}$,

(4) $(\rho \vee \upsilon)(\lozenge) = \rho(\lozenge) \vee \upsilon(\lozenge) = \max\{\rho(\lozenge), \upsilon(\lozenge)\}$ for all $\lozenge \in \mathcal{T}$,

(5) $\rho \subseteq \upsilon$ if $\rho(\lozenge) \leq \upsilon(\lozenge)$ for all $\lozenge \in \mathcal{T}$,

(6) $(\rho \cap \upsilon)(\lozenge) = \min\{\rho(\lozenge), \upsilon(\lozenge)\}$ and $(\rho \cup \upsilon)(\lozenge) = \max\{\rho(\lozenge), \upsilon(\lozenge)\}$ for all $\lozenge \in \mathcal{T}$.

(7) the *support* of ρ instead of $supp(\rho) = \{\lozenge \in \mathcal{T} \mid \rho(\lozenge) \neq 0\}$ For the symbol $\rho \leq \upsilon$, we mean $\upsilon \geq \rho$.

For $z \in \mathcal{T}$ and $t \in Int$, a fuzzy point (Fp) x_t of a set $\mathcal{T}$ is a fuzzy set of $\mathcal{T}$ defined by

$$x_t(z) = \begin{cases} t & \text{if } z = x \\ 0 & \text{if } z \neq x \end{cases}$$

Definition 2.3: [16] An **intuitionsic fuzzy set** (IF set) $\mathfrak{T}$ in a nonempty set $\mathcal{T}$ is an object having the form

$$\mathfrak{T} := \{(z, \rho_{\mathfrak{T}}(z), \upsilon_{\mathfrak{T}}(z)) \mid z \in \mathcal{T}\},$$

where $\rho_{\mathfrak{T}} : \mathcal{T} \to Int$ is the degree of membership and $\upsilon_{\mathfrak{T}} : \mathcal{T} \to Int$ is the degree of non-membership such that $0 \leq \rho_{\mathfrak{T}}(z) + \upsilon_{\mathfrak{T}}(z) \leq 1$ for all $z \in \mathcal{T}$.

We will use the symbol $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}})$ for the IF set $\mathfrak{T} = \{(z,\ \rho_{\mathfrak{T}}(z),\ \upsilon_{\mathfrak{T}}(z)) \mid z \in T\}$.

Definition 2.4: The **picture fuzzy set** (PF set) $\mathfrak{T}$ on a set $\mathcal{T}$ is defined the form

$$\mathfrak{T} := \{(z, \rho_{\mathfrak{T}}(z), \upsilon_{\mathfrak{T}}(z), \delta_{\mathfrak{T}}(z)) \mid z \in T\},$$

where $\rho_{\mathfrak{T}} : \mathcal{T} \to Int$ is the degree of positive membership, $\upsilon_{\mathfrak{T}} : \mathcal{T} \to Int$ is the degree of neutral membership and $\delta_{\mathfrak{T}} : \mathcal{T} \to Int$ is the degree of negative membership such that $0 \leq \rho_{\mathfrak{T}}(z) + \upsilon_{\mathfrak{T}}(z) + \delta_{\mathfrak{T}}(z) \leq 1$ for all $z \in \mathcal{T}$.

We shall use the symbol $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ for the PF set $_{\mathfrak{T}} = \{(z,\ \rho_{\mathfrak{T}}(z),\ \upsilon_{\mathfrak{T}}(z),\ \delta_{\mathfrak{T}}(z)) \mid z \in \mathcal{T}\}$.

For PF sets $\mathfrak{T}_1 = (\rho_{\mathfrak{T}_1}, \upsilon_{\mathfrak{T}_1}, \delta_{\mathfrak{T}_1})$ and $\mathfrak{T}_2 = (\rho_{\mathfrak{T}_2}, \upsilon_{\mathfrak{T}_2}, \delta_{\mathfrak{T}_2})$ of $\mathcal{T}$, defined the relation as follows:

(1) $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$ if and only if $\rho_{\mathfrak{T}_1}(z) \leq \rho_{\mathfrak{T}_2}(z)$, $\upsilon_{\mathfrak{T}_1}(z) \geq \upsilon_{\mathfrak{T}_2}(z)$ and $\delta_{\mathfrak{T}_1}(z) \leq \delta_{\mathfrak{T}_2}(z)$ for all $z \in \mathcal{T}$,

(2) $\mathfrak{T}_1 = \mathfrak{T}_2$ if and only if $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$ and $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$,

(3) $\mathfrak{T}_1 \cap \mathfrak{T}_2 = \{(\rho_{\mathfrak{T}_1} \wedge \rho_{\mathfrak{T}_2})(z), (\upsilon_{\mathfrak{T}_1} \vee \upsilon_{\mathfrak{T}_2})(z), (\delta_{\mathfrak{T}_1}(z) \vee \delta_{\mathfrak{T}_2}(z)) \mid z \in \mathcal{T}\}$,

(4) $\mathfrak{T}_1 \cup \mathfrak{T}_2 = \{(\rho_{\mathfrak{T}_1} \vee \rho_{\mathfrak{T}_2})(z), (\upsilon_{\mathfrak{T}_1} \wedge \upsilon_{\mathfrak{T}_2})(z), (\delta_{\mathfrak{T}_1}(h) \wedge \delta_{\mathfrak{T}_2}(z)) \mid z \in \mathcal{T}\}$.

For $z \in \mathcal{T}$, define $F_z = \{(u, w) \in \mathcal{T} \times \mathcal{T} \mid z = uw\}$.

For PF sets $\mathfrak{T}_1 = (\rho_{\mathfrak{T}_1},, \upsilon_{\mathfrak{T}_1} \delta_{\mathfrak{T}_1})$ and $\mathfrak{T}_2 = (\rho_{\mathfrak{T}_2},, \upsilon_{\mathfrak{T}_2} \delta_{\mathfrak{T}_2})$, defined the products $\mathfrak{T}_1 \circ \mathfrak{T}_2 = (\rho_{\mathfrak{T}_1} \circ \rho_{\mathfrak{T}_2}, \upsilon_{\mathfrak{T}_1} \circ \upsilon_{\mathfrak{T}_2}, \delta_{\mathfrak{T}_1} \circ \delta_{\mathfrak{T}_2})$ as follows:

$$(\rho_{\mathfrak{T}_1} \circ \rho_{\mathfrak{T}_2})(h) = \begin{cases} \bigvee\limits_{(u,w) \in F_h} \{\rho_{\mathfrak{T}_1}(u) \wedge \rho_{\mathfrak{T}_2}(w)\} & \text{if} \quad F_h \neq \emptyset, \\ 0 & \text{if otherwise.} \end{cases}$$

$$(\upsilon_{\mathfrak{T}_1} \circ \upsilon_{\mathfrak{T}_2})(h) = \begin{cases} \bigwedge_{(y,x)\in F_h} \{\upsilon_{\mathfrak{T}_1}(y) \vee \upsilon_{\mathfrak{T}_2}(z)\} & \text{if } \quad F_h \neq \emptyset, \\ 0 & \text{if otherwise.} \end{cases}$$

and

$$(\delta_{\mathfrak{T}_1} \circ \delta_{\mathfrak{T}_2})(h) = \begin{cases} \bigwedge_{(u,w)\in F_h} \{\delta_{\mathfrak{T}_1}(u) \vee \delta_{\mathfrak{T}_2}(w)\} & \text{if } \quad F_h \neq \emptyset, \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.5: [19] Let $\mathcal{L}$ be a nonempty set of a set $\mathcal{T}$. A **picture characteristic function** $\chi_{\mathcal{L}} = (\rho_{\chi_{\mathcal{L}}}, \upsilon_{\chi_{\mathcal{L}}}, \delta_{\chi_{\mathcal{L}}})$

$$\rho_{\chi_{\mathcal{L}}}(\Diamond) := \begin{cases} 1 & \Diamond \in \mathcal{L}, \\ 0 & \Diamond \notin \mathcal{L}, \end{cases}$$

and

$$\upsilon_{\chi_{\mathcal{L}}}(\Diamond) := \begin{cases} 0 & \Diamond \in \mathcal{L}, \\ 1 & \Diamond \notin \mathcal{L}, \end{cases}$$

and

$$\delta_{\chi_{\mathcal{L}}}(\Diamond) := \begin{cases} 0 & \Diamond \in \mathcal{L}, \\ 1 & \Diamond \notin \mathcal{L}, \end{cases}$$

To simplify matters, we will employ the symbol $\chi_{\mathcal{L}} = (\delta_{\chi_{\mathcal{L}}}, \rho_{\chi_{\mathcal{L}}})$ for the IF set $\chi_{\mathcal{L}} := \{(\Diamond, \rho_{\chi_{\mathcal{L}}}(\Diamond), \upsilon_{\chi_{\mathcal{L}}}(\Diamond), \delta_{\chi_{\mathcal{L}}}(\Diamond),) \mid \Diamond \in \mathcal{T}\}$.

Lemma 2.6: *Let χ_L and χ_L be any two PF sets of a SG $\mathcal{T}$. Then, the following properties hold:*

(1) $\chi_{K\cap L} = \chi_L \cap \chi_L$.

(2) $\chi_{KL} = \chi_L \circ \chi_L$.

Definition 2.7: A PF set $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ on a SG $\mathcal{T}$ is called a **picture fuzzy subsemigroup** (PF-SSG) of $\mathcal{T}$ if it satisfies the following conditions:

(1) $\rho_{\mathfrak{T}}(zr) \geq \rho_{\mathfrak{T}}(z) \wedge \rho_{\mathfrak{T}}(r)$,

(2) $\upsilon_{\mathfrak{T}}(zr) \leq \upsilon_{\mathfrak{T}}(z) \vee \upsilon_{\mathfrak{T}}(r)$,

(3) $\delta_{\mathfrak{T}}(zr) \leq \delta_{\mathfrak{T}}(z) \vee \delta_{\mathfrak{T}}(r)$.

for all $z, r \in \mathcal{T}$.

Definition 2.8: A PF set $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_F, \delta_{\mathfrak{T}})$ on a SG $\mathcal{T}$ is called a **picture fuzzy l Id** (PF-l Id) [picture fuzzy r Id (PF-r Id)] of $\mathcal{T}$ if it satisfies the following conditions:

(1) $\rho_{\mathfrak{T}}(zr) \geq \rho_{\mathfrak{T}}(r)$ $[\rho_{\mathfrak{T}}(zr) \geq \rho_{\mathfrak{T}}(z)]$,

(2) $\upsilon_{\mathfrak{T}}(zr) \leq \upsilon_{\mathfrak{T}}(r)$ $[\upsilon_{\mathfrak{T}}(zr) \leq xi_{\mathfrak{T}}(z)]$,

(3) $\delta_{\mathfrak{T}}(zr) \leq \delta_{\mathfrak{T}}(r)$ $[\delta_{\mathfrak{T}}(zr) \leq \delta_{\mathfrak{T}}(z)]$.

for all $z, r \in \mathcal{T}$.

3. Main Results

Within this segment, we delineate the concept of PFA-Ids and investigate their properties within SGs.

Definition 3.1: A PF set $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ of a SG $\mathcal{T}$ is called a PF almost left Id (PFA-l Id) of $\mathcal{T}$ if $(x_t \circ \rho_{\mathfrak{T}}) \wedge \rho_{\mathfrak{T}} \neq 0$, $(x_t \circ \upsilon_{\mathfrak{T}}) \vee \upsilon_{\mathfrak{T}} \neq 0$ and $(x_t \circ \delta_{\mathfrak{T}}) \vee \delta_{\mathfrak{T}} \neq 0$ for any Fp $x_t \in \mathcal{T}$.

Definition 3.2: A PF set $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ of a SG $\mathcal{T}$ is called a PF almost right Id (PFA-r Id) of $\mathcal{T}$ if $(\rho_{\mathfrak{T}} \circ x_t) \wedge \rho_{\mathfrak{T}} \neq 0$, $(\upsilon_{\mathfrak{T}} \circ x_t) \vee \upsilon_{\mathfrak{T}} \neq 0$ and $(\delta_{\mathfrak{T}} \circ x_t) \vee \delta_{\mathfrak{T}} \neq 0$ for any Fp $x_t \in \mathcal{T}$.

Example 3.3: Every PF-l Id of a SG $\mathcal{T}$ is a PFA-l Id (PFA-r Id) of $\mathcal{T}$.

Theorem 3.4: *If $\mathfrak{T}_1 = (\rho_{\mathfrak{T}_1}, \upsilon_{\mathfrak{T}_1}, \delta_{\mathfrak{T}_1})$ is a PFA-l Id (PFA-r Id) of a SG $\mathcal{T}$ and $\mathfrak{T}_2 = (\rho_{\mathfrak{T}_2}, \upsilon_{\mathfrak{T}_2}, \delta_{\mathfrak{T}_2})$ is a PF subset of $\mathcal{T}$ such that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$, then $\mathfrak{T}_2$ is a PFA-l Id (PFA-r Id) of $\mathcal{T}$.*

Proof: Suppose that $\mathfrak{T}_1 = (\rho_{\mathfrak{T}_1}, \upsilon_{\mathfrak{T}_1}, \delta_{\mathfrak{T}_1})$ is a PFA-l Id of a SG $\mathcal{T}$ and $\mathfrak{T}_2 = (\rho_{\mathfrak{T}_2}, \upsilon_{\mathfrak{T}_2}, \delta_{\mathfrak{T}_2})$ is a PF subset of $\mathfrak{T}$, $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Then for any Fp $x_t \in T$,

$$(x_t \circ \rho_{\mathfrak{T}_1}) \wedge \rho_{\mathfrak{T}_1} \neq 0, \ (x_t \circ \upsilon_{\mathfrak{T}_1}) \vee \upsilon_{\mathfrak{T}_1} \neq 0 \text{ and } (x_t \circ \delta_{\mathfrak{T}_1}) \vee \delta_{\mathfrak{T}_1} \neq 0. \text{ Thus,}$$

$$0 \neq (x_t \circ \rho_{\mathfrak{T}_1}) \wedge \rho_{\mathfrak{T}_1} \subseteq (x_t \circ \rho_{\mathfrak{T}_2}) \wedge \rho_{\mathfrak{T}_2},$$

$$0 \neq (x_t \circ \upsilon_{\mathfrak{T}_1}) \vee \upsilon_{\mathfrak{T}_1} \subseteq (x_t \circ \upsilon_{\mathfrak{T}_2}) \vee \upsilon_{\mathfrak{T}_2},$$

and

$$0 \neq (x_t \circ \delta_{\mathfrak{T}_1}) \vee \delta_{\mathfrak{T}_1} \subseteq (x_t \circ \delta_{\mathfrak{T}_2}) \vee \delta_{\mathfrak{T}_2}.$$

Hence $(x_t \circ \rho_{\mathfrak{T}_2}) \wedge \rho_{\mathfrak{T}_2} \neq 0$, $(x_t \circ \upsilon_{\mathfrak{T}_2}) \vee \upsilon_{\mathfrak{T}_2} \neq 0$ and $(x_t \circ \delta_{\mathfrak{T}_2}) \vee \delta_{\mathfrak{T}_2} \neq 0$. Therefore, $\mathfrak{T}_2$ is a PFA-Id of $\mathcal{T}$.

Theorem 3.5: *Let $\mathfrak{T}_1 = (\rho_{\mathfrak{T}_1}, \upsilon_{\mathfrak{T}_1}, \delta_{\mathfrak{T}_1})$ and $\mathfrak{T}_2 = (\rho_{\mathfrak{T}_2}, \upsilon_{\mathfrak{T}_2}, \delta_{\mathfrak{T}_2})$ be PFA-l Id (PFA-r Id)s of a SG $\mathcal{T}$. Then $\mathfrak{T}_1 \cup \mathfrak{T}_2$ is also a PFA-l Id (PFA-r Id) of $\mathcal{T}$.*

Proof: Since $\mathfrak{T}_1 \subseteq \mathfrak{T}_1 \cup \mathfrak{T}_2$, by Theorem 3.4, $\mathfrak{T}_1 \cup \mathfrak{T}_2$ is a PFA-l Id of $\mathcal{T}$.

Theorem 3.6: *Let L be a nonempty subset of a SG $\mathcal{T}$. Then L is an A-l Id (A-r Id) of $\mathcal{T}$ if and only if $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-l Id (PFA-r Id) of $\mathcal{T}$.*

Proof: Let L is an A-l Id of $\mathcal{T}$. Then $uL \cap L \neq \varnothing$ for all $u \in \mathcal{T}$. Thus, there exists $c \in \mathcal{T}$ such that $c \in uL$ and $c \in L$. Let $u \in \mathcal{T}$ and $t \in (0,1]$. Then $(x_t \circ \rho_{\chi_L})(c) \neq 0$, $(x_t \circ \upsilon_{\chi_L})(c) \neq 0$, $(x_t \circ \delta_{\chi_L})(c) \neq 0$ and $\rho_{\chi_L}(c) = 1$, $\upsilon_{\chi_L}(c) = 0$, $\delta_{\chi_L}(c) = 0$. Thus, $((x_t \circ \rho_{\chi_L}) \wedge \rho_{\chi_L})(c) \neq 0$, $((x_t \circ \upsilon_{\chi_L}) \vee \upsilon_{\chi_L})(c) \neq 0$ and $((x_t \circ \delta_{\chi_L}) \vee \delta_{\chi_L})(c) \neq 0$ so $(x_t \circ \rho_{\chi_L}) \wedge \rho_{\chi_L} \neq 0$, $(x_t \circ \upsilon_{\chi_L}) \vee \upsilon_{\chi_L} \neq 0$ and $(x_t \circ \delta_{\chi_L}) \vee \delta_{\chi_L} \neq 0$ Hence, $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-l Id of $\mathcal{T}$.

Conversely, let $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-l Id of $\mathcal{T}$ and let $x \in \mathcal{T}$ and $t \in (0,1]$. Then $(x_t \circ \rho_{\chi_L}) \wedge \rho_{\chi_L} \neq 0$, $(x_t \circ \upsilon_{\chi_L}) \vee \upsilon_{\chi_L} \neq 0$ and $(x_t \circ \delta_{\chi_L}) \vee \delta_{\chi_L} \neq 0$ Thus there exists $c \in \mathcal{T}$ such that $((x_t \circ \rho_{\chi_L}) \wedge \rho_{\chi_L})(c) \neq 0$, $((x_t \circ \upsilon_{\chi_L}) \vee \upsilon_{\chi_L})(c) \neq 0$ and $((x_t \circ \delta_{\chi_L}) \vee \delta_{\chi_L})(c) \neq 0$. It implies that $(x_t \circ \rho_{\chi_L})(c) \neq 0$, $(x_t \circ \upsilon_{\chi_L})(c) \neq 0$, $(x_t \circ \delta_{\chi_L})(c) \neq 0$ and $\rho_{\chi_L}(c) = 1$, $\upsilon_{\chi_L}(c)$ and $\delta_{\chi_L}(c)$ are equal to zero. Thus, $c \in uL$ and $c \in L$. Hence, $c \in uL \cap L$. So $uL \cap L \neq \varnothing$. We conclude that L is an A-l Id of $\mathcal{T}$.

Theorem 3.7: *Let $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ be a PF subset of a SG $\mathcal{T}$. Then $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ is a PFA-l Id (PFA-r Id) of $\mathcal{T}$ if and only if $supp(\mathfrak{T})$ is an A-l Id (A-r Id) of $\mathcal{T}$.*

Proof: Let $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ is a PFA-l Id of $\mathcal{T}$, $u \in \mathcal{T}$ and $t \in (0,1]$. Then $(x_t \circ \rho_{\mathfrak{T}}) \wedge \rho_{\mathfrak{T}} \neq 0$, $(x_t \circ \upsilon_{\mathfrak{T}}) \vee \upsilon_{\mathfrak{T}} \neq 0$ and $(x_t \circ \delta_{\mathfrak{T}}) \vee \delta_{\mathfrak{T}} \neq 0$. Thus, there exists $z \in \mathcal{T}$ such that $((x_t \circ \rho_{\mathfrak{T}}) \wedge \rho_{\mathfrak{T}})(z) \neq 0$, $(x_t \circ \upsilon_{\mathfrak{T}}) \vee \upsilon_{\mathfrak{T}})(z) \neq 0$ and

$((x_t \circ \delta_{\mathfrak{T}}) \vee \delta_{\mathfrak{T}})\,(z) \neq 0$. So $\rho_{\mathfrak{T}}(z)$, $\upsilon_{\mathfrak{T}}(z)$ and $\delta_{\mathfrak{T}}(z)$ are not equal zero. There exists $w \in \mathcal{T}$ such that $z = xw$ and $\rho_{\mathfrak{T}}(w) \neq 0$, and $\upsilon_{\mathfrak{T}}(w) \neq 0$ and $\delta_{\mathfrak{T}}(w) \neq 0$. Thus $((x_t \circ \lambda_{supp(\rho_{\mathfrak{T}})}) \wedge \lambda_{supp(\rho_{\mathfrak{T}})})\,(z) \neq 0$, $((x_t \circ \lambda_{supp(\upsilon_{\mathfrak{T}})}) \vee \lambda_{supp(\upsilon_{\mathfrak{T}})})$ $(z) \neq 0$ and $((x_t \circ \lambda_{supp(\delta_{\mathfrak{T}})}) \vee \lambda_{supp(\delta_{\mathfrak{T}})})\,(z) \neq 0$. Hence, $(x_t \circ \lambda_{supp(\rho_{\mathfrak{T}})}) \wedge \lambda_{supp(\rho_{\mathfrak{T}})}$ $\neq 0$, $(x_t \circ \lambda_{supp(\upsilon_{\mathfrak{T}})}) \vee \lambda_{supp(\upsilon_{\mathfrak{T}})} \neq 0$ and $(x_t \circ \lambda_{supp(\delta_{\mathfrak{T}})}) \vee \lambda_{supp(\delta_{\mathfrak{T}})} \neq 0$. Therefore, $\lambda_{supp(\mathfrak{T})}$ is a PFA-l Id of $\mathcal{T}$. Which it implies that $supp(\mathfrak{T})$ is an A-l Id of $\mathcal{T}$ by Theorem 3.6.

Conversely, let $supp(\mathfrak{T})$ is an A-l Id of $\mathcal{T}$. From Theorem 3.6, $\lambda_{supp(\mathfrak{T})}$ is a PFA-l Id of $\mathcal{T}$. Then for any Fps $x_t \in \mathcal{T}$, $(x_t \circ \lambda_{supp(\rho_{\mathfrak{T}})}) \wedge \lambda_{supp(\rho_{\mathfrak{T}})} \neq 0$, $(x_t \circ \lambda_{supp(\upsilon_{\mathfrak{T}})}) \vee \lambda_{supp(\upsilon_{\mathfrak{T}})} \neq 0$ and $(x_t \circ \lambda_{supp(\delta_{\mathfrak{T}})}) \vee \lambda_{supp(\delta_{\mathfrak{T}})} \neq 0$. Thus, there exists $c \in \mathcal{T}$ such that $((x_t \circ \lambda_{supp(\rho_{\mathfrak{T}})}) \wedge \lambda_{supp(\rho_{\mathfrak{T}})})\,(c) \neq 0$, $((x_t \circ \lambda_{supp(\upsilon_{\mathfrak{T}})}) \vee$ $\lambda_{supp(\upsilon_{\mathfrak{T}})})\,(c) \neq 0$ and $((x_t \circ \lambda_{supp(\delta_{\mathfrak{T}})}) \vee \lambda_{supp(\delta_{\mathfrak{T}})})\,(c) \neq 0$. Hence, $((x_t \circ \rho_{\mathfrak{T}}) \wedge \rho_{\mathfrak{T}})$ $(c) \neq 0$, $(x_t \circ \upsilon_{\mathfrak{T}}) \vee \upsilon_{\mathfrak{T}})\,(c) \neq 0$ and $((x_t \circ \delta_{\mathfrak{T}}) \vee \delta_{\mathfrak{T}})\,(c) \neq 0$. So there exists $w \in \mathcal{T}$ such that $z = xw$ and $\rho_{\mathfrak{T}}(w)$, $\upsilon_{\mathfrak{T}}(w)$, $\delta_{\mathfrak{T}}(w)$, $\rho_{\mathfrak{T}}(z)$, $\upsilon_{\mathfrak{T}}(z)$ and $\delta_{\mathfrak{T}}(z)$ are not equal to zero implies $(x_t \circ \rho_{\mathfrak{T}}) \wedge \rho_{\mathfrak{T}} \neq 0$, $(x_t \circ \upsilon_{\mathfrak{T}}) \vee \upsilon_{\mathfrak{T}} \neq 0$ and $(x_t \circ \delta_{\mathfrak{T}}) \vee \delta_{\mathfrak{T}} \neq 0$. Therefore, $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ is a PFA-l Id of $\mathcal{T}$.

Next, we explore minimal PFA-l Id (Mn PFA-r Id) Ids in SGs and examine the connections between minimal A-l Id (Mn A-rIds) and minimal PFA-l Ids (Mn PFA-r Ids) within SGs.

Definition 3.8: An A-l Id (A-rIds) K of a SG $\mathcal{T}$ is called minimal (Mn) if for any A-l Id (Mn A-rIds) M of $\mathcal{T}$ if whenever $M \subseteq K$, then $M = K$.

Definition 3.9: A PFA-l Id (PFA-r Id) $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ of a SG $\mathcal{T}$ is called minimal (Mn) if for any PFA-l Id (PFA-r Id) $\mathfrak{T}_1 = (\rho_{\mathfrak{T}_1}, \upsilon_{\mathfrak{T}_1}, \delta_{\mathfrak{T}_1})$ of $\mathcal{T}$ if whenever $\mathfrak{T}_1 \subseteq \mathfrak{T}$, then $supp(\mathfrak{T}_1) = supp(\mathfrak{T})$.

Theorem 3.10: *Let L be a nonempty subset of a SG $\mathcal{T}$. Then L is a Mn A-l Id (A-r Id) of $\mathcal{T}$ if and only if $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a Mn PFA-l Id (PFA-r Id) of $\mathcal{T}$.*

Proof: Let L is an Mn A-l Id of $\mathcal{T}$. Then L is an A-l Id of $\mathcal{T}$. Thus, from Theorem 3.6, $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-Id of $\mathcal{T}$. Let $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ be a PFA-l Id of $\mathcal{T}$ such that $\mathfrak{T} \subseteq \chi_L$ Then $supp(\mathfrak{T}) \subseteq supp(\chi_L) = L$. Form Theorem 3.7, $supp(\mathfrak{T})$ is an A-l Id of $\mathcal{T}$. Since L is an Mn we have $supp(\mathfrak{T})$ $= L = supp(\chi_L)$. Therefore, $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is an Mn PFA-l Id of $\mathcal{T}$.

Conversely, let $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is an Mn PFA-l Id of $\mathcal{T}$. Then $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-l Id of $\mathcal{T}$. Thus, from Theorem 3.6, L is an A-l Id of $\mathcal{T}$. Let M be an A-l Id of $\mathcal{T}$ such that $M \subseteq L$. Then $\chi_M = (\rho_{\chi_M}, \upsilon_{\chi_M}, \delta_{\chi_M})$ is a PFA-l Id of $\mathcal{T}$ such that $\chi_M \subseteq \chi_L$. Thus, $supp(\chi_M) \subseteq supp(\chi_L)$. Since $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is an Mn PFA-l Id of $\mathcal{T}$ we have $supp(\chi_M) = supp(\chi_L)$. Thus, $M = supp(\chi_M) = supp(\chi_L) = L$. Hence, L is an Mn A-l Id of $\mathcal{T}$.

Corollary 3.11: Let $\mathcal{T}$ be a SG. Then $\mathcal{T}$ has no proper A-Id if and only if for all PFA-Ids $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ of $\mathcal{T}$, $supp(\mathfrak{T}) = \mathcal{T}$.

4. Prime PFA-Ids in SGs

Within this section, we elucidate the concept of prime (semiprime, strongly prime) A-Id in SGs. We prove connection between prime (semiprime, strongly prime) A-Ids and prime (semiprime) semiprime PFA-Ids in SGs.

Definition 4.1: Let $\mathcal{T}$ be a SG.

- An A-Id L of $\mathcal{T}$ is called *prime* (PA-Id) if for any A-Ids R and G of $\mathcal{T}$ such that $RG \subseteq L$ implies that $R \subseteq L$ or $G \subseteq L$.
- An A-Id L of $\mathcal{T}$ is called *semiprime* (SA-Id) if for any A-Id R of $\mathcal{T}$ such that $R^2 \subseteq L$ implies that $R \subseteq L$.
- An A-Id L of $\mathcal{T}$ is called *strongly prime* (SPA-Id) if for any A-Ids R and G of $\mathcal{T}$ such that $RG \cap GR \subseteq L$. implies that $R \subseteq L$ or $G \subseteq L$.

Definition 4.2: *Let $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ be a PFA-Id of a SG $\mathcal{T}$ is*

1. a *PF prime A-Id* (PFPA-Id) of $\mathcal{T}$ if for any A-Ids $\mathfrak{K} = (\rho_{\mathfrak{K}}, \upsilon_{\mathfrak{K}}, \delta_{\mathfrak{K}})$ and $\mathfrak{R} = (\rho_{\mathfrak{R}}, \upsilon_{\mathfrak{R}}, \delta_{\mathfrak{R}})$ of $\mathfrak{T}$, $\mathfrak{K} \circ \mathfrak{R} \subseteq \mathfrak{T}$ implies that $\mathfrak{K} \subseteq \mathfrak{T}$ or $\mathfrak{R} \subseteq \mathfrak{T}$,

2. a *PF semiprime A-Id* (PFSA-Id) of $\mathcal{T}$ if for any A-Id $\mathfrak{K}=(\rho_{\mathfrak{K}},\upsilon_{\mathfrak{K}},\delta_{\mathfrak{K}})$ of $\mathcal{T}$, $\mathfrak{K}^2\subseteq\mathfrak{T}$ implies that $\mathfrak{K}\subseteq\mathfrak{T}$,
3. a *PF strongly prime A-Id* (PFSPA-Id) of $\mathcal{T}$ if for any A-Id $\mathfrak{K}=(\rho_{\mathfrak{K}},\upsilon_{\mathfrak{K}},\delta_{\mathfrak{K}})$ and $\mathfrak{R}=(\rho_{\mathfrak{R}},\upsilon_{\mathfrak{R}},\delta_{\mathfrak{R}})$ of $\mathcal{T}$, $(\mathfrak{K}\circ\mathfrak{R})\cap(\mathfrak{R}\circ\mathfrak{K})\subseteq\mathfrak{T}$ implies that $\mathfrak{K}\subseteq\mathfrak{T}$ or $\mathfrak{R}\subseteq\mathfrak{T}$.

It is clearly, every PFSPA-Id of a SG is a PFPA-Id, and every PFPA-Id of a SG is a PFSA-Id.

This theorems follows we study relation between PA-Ids (SA-Ids, SPA-Ids) and PFPA-Ids (PFSA-Ids, PFSPA-Ids) in SGs.

Theorem 4.3: *Let L be a nonempty subset of a SG $\mathcal{T}$. Then L is a PA-Id (SA-Id) of $\mathcal{T}$ if and only if $\chi_L=(\rho_{\chi_L},\upsilon_{\chi_L},\delta_{\chi_L})$ is a PFPA-Id (PFSA-Id) of $\mathcal{T}$.*

Proof: Let L is a *PA* Id of a SG $\mathcal{T}$. Then L is an A-Id of $\mathcal{T}$. Thus, from Theorem 3.3, $\chi_L=(\rho_{\chi_L},\upsilon_{\chi_L},\delta_{\chi_L})$ is a PFA-Id of $\mathcal{T}$. Let $\mathfrak{T}=(\rho_{\mathfrak{T}},\upsilon_{\mathfrak{T}},\delta_{\mathfrak{T}})$ and $\mathfrak{R}=(\rho_{\mathfrak{R}},\upsilon_{\mathfrak{R}},\delta_{\mathfrak{R}})$ be PFA-Id such that $\rho_{\mathfrak{T}}\circ\rho_{\mathfrak{R}}\leq\rho_{\chi_L}$, $\upsilon_{\mathfrak{T}}\circ\upsilon_{\mathfrak{R}}\geq\upsilon_{\chi_L}$ and $\delta_{\mathfrak{T}}\circ\delta_{\mathfrak{R}}\geq\delta_{\chi_L}$. Let $\rho_{\mathfrak{T}}\not\leq\rho_{\chi_L}$, $\upsilon_{\mathfrak{T}}\not\geq\upsilon_{\chi_L}$ and $\delta_{\mathfrak{T}}\not\geq\delta_{\chi_L}$ or $\rho_{\mathfrak{R}}\not\leq\rho_{\chi_L}$, $\upsilon_{\mathfrak{R}}\not\geq\upsilon_{\chi_L}$ and $\delta_{\mathfrak{R}}\not\geq\delta_{\chi_L}$. Then there exist $h,r\in T$ such that $\rho_{\mathfrak{T}}(h)$, $\upsilon_{\mathfrak{T}}(h)$, $\delta_{\mathfrak{T}}(h)$, $\rho_{\mathfrak{R}}(r)$, $\upsilon_{\mathfrak{R}}(r)$, $\delta_{\mathfrak{R}}(r)$ are not equal to zero. While $\rho_{\chi_L}(r)$, $\upsilon_{\chi_L}(r)$ and $\delta_{\chi_L}(r)$ are equal to zero. Thus, $h\in supp(\mathfrak{T})$ and, but $h,r\notin L$. So $supp(\mathfrak{T})\nsubseteq L$ and $supp(\mathfrak{R})\nsubseteq L$. Since $supp(\mathfrak{T})$ and $supp(\mathfrak{R})$ are A-Ids of $\mathcal{T}$ we have $supp(\mathfrak{T})\ supp(\mathfrak{R})\nsubseteq L$. Thus, there exists $v=de$ for some $d\in supp(\mathfrak{T})$ and $e\in supp(\mathfrak{R})$ such that $v\in L$. Hence $\rho_{\chi_L}(v)=0$, and $\upsilon_{\chi_L}(v)=0$ and $\delta_{\chi_L}(v)=0$ implies that $(\rho_{\mathfrak{T}}\circ\rho_{\mathfrak{R}})(v)=0$, $(\upsilon_{\mathfrak{T}}\circ\upsilon_{\mathfrak{R}})(v)=0$ and $(\delta_{\mathfrak{T}}\circ\delta_{\mathfrak{R}})(v)=0$. Since $\rho_{\mathfrak{T}}\circ\rho_{\mathfrak{R}}\leq\rho_{\chi_L}$, $\upsilon_{\mathfrak{T}}\circ\upsilon_{\mathfrak{R}}\geq\upsilon_{\chi_L}$. we have $d\in supp(\mathfrak{T})$ and $e\in supp(\mathfrak{R})$. Thus, $\upsilon_{\mathfrak{T}}(d)$, $\delta_{\mathfrak{T}}(d)$, $\rho_{\mathfrak{R}}(e)$, $\upsilon_{\mathfrak{R}}(e)$, $\delta_{\mathfrak{R}}(e)$ are equal to zero. It implies that

$$(\rho_{\mathfrak{T}}\circ\rho_{\mathfrak{R}})(v)=\bigvee_{(de)\in F_v}\{\rho_{\mathfrak{T}}(d)\wedge\rho_{\mathfrak{R}}(e)\}\neq 0,$$

$$(\upsilon_{\mathfrak{T}}\circ\upsilon_{\mathfrak{R}})(v)=\bigwedge_{(de)\in F_v}\{\upsilon_{\mathfrak{T}}^{m}(d)\vee\upsilon_{\mathfrak{R}}(e)\}\neq 0$$

and

$$(\delta_{\mathfrak{T}} \circ \delta_{\mathfrak{R}})(v) = \bigwedge_{(de)\in F_v} \{\delta_{\mathfrak{T}}(d) \vee \delta_{\mathfrak{R}}(e)\} \neq 0.$$

It is a contradiction so $\rho_{\mathfrak{T}} \leq \rho_{\chi_L}$, $\upsilon_{\mathfrak{T}} \geq \upsilon_{\chi_L}$ and $\delta_{\mathfrak{T}} \geq \delta_{\chi_L}$ or $\rho_{\mathfrak{R}} \leq \rho_{\chi_L}$, $\upsilon_{\mathfrak{R}} \geq \upsilon_{\chi_L}$ and $\delta_{\mathfrak{R}} \geq \delta_{\chi_L}$ Therefore $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFPA-Id of $\mathcal{T}$.

Conversely, let $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFPA-Id of $\mathcal{T}$. Then $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-Id of $\mathcal{T}$. Thus by Theorem 3.6, L is an A-Id of $\mathcal{T}$. Let F and R be A-Ids of $\mathcal{T}$ such that $FR \subseteq L$. Then $\chi_F = (\rho_{\chi_F}, \upsilon_{\chi_F}, \delta_{\chi_F})$ and $\chi_R = (\rho_{\chi_R}, \upsilon_{\chi_R}, \delta_{\chi_R})$ are PFA-Ids of $\mathcal{T}$. By Lemma 2.6 $\rho_{\chi_F} \circ \rho_{\chi_R} = \rho_{\chi_{FR}} \leq \rho_{\chi_L}$, $\upsilon_{\chi_F} \circ \upsilon^n_{\chi_R} = \upsilon_{\chi_{FR}} \geq \upsilon_{\chi_L}$ and $\delta_{\chi_F} \circ \delta_{\chi_R} = \delta_{\chi_{FR}} \geq \delta_{\chi_L}$. By assumption, $\rho_{\chi_F} \leq \rho_{\chi_L}$, $\upsilon_{\chi_F} \geq \upsilon_{\chi_L}$ and $\delta_{\chi_F} \geq \delta_{\chi_L}$ or $\rho_{\chi_R} \leq \rho_{\chi_R}$, $\upsilon_{\chi_R} \geq \upsilon_{\chi_R}$ and $\delta_{\chi_R} \geq \delta_{\chi_R}$. Thus, $F \subseteq L$ or $R \subseteq L$. Therefore L is a PA-Id of $\mathcal{T}$.

Theorem 4.4: *Let L be a nonempty subset of a SG $\mathcal{T}$. Then L is a SPA-Id of $\mathcal{T}$ if and only if $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFSPA-Id of $\mathcal{T}$.*

Proof: Let L is a SPA-Id of a SG $\mathcal{T}$. Then L is an A-Id of $\mathcal{T}$. Thus by Theorem 3.6, $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA-Id of $\mathcal{T}$. Let $\mathfrak{T} = (\rho_{\mathfrak{T}}, \upsilon_{\mathfrak{T}}, \delta_{\mathfrak{T}})$ and $\mathfrak{R} = (\rho_{\mathfrak{R}}, \upsilon_{\mathfrak{R}}, \delta_{\mathfrak{R}})$ be PFA-Id such that $(\rho_{\mathfrak{T}} \circ \rho_{\mathfrak{R}}) \cap (\rho_{\mathfrak{R}} \circ \rho^m_{\mathfrak{T}}) \subseteq \rho_{\chi_K}$, $(\upsilon_{\mathfrak{T}} \circ \upsilon^n_{\mathfrak{R}}) \cup (\upsilon_{\mathfrak{R}} \circ \upsilon_{\mathfrak{F}}) \subseteq \upsilon_{\chi_K}$ and $(\delta_{\mathfrak{T}} \circ \delta_{\mathfrak{R}}) \cup (\delta_{\mathfrak{R}} \circ \delta_{\mathfrak{T}}) \subseteq \delta_{\chi_K}$. Assume that $\rho_{\mathfrak{R}} \nsubseteq \rho_{\chi_L}$ or $\rho_{\mathfrak{R}} \nsubseteq \rho_{\chi_L}$, $\upsilon_{\mathfrak{R}} \nsubseteq \upsilon_{\chi_L}$ or $\upsilon_{\mathfrak{R}} \nsubseteq \upsilon_{\chi_L}$ and $\delta_{\mathfrak{R}} \nsubseteq \delta_{\chi_L}$ or $\delta_{\mathfrak{R}} \nsubseteq \delta_{\chi_L}$. Then there exist $h, r \in \mathcal{T}$ such that $\vartheta_{\mathfrak{F}}(h)$, $\xi_{\mathfrak{F}}(h)$, $\delta_{\mathfrak{F}}(h)$, $\vartheta_{\mathfrak{R}}(r)$, $\xi_{\mathfrak{R}}(r)$ and $\delta_{\mathfrak{R}}(r)$ are not equal to zero. While $\vartheta_{\chi_L}(r)$, $\xi_{\chi_L}(r)$ and $\delta_{\chi_L}(r)$ are equal to zero. Thus, $h \in supp(\mathfrak{F})$ and $r \in supp(\mathfrak{R})$, but $h, r \notin L$. So $supp(\mathfrak{F}) \nsubseteq L$ and $supp(\mathfrak{R}) \nsubseteq L$. Hence, there exists $p \in [supp(\mathfrak{F})\ supp(\mathfrak{R})] \cap [supp(\mathfrak{R})\ supp(\mathfrak{F})]$ such that $p \notin L$. Since $p \in supp(\mathfrak{F})\ supp(\mathfrak{R})$ and $p \in supp(\mathfrak{R})\ supp(\mathfrak{F})$ so $p = d_1 e_1$ and $p = e_2 d_2$ for some $d_1, d_2 \in supp(\mathfrak{F})$ and for some $e_1, e_2 \in supp(\mathfrak{R})$. Thus

$$(\rho_{\mathfrak{F}} \circ \rho_{\mathfrak{R}})(p) = \bigvee_{(d_1 e_1)\in F_v} \{\rho_{\mathfrak{F}}(d_1) \wedge \rho_{\mathfrak{R}}(e_1)\} \neq 0,$$

$$(\upsilon_{\mathfrak{F}} \circ \upsilon_{\mathfrak{R}})(p) = \bigwedge_{(d_1 e_1)\in F_v} \{\upsilon\mathfrak{F}(d_1) \vee \xi^n_{\mathfrak{R}}(e_1)\} \neq 0$$

and

$$(\delta_{\mathfrak{F}} \circ \delta_{\mathfrak{R}})(p) = \bigwedge_{(d_1 e_1) \in F_v} \{\delta_{\mathfrak{F}}(d_1) \vee \delta_{\mathfrak{R}}(e_1)\} \neq 0.$$

So $(\rho_{\mathfrak{F}} \circ \rho_{\mathfrak{R}})\ (p) \neq 0$, $(\upsilon_{\mathfrak{F}} \circ \upsilon_{\mathfrak{R}})\ (p) \neq 0$ and $(\delta_{\mathfrak{F}} \circ \delta_{\mathfrak{R}})\ (p) \neq 0$. Similarly $(\rho_{\mathfrak{R}} \circ \rho_{\mathfrak{F}})\ (p) \neq 0$, $(\upsilon_{\mathfrak{R}} \circ \upsilon_{\mathfrak{F}})\ (p) \neq 0$ and $(\delta_{\mathfrak{R}} \circ \delta_{\mathfrak{F}})\ (p) \neq 0$. It is a contradiction so $(\rho_{\mathfrak{F}} \circ \rho_{\mathfrak{R}})\ (p) = 0$, $(\upsilon_{\mathfrak{F}} \circ \upsilon_{\mathfrak{R}})\ (p) = 0$ and $(\delta_{\mathfrak{F}} \circ \delta_{\mathfrak{R}})\ (p) = 0$. Similarly $(\rho_{\mathfrak{R}} \circ \rho_{\mathfrak{F}})\ (p) = 0$, $(\upsilon_{\mathfrak{R}} \circ \upsilon_{\mathfrak{F}})\ (p) = 0$ and $(\delta_{\mathfrak{R}} \circ \delta_{\mathfrak{F}})\ (p) = 0$. Hence, $\rho_{\mathfrak{R}} \subseteq \rho_{\chi_L}$ or $\rho_{\mathfrak{R}} \subseteq \rho_{\chi_L}$, $\upsilon_{\mathfrak{R}} \subseteq \upsilon_{\chi_L}$ or $\upsilon_{\mathfrak{R}} \subseteq \upsilon_{\chi_L}$ and $\delta_{\mathfrak{R}} \subseteq \delta_{\chi_L}$ or $\delta_{\mathfrak{R}} \subseteq \delta_{\chi_K}$. Therefore, $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFSPA-Id of $\mathcal{T}$.

Conversely, let $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFSPA-Id of $\mathcal{T}$. Then $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is a PFA Id of $\mathcal{T}$. Thus, from Theorem 3.6, $\chi_L = (\rho_{\chi_L}, \upsilon_{\chi_L}, \delta_{\chi_L})$ is an A-Id of $\mathcal{T}$. Let Z and R be A-Ids of $\mathcal{T}$ such that $ZR \cap RZ \subseteq L$. Then $\chi_Z = (\rho_{\chi_Z}, \upsilon_{\chi_Z}, \delta_{\chi_Z})$ and $\chi_R = (\rho_{\chi_R}, \upsilon_{\chi_R}, \delta_{\chi_R})$ are PFA-Ids of $\mathcal{T}$. By Lemma 2.6 $\rho_{\chi_Z} \circ \rho_{\chi_R} = \rho_{\chi_{ZR}} \leq \rho_{\chi_L}$, $\upsilon_{\chi_Z} \circ \upsilon_{\chi_R} = \upsilon_{\chi_{ZR}} \geq \upsilon_{\chi_L}$ and $\delta_{\chi_Z} \circ \delta_{\chi_R} = \delta_{\chi_{ZR}} \geq \delta_{\chi_L}$. By assumption, $\rho_{\chi_Z} \leq \rho_{\chi_L}$, $\upsilon_{\chi_T} \geq \upsilon_{\chi_L}$ and $\delta_{\chi_Z} \geq \delta_{\chi_L}$ or $\rho_{\chi_R} \leq \rho_{\chi_L}$, $\upsilon_{\chi_R} \geq \upsilon_{\chi_L}$ and $\delta_{\chi_R} \geq \delta_{\chi_L}$. Thus $Z \subseteq L$ or $R \subseteq L$. We conclude that L is a PA-Id of $\mathcal{T}$.

Conclusion

Within this manuscript, we introduce the concept of PFA-Ids in SGs and investigate their properties. Moreover, we delineate the connection between PFA-Ids and A-Ids. In the future, we plan to explore other types of PFA-Ids in SGs or within the algebraic context.

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