

On k -super graceful graphs with extremal maximum vertex degree

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Abstract

Let $G(V, E)$ be a simple and loopless graph with $|V|$ vertices and $|E|$ edges. For $k \geq 1$, a graph G is called k -super graceful if there is a bijective labeling $f: V \cup E \rightarrow \{i \mid k \leq i \leq k + |V| + |E| - 1\}$ with $f(vu) = |f(v) - f(u)|$ for every edge $vu \in E$. In this paper, we study the existence of k -super graceful labeling of certain graphs with extremal maximum vertex degree.

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1. Introduction

In this paper, we only consider simple and loopless graph $G = (V, E)$ that has $|V| = p$ vertices and $|E| = q \geq 1$ edges. We say G is a (p, q) -graph. Let $[a, b] = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$, here $\mathbb{Z}$ is the set of integers. All notation not defined in this paper can be found in [3]. Rosa [13] defined a graceful labeling of G as an injective vertex labeling function $f: V \rightarrow [0, q]$ such that the induced edge-labeling function $f^*(vu) = |f(v) - f(u)|$ for each $vu \in E$ is also injective. The following conjecture has since then become one of the most famous unsolved graph labeling problems.

Conjecture. *All trees are graceful.*

Since then, there have been more than 1500 research papers on graph labelings (see the dynamic survey by Gallian [5]). Interested readers may refer to [2, 4] for two variations on graceful labelings.

In [1], Bange et al. defined a k -sequentially additive labeling f of a graph G as a bijection from $V \cup E$ to $[k, k + p + q - 1]$ such that for each edge $vu \in E$, $f(vu) = f(u) + f(v)$. They conjectured that all trees admit a 1-sequentially additive labeling. More results on k -sequentially additive labeling can be found in [6, 7, 10].

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In [9], the concept of k -super graceful labeling was introduced. This was referred to as a k -sequential labeling in [15] that we are only aware of after the completion of this paper.

Definition 1.1: Let G be a (p, q) -graph. Suppose $f: V \cup E \rightarrow [k, k + p + q - 1]$, where $k \geq 1$. If f is a bijection and $f(vu) = |f(v) - f(u)|$ for each edge vu of G , then f is a k -super graceful labeling of G and G is k -super graceful.

This is a generalization of a super graceful labeling defined in [11,12]. Properties of k -super gracefulness and sufficient conditions on the existence of k -super graceful labeling of various bipartite and tripartite graphs with arbitrarily large maximum vertex degree have been investigated [9]. Motivated by this, we shall study the existence of k -super gracefulness labeling of certain graphs with extremal vertex degree of G . In Section 2, basic properties are obtained. In Sections 3 and 4, we determined the k -super gracefulness of graphs G with $\Delta(G) \in \{1, 2\}$. Unless stated otherwise, we assume G has p vertices and q edges.

2. Basic Properties

Let $G + H$ be the disjoint union of graphs G and H . Let $nG = \overbrace{G + \dots + G}^{n \text{ times}}$. We say G is k -super graceful with edge-label set A if A contains all the edge labels. The following 3 results were obtained in [9].

Theorem 2.1: For $k \geq 1$, if G is k -super graceful, then integers in $[p + q, p + q + k - 1]$ must be vertex labels of k mutually non-adjacent vertices of the labeling. Moreover, no two of integers in $[k, 2k]$ are vertex labels of adjacent vertices.

Corollary 2.2: Let α be the independence number of G . If G is k -super graceful, then $k \in [1, \alpha]$. Moreover, the upper bound is sharp.

Lemma 2.3: Let H be a graph with q edges. If H is super-graceful with edge-label set $[1, q]$ and G is $(q + 1)$ -super graceful, then $G + H$ is super graceful.

The following three lemmas can be proved similarly.

Lemma 2.4: If there is a sequence of graphs $G_i, i = 1, 2, \dots, t, t \geq 2$, such that G_i is k_i -super graceful with largest label r_i and $k_{i+1} = r_i + 1$ for $1 \leq i \leq t - 1$, then $\sum_{i=j}^t G_i$ is k_j -super graceful for $j \geq 1$.

Lemma 2.5: Let $G_1, \dots, G_t$ be a sequence of graphs, where the size of G_i is q_i , $t \geq 2$. Suppose G_i is k_i -super graceful whose edges are labeled by the last q_i

labels and that $k_{i+1} = k_i + q_i$, $1 \leq i \leq t-1$. Then for $s \leq t$, $\sum_{i=j}^s G_i$ is k_j -super graceful with edge-label set $[k_j, k_j + \sum_{i=j}^s q_i - 1]$.

Conversely, if $G = \sum_{i=1}^t G_i$ has q edges and admits a k -super graceful labeling with edge-label set $[k, k + q - 1]$ such that vertex labels of each G_i form a sequence of consecutive integers, then each G_i is k_i -super graceful for some k_i . Moreover, the corresponding edge-label set is $[k_i, k_i + q_i - 1]$.

Lemma 2.6: Suppose H is a graph with p vertices and q edges. If $f: V(H) \rightarrow [t, t + p - 1]$, $t \geq 1$, is a bijection such that $f(vu) = |f(v) - f(u)|$ for every edge vu of H with edge-label set $[k, k + q - 1]$, then H is k -super graceful. Moreover, G is $(k + q)$ -super graceful implies that $G + H$ is k -super graceful.

For connected graphs, P_{n+1} and $K(1, n)$ of size $n + 1$ are super graceful with edge-label set $[1, n]$, where $n \geq 1$. For disconnected graphs, we may apply Lemma 2.5 to construct k -super graceful graphs of size q whose edges are labeled by the smallest q labels.

Example 2.1: Clearly, K_2 is super graceful with edge label 1 and $4K_2$ is 2-super graceful with edge-label set $[2, 5]$ and the corresponding end-vertex labels (7,9), (10,13), (8,12), (6,11). By Lemma 2.5, $5K_2$ is super graceful with edge-label set $[1, 5]$.

Also, P_3 is super graceful with edge-label set $\{1, 2\}$ and $5K_2$ is 3-super graceful with edge-label set $[3, 7]$ and the corresponding end-vertex labels (11,14), (9,13), (12,17), (10,16), (8,15). Hence, $P_3 + 5K_2$ is super graceful with edge-label set $[1, 7]$.

Lemma 2.7: If G is k -super graceful with all vertices labeled by odd integers or all vertices labeled by even integers, then $k = 1$.

Proof: From Theorem 2.1 we know that the k largest labels are vertex labels. So $k = 1$.

Now consider a k -super graceful graph G such that the corresponding edge labels are all even integers. Suppose there are a odd vertex labels and b even vertex labels. Thus $p = a + b$. By the assumption $a \geq 2$ and $b \neq 1$. Suppose G has ω components. Since there is no isolated vertex, $q \geq p - \omega \geq p - q$. Hence $2q \geq p$. Suppose $2q = p$, then $\omega = q$. This implies that $G \cong qK_2$. We will study this case in the next section.

Now we assume that $a + b = p < 2q$ and $q \geq 2$. It is clearly that $b + q - 1 \leq a \leq b + q + 1$. This implies $2a \leq a + b + q + 1 \leq 3q$ and $2b \leq 2a - 2q + 2 \leq 3q - 2q + 2 = q + 2$. Hence $a \leq \lfloor \frac{3q}{2} \rfloor$ and $b \leq \min\{\lfloor \frac{q+2}{2} \rfloor, 2q - a - 1\}$.

For $q=2$, since $a \geq 2$ and $b \neq 1$, we have $b=0$. So $(a,b) \in \{(2,0), (3,0)\}$. For $(a,b) = (2,0)$, the graph is not simple. For $(a,b) = (3,0)$, $G \cong P_3$. By Lemma 2.7, $k=1$ and the consecutive vertex labels must be 3,5,1 or 3,1,5.

Now we consider $q \geq 3$. When $b=0$, then $k=1$, $a=p$ and the largest label is $a+q$ which is odd. Since the number of odd labels is one more than that of even labels in $[1, p+q] = [1, a+q]$, we get $a=q+1$. So G has $q+1$ vertices and q edges.

As examples, one may find from [8, Theorems 4.1 & 4.2] that the $(q+1, q)$ -graphs, caterpillar and spider graph are super graceful with all odd vertex labels.

Theorem 2.8: Suppose G is k -super graceful with all odd vertex labels. Then G is a $(q+1, q)$ -graph and $k=1$, where $q \geq 1$.

Proof: Since all vertices labeled by odd integers, all edges labeled by even integers. Combining the above discussions we have the theorem.

Theorem 2.9: For a $(q+1, q)$ -graph G , we have the following equivalent statements:

- (i) G admits a super graceful labeling with edge-label set $[1, q]$.
- (ii) G is graceful.
- (iii) G is super graceful with all odd vertex labels.

Proof: [(i) $\Rightarrow$ (ii)] Let g be a super graceful labeling of G with edge-label set $[1, q]$. For each $v \in V$, define $h(v)$

$$(4) \quad f(v_{2r+1}u_{2r+2}) = 2, f(v_{2i-1}u_{2i}) = 8r+6-8i \text{ and } f(v_{2i}u_{2i+1}) = 8r+10-8i, 1 \leq i \leq r.$$

Now, we can easily check that $h(vu) = g(vu)$ for each $vu \in E$. Since g is bijection and $g(E) = [1, q]$, h is a bijection. Hence h is a graceful labeling of G .

[(ii) $\Rightarrow$ (iii)] Let h be a graceful labeling of G . For each vertex v and edge vu of G , define $f(v) = 2h(v) + 1$ and $f(vu) = 2h(vu)$. Clearly, f is a super graceful labeling of G such that all vertex labels are odd integers.

[(iii) $\Rightarrow$ (i)] Let f be a super graceful labeling of G such that all vertex labels are odd integers. For each vertex v and edge vu of G , define $g(v) = (f(v)-1) / 2+q+1$ and $g(vu) = f(vu) / 2$. Clearly, g is a super graceful labeling of G with edge-label set $[1, q]$.

Consider a graph G with a vertex v . Let $P_n = u_1 \cdots u_n$ be a path of order $n \geq 2$. Construct graph G_n^v by merging v with u_1 .

Theorem 2.10: *Suppose G is a super graceful $(q+1, q)$ -graph with all vertices labeled by odd integers, where $q \geq 1$. Let v be the vertex labeled by 1, then G_n^v is super graceful with all odd vertex labels and u_n is labeled by 1.*

Proof: Here we only show the case when $n = 2$. For $n \geq 3$, we can proceed by induction.

Suppose G admits a super graceful labeling h with all odd vertex integers and $h(v) = 1$. Define a labeling g on G_2^v by $g(x) = 2q + 4 - h(x)$ for vertex x of G , $g(u_2) = 1$ and $g(xy) = |g(x) - g(y)|$. One can verify that g is a super graceful labeling.

Corollary 2.11: *If G is a super graceful $(q+1, q)$ -graph such that all vertex labels are odd integers, then G admits two super graceful labelings f and g such that $g(u) = 2q - f(u)$ and $g(vu) = f(vu)$ for each vertex v and edge vu of G .*

The following conjecture is equivalent to the Graceful Tree Conjecture.

Conjecture 2.1: *Each tree admits a super graceful labeling with all odd vertex labels.*

Problem 2.1: Suppose $k \geq 1$. Characterize k -super graceful graphs with all odd vertex labels.

Theorem 2.12: *A graph G is k -super graceful with all odd edge labels if and only if $k = 1$ and $G = K(1, q)$, $q \geq 1$.*

Proof: (Sufficiency) Suppose $G = K(1, q)$. We first label the central vertex of G by $2q + 1$. We then label the edges by $1, 3, 5, \dots, 2q - 1$, the corresponding end-vertices by $2q, 2q - 2, 2q - 4, \dots, 2$, respectively. We have a required labeling.

(Necessity) Observe that any two adjacent vertex labels must have different parity. Hence, G is bipartite. Suppose G has q edges. Let G have partite sets A, B such that all vertices in A have odd labels and all vertices in B have even labels. Since $q \geq 1$, $|A| = a \geq 1$ and $|B| \geq 1$. Suppose k is odd. We have $|B| = a + q - i \geq 1$, $i = 0, 1$. Note that $|B| \leq q$ which implies that $a + q - i \leq q$. So $a \leq i$. Hence, $a = i = 1$. Consequently, G must be a star $K(1, q)$, $q \geq 1$ with largest label $2q + k$ which is odd. If $k \geq 3$, by Theorem

4.2 in [9], we know k is the central vertex label of $K(1, q)$. This means $2q + k$ must be an edge label, contradicting Theorem 2.1. Hence, $k = 1$.

Suppose k is even. Now $|B| = a + q + i \geq 1$, $i = 0, 1$. Again, $|B| \leq q$ implies that $a + i \leq 0$, a contradiction.

It was shown in [9, 11] the complete bipartite graph $K(m, n)$ is 1-, m - and n -super graceful for all $n \geq m \geq 2$ whereas $K(1, q)$ is k -super graceful if and only if $q \equiv 0 \pmod{k}$.

Corollary 2.13: *There exists k -super graceful graph G with $\Delta(G) = k$ for each $k \geq 1$.*

3. $\Delta(G) = 1$

Obviously, for $n \geq 1$, $G = nK_2$ if and only if $\Delta(G) = 1$. In this section, we shall say the i -th component of nK_2 has edge $e_i = u_i v_i$, $1 \leq i \leq n$.

Lemma 3.1: *Suppose $k, n \geq 1$. If nK_2 is k -super graceful, then (i) $n \equiv 0 \pmod{4}$, or (ii) $n \equiv 1 \pmod{4}$ and $k \equiv 1 \pmod{2}$, or (iii) $n \equiv 3 \pmod{4}$ and $k \equiv 0 \pmod{2}$. Moreover, $n \geq 2k - 1$.*

Proof: Since nK_2 is k -super graceful, the labels are from k to $k + 3n - 1$. Without loss of generality, assume that $f(v_i) = f(u_i) + f(e_i)$ for $1 \leq i \leq n$. Hence, $\sum_{i=1}^n f(v_i) = \sum_{i=1}^n f(u_i) + \sum_{i=1}^n f(e_i)$. Therefore, $2\sum_{i=1}^n f(v_i) = \sum_{i=1}^n f(v_i) + \sum_{i=1}^n f(u_i) + \sum_{i=1}^n f(e_i) = 3n(2k + 3n - 1) / 2$ is even. So

$$3n(2k + 3n - 1) \equiv 0 \pmod{4} \Leftrightarrow n(2k + n + 1) \equiv 0 \pmod{4}.$$

It is equivalent to (i) $n \equiv 0 \pmod{4}$, or (ii) $n \equiv 1 \pmod{4}$ and $k \equiv 1 \pmod{2}$, or (iii) $n \equiv 3 \pmod{4}$ and $k \equiv 0 \pmod{2}$. Moreover, since $3n(3n + 2k - 1) / 4 = \sum_{i=1}^n f(u_i) \leq \sum_{i=1}^n (2n + k - 1 + i) = n(5n + 2k - 1) / 2$, $n \geq 2k - 1$.

So as a corollary of Lemma 3.1, if nK_2 is super graceful, then $n \equiv 0, 1 \pmod{4}$. The converse is also true (see Corollary 3.7).

We first consider k -super graceful nK_2 with all even edge labels. Suppose there are s edges with odd end-vertex labels. Thus there are $2n - 2s$ vertices with even labels. Hence $-1 \leq 2s - (3n - 2s) \leq 1$.

Suppose such a labeling for nK_2 exists, we must have triples (a_i, b_i, c_i) with $b_i = c_i - a_i$ such that the b_i 's are distinct even numbers and the a_i 's and c_i 's are distinct odd numbers from 1 to $3n$. If $n = 4t \geq 4$, from the above discussion, $4s = 3n = 12t$ and hence $1 \leq i \leq 3t$. We are left with $3t$ even numbers, say d_j, e_j, f_j ($1 \leq j \leq t$) such have $f_j = d_j + e_j$. Observe that

$\sum(d_j + e_j + f_j) \equiv 0 \pmod{4}$. Similarly, we can also find such a labeling for $n = 4t + 1 \geq 5$ with $s = 3t + 1$.

Theorem 3.2: For $t \geq 1$, if $(4t)K_2$ is super graceful with all even edge labels, then so is $(4t + 1)K_2$.

Proof: Write $(4t)K_2 = G + H$, where $G \cong (3t)K_2$ and $H \cong tK_2$ such that G has all odd vertex labels and H has all even vertex label(s). Now, the vertices of $(4t + 1)K_2 \cong G + H + K_2$ is labeled as follows:

For each original vertex labels of G , add 2 to it. Keep the labels of H ; and label the vertices of the last K_2 by 1 and $12t + 3$.

Now, odds from 3 to $12t + 1$ appear in G ; evens from 2 to $12t$ appear in $G + H$. Only 1, $12t + 2$ and $12t + 3$ are not involved in $G + H$, but they appear in K_2 . Hence $(4t + 1)K_2$ is super graceful.

Example 3.1: For $n = 4, 5, 8, 9$, the following labelings show that nK_2 is super graceful with all their edges labeled by even numbers.

(1) $n = 4$:

- a. Edge labels are 2, 4, 6, 10; adjacent vertex labels are (5, 7), (8, 12), (3, 9), (1, 11). So, the corresponding triples (a_i, b_i, c_i) 's are (5, 2, 7), (3, 6, 9), (1, 10, 11) and (d_1, e_1, f_1) is (8, 4, 12).
- b. Edge labels are 2, 4, 6, 8; adjacent vertex labels are (10, 12), (5, 9), (1, 7), (3, 11). So, the corresponding triples (a_i, b_i, c_i) 's are (5, 4, 9), (1, 6, 7), (3, 8, 11) and (d_1, e_1, f_1) is (10, 2, 12).

(2) $n = 5$: Edge labels are 2, 4, 6, 10, 14; adjacent vertex labels are (7, 9), (8, 12), (5, 11), (3, 13), (1, 15). So, the corresponding triples (a_i, b_i, c_i) 's are (7, 2, 9), (5, 6, 11), (3, 10, 13), (1, 14, 15) and (d_1, e_1, f_1) is.

(3) $n = 8$: Edge labels are 2, 4, 6, 8, 10, 12, 14, 18; adjacent vertex labels are (20, 22), (7, 11), (15, 21), (16, 24), (3, 13), (5, 17), (9, 23), (1, 19). So, the corresponding triples (a_i, b_i, c_i) 's are (7, 4, 11), (15, 6, 21), (3, 10, 13), (5, 12, 17), (9, 14, 23), (1, 18, 19) and (d_i, e_i, f_i) 's are (20, 2, 22), (16, 8, 24).

(4) $n = 9$: Edge labels are 2, 4, 6, 8, 10, 12, 18, 20, 24; adjacent vertex labels are (14, 16), (22, 26), (9, 15), (11, 19), (7, 17), (13, 25), (5, 23), (1, 21), (3, 27). So, the corresponding triples (a_i, b_i, c_i) 's are (9, 6, 15), (11, 8, 19), (7, 10, 17), (13, 12, 25), (5, 18, 23), (1, 20, 21), (3, 24, 27) and (d_i, e_i, f_i) 's are (14, 2, 16), (22, 4, 26).

It is clear that $3K_2$ is not 2-super graceful with all even edge labels. But $3K_2$ is 2-super graceful with edge labels are 6,4,7 and adjacent vertex labels are (2,8), (4,9), (3,10), respectively.

Problem 3.1: For all possible n and k , obtain necessary and sufficient condition such that nK_2 is k -super graceful with all even edge labels.

Lemma 3.3: For $n \geq 1$, nK_2 is k -sequentially additive if and only if it is k -super graceful.

Proof: Let f be a k -super graceful labeling of nK_2 . For each K_2 component with vertex labels $f(v) > f(u)$ and edge label $f(vu)$, we have $f(v) - f(u) = f(vu)$ if and only if $f(u) + f(vu) = f(v)$. Define g such that $g(u) = f(u)$, $g(v) = f(vu)$ and $g(vu) = f(v)$. Clearly g is a k -sequentially additive labeling of nK_2 . By a similar argument, we have the converse.

Lemma 3.1 implies that for $n \equiv 2, 3 \pmod{4}$, nK_2 is not n -sequentially additive. In fact,

Theorem 3.4: The graph nK_2 is n -super graceful with $n \geq 1$, and hence n -sequentially additive, if and only if $n = 1$.

Proof: By Lemma 3.1, nK_2 is n -super graceful implies that $n \geq 2n - 1$. Thus, $n = 1$. The converse is clear.

Definition 3.1: For $n, k \geq 1$, a sequence $\{(a_i, b_i)\}_{i=1}^n$ with $a_i - b_i = k + i - 1$ for mutually distinct $a_i, b_i \in [1, 2n]$ is a k -Skolem sequence of length n . A 1-Skolem sequence is also called a Skolem sequence (see [14]).

Lemma 3.5: The graph nK_2 is k -super graceful with edge-label set $[k, k + n - 1]$ if and only if a k -Skolem sequence of length n exists.

Proof: (Sufficiency) From a k -Skolem sequence $\{(a_i, b_i)\}_{i=1}^n$, let $a_i + k + n - 1$ and $b_i + k + n - 1$ be vertex labels of the i -th component of nK_2 , we get a k -super graceful labeling of nK_2 with edge-label set $[k, k + n - 1]$.

(Necessity) If nK_2 is k -super graceful with edge-label set $[k, k + n - 1]$, then it has vertex-label set $[k + n, k + 3n - 1]$. Subtracting $k + n - 1$ from each vertex label gives a k -Skolem sequence of length n .

Lemma 3.6 ([14]): A necessary and sufficient condition for the existence of a Skolem sequence of length n is $n \equiv 0, 1 \pmod{4}$.

Corollary 3.7: For $n \geq 1$, a necessary and sufficient condition for nK_2 being super graceful is $n \equiv 0, 1 \pmod{4}$. Moreover, it is possible to have $[1, n]$ as the edge-label set.

Proof: It follows from Lemmas 3.5 and 3.6 by extending a Skolem sequence (a_i, b_i) to sequence of triples $(a_i + n, i, b_i + n)$ ($i \in [1, n]$) induces a super graceful labeling of nK_2 with edge-label set $[1, n]$.

Theorem 3.8: For $n, k \geq 1$, nK_2 is k -super graceful with the largest n labels assigned to mutually non-adjacent vertices if and only if $n = 2k - 1$. Moreover, $(2k - 1)K_2$ is k -super graceful with edge-label set $[k, 3k - 2]$.

Proof: (Sufficiency) Suppose nK_2 is k -super graceful with the largest n labels assigned to mutually non-adjacent vertices. We then have $\sum_{i=k}^{k+2n-1} i = \sum_{j=k+2n}^{k+3n-1} j$. Solving the equation gives $n = 2k - 1$.

(Necessity) It suffices to find a k -Skolem sequence $\{(a_i, b_i)\}$ of length $2k - 1$. It is presented below:

range of i	a_i	b_i	$a_i - b_i$
$[1, k]$	$2k - 2 + 2i$	$k - 1 + i$	$k - 1 + i$
$[k + 1, 2k - 1]$	$2i - 1$	$i - k$	$k - 1 + i$

By Lemma 3.5, $(2k - 1)K_2$ is k -super graceful with edge-label set $[k, 3k - 2]$.

Corollary 3.9: If nK_2 is k -super graceful with edge-label set $[k, k + n - 1]$, then $(3n + 2k - 1)K_2$ is also k -super graceful with edge-label set $[k, 3n + 3k - 2]$.

Proof: By Theorem 3.8, we know that $(2n + 2k - 1)K_2$ is $(n + k)$ -super graceful with edge-label set $[n + k, 3n + 3k - 2]$. By Lemma 2.5 $(3n + 2k - 1)K_2$ is k -super graceful with edge-label set $[k, 3n + 3k - 2]$.

Corollary 3.10: If $n = (2k - 1)(3^r - 1) / 2$ for $r \geq 1$, then nK_2 is k -super graceful with edge-label set $[k, k + n - 1]$.

Proof: The corollary follows by applying Corollary 3.9 and mathematical induction on r .

Corollary 3.11: (i) There is a k -Skolem sequence of length $2k - 1$ for each $k \geq 1$. (ii) For each $k \geq 1$, $(2k - 1)K_2$ is k -sequentially additive with edge-label set $[5k - 2, 7k - 4]$. (iii) There exists k -super graceful graph G of size $n = 2k - 1$ with

edge-label set $[k, k + n - 1]$ for each $k \geq 1$. Particularly, there are infinitely many k -super graceful 1-regular graphs for each $k \geq 1$.

Theorem 3.12: Suppose $n \leq 84$. A necessary and sufficient condition for nK_2 to be 2-super graceful with edge-label set $[2, n + 1]$ is $n \equiv 0, 3 \pmod{4}$.

Proof: The necessity follows from Lemma 3.1. By Lemma 3.5, it suffices to prove the sufficiency by constructing a 2-Skolem sequence of length n .

Let $n = 4r - 1$ for $1 \leq r \leq 21$.

For $r = 1$, a 2-Skolem sequence of length 3 is given by $(4, 2)$, $(6, 3)$ and $(5, 1)$.

For $r = 2$, a 2-Skolem sequence of length 7 is given by $(10, 8)$, $(4, 1)$, $(6, 2)$, $(14, 9)$, $(13, 7)$, $(12, 5)$, $(11, 3)$.

For $r = 3$, a 2-Skolem sequence of length 11 is given by $(14, 12)$, $(4, 1)$, $(6, 2)$, $(8, 3)$, $(16, 10)$, $(22, 15)$, $(21, 13)$, $(20, 11)$, $(19, 9)$, $(18, 7)$, $(17, 5)$.

For $r \geq 4$, we define a partial 2-Skolem sequence $\{(a_i, b_i)\}$ of length $4r - 1$ by

$$(a_i, b_i) = \begin{cases} (2i, i-1) & \text{for } 2 \leq i \leq 4 \text{ or even } i \text{ in the range } 6 \leq i \leq 2r-2; \\ (10r-2-i, 10r-3-2i) & 2r \leq i \leq 4r-1. \end{cases}$$

It is easy to see that the difference $a_i - b_i = i + 1$. Up to now, the values of a_i 's cover the range $\{4, 6, 8\} \cup \{4s \mid 3 \leq s \leq r-1\} \cup [6r - 1, 8r - 2]$, the values of b_i 's cover the range $\{1, 2, 3\} \cup \{2s-1 \mid 3 \leq s \leq 3r-1\}$. That is, the set of unassigned integers is $\{4l+2 \mid 2 \leq l \leq r-2\} \cup \{2l \mid 2r-1 \leq l \leq 3r-1\}$ which will be assigned to a_i and b_i for odd integers i in the range $\{1\} \cup [5, 2r - 1]$. The difference $(a_i - b_i)$'s cover the range $\{3, 4, 5\} \cup \{2s-1 \mid 4 \leq s \leq r\} \cup [2r + 1, 4r]$. That is, the set of missing differences is $\{2\} \cup \{2l \mid 3 \leq l \leq r\}$. Ad hoc assignments for the remaining (a_i, b_i) 's are shown below.

r	(a_i, b_i)	(a_i, b_i) for odd i in $[5, 2r - 1]$
4	(20, 18)	(16, 10), (22, 14)
5	(28, 26)	(24, 18), (22, 14), (20, 10)
6	(32, 30)	(34, 28), (26, 18), (24, 14), (22, 10)
7	(36, 34)	(38, 32), (26, 18), (40, 30), (22, 10), (28, 14)
8	(42, 40)	(38, 32), (18, 10), (44, 34), (26, 14), (36, 22), (46, 30)
9	(52, 50)	(48, 42), (46, 38), (44, 34), (22, 10), (40, 26), (30, 14), (36, 18)
10	(42, 40)	(44, 48), (18, 10), (56, 46), (26, 14), (58, 44), (38, 22), (52, 34), (50, 30)
11	(64, 62)	(54, 48), (58, 50), (52, 42), (30, 18), (60, 46), (26, 10), (56, 38), (34, 14), (44, 22)

Using computer searching, we have also obtained at least one such pairings for $12 \leq r \leq 21$. In fact, our computer search shows that there are altogether 189 different pairings for $r = 11$. For $r = 20$, there are at least 5657 different pairings.

Let $n = 4r$ for $1 \leq r \leq 21$. From the 2-Skolem sequence $\{(a_i, b_i)\}_{i=1}^{4r-1}$ of length $4r-1$ defined above, we define

$$(A_i, B_i) = \begin{cases} (a_i, b_i) & \text{if } 1 \leq i \leq 2r-1, \\ (a_i + 2, b_i + 2) & \text{if } 2r \leq i \leq 4r-1, \\ (6r, 2r-1) & \text{if } i = 4r. \end{cases}$$

One may easily check that $\{(A_i, B_i)\}_{i=1}^{4r}$ is a 2-Skolem sequence of length $4r$.

Clearly, nK_2 is 2-super graceful for all $n \equiv 0, 3 \pmod{4}$ if the following conjecture holds. Moreover, for $n \equiv 0 \pmod{4}$, we can get a labeling with all even edge labels.

Conjecture 3.1: For $r \geq 22$, there exist $r-1$ pairs of integers using $\{4l+2 \mid 2 \leq l \leq r-2\} \cup \{2l \mid 2r-1 \leq l \leq 3r-1\}$ completely such that the set of differences of the pairs is $\{2\} \cup \{2l \mid 3 \leq l \leq r\}$.

By Corollary 3.9 and Theorem 3.12, we obtain infinitely many 2-super graceful 1-regular graphs recursively. By Lemma 3.1, it is easy to show that $4K_2$ is not 3-super graceful but $8K_2$ is 3-super graceful given by the triples $(10, 3, 13)$, $(18, 4, 22)$, $(16, 5, 21)$, $(14, 6, 20)$, $(19, 7, 26)$, $(17, 8, 25)$, $(15, 9, 24)$, $(12, 11, 23)$. Consequently, $3K_2$ is k -super graceful if and only if $k = 2$; $4K_2$ is k -super graceful if and only if $k = 1, 2$; and $5K_2$ is k -super graceful if and only if $k = 1, 3$.

We can also show that nK_2 is 3-super graceful for $n = 8, 9, 12, 13$. For $n = 8$, the triples are $(13, 3, 16)$, $(14, 4, 18)$, $(21, 5, 26)$, $(19, 6, 25)$, $(17, 7, 24)$, $(15, 8, 23)$, $(11, 9, 20)$, $(12, 10, 22)$. For $9K_2$, the triples are $(24, 5, 29)$, $(22, 6, 28)$, $(20, 7, 27)$, $(17, 9, 26)$, $(15, 10, 25)$, $(12, 11, 23)$, $(13, 8, 21)$, $(16, 3, 19)$, $(14, 4, 18)$. For $n = 12$, the triples are $(18, 3, 21)$, $(19, 4, 23)$, $(20, 5, 25)$, $(32, 6, 38)$, $(30, 7, 37)$, $(28, 8, 36)$, $(26, 9, 35)$, $(24, 10, 34)$, $(22, 11, 33)$, $(15, 12, 27)$, $(16, 13, 29)$, $(17, 14, 31)$ (where the last three triples may be replaced by $(14, 13, 27)$, $(17, 12, 29)$, $(16, 15, 31)$). For $13K_2$, the triples are $(36, 5, 41)$, $(34, 6, 40)$, $(32, 7, 39)$, $(30, 8, 38)$, $(22, 15, 37)$, $(21, 14, 35)$, $(20, 13, 33)$, $(19, 12, 31)$, $(18, 11, 29)$, $(24, 4, 28)$, $(17, 10, 27)$, $(23, 3, 26)$, $(16, 9, 25)$. Note that each labeling above gives the edge-label set $[3, n+2]$. Moreover, $12K_2$ is 5-super graceful with triples $(32, 8, 40)$, $(30, 9, 39)$, $(28, 10, 38)$, $(22, 15, 37)$, $(20, 16, 36)$, $(18, 17, 35)$, $(23, 11, 34)$,

(21,12,33), (25,6,31), (24,5,29), (14,13,27), (19,7,26) but the edge-label set is not [5, 16].

Conjecture 3.2: Every k -super graceful nK_2 admits a labeling with edge-label set $[k, k + n - 1]$.

4. $\Delta(G) = 2$

Clearly, every component of G is a cycle or a path if and only if $\Delta(G) = 2$. The following follows directly from the definition.

Lemma 4.1: In any k -super graceful labeling of a graph containing a C_3 component, the largest edge label is the sum of the other two edge labels.

In [6], the authors showed that 2-regular graphs nC_3 is simply sequentially additive (SSA) for all $n \geq 1$ and nC_4 is SSA if and only if $n \equiv 0, 1 \pmod{3}$. However,

Theorem 4.2: For $k \geq 1$, $k = 1$ a necessary and sufficient condition for kC_3 to be k -super graceful.

Proof: Let $u_{i,1}, u_{i,2}$ and $u_{i,3}$ be the three vertices of the i -th component of kC_3 . Since every cycle is super graceful, we immediately have C_3 is super graceful. Suppose f is a k -super graceful labeling of kC_3 , we may assume $k \geq 2$. By Theorem 2.1, without loss of generality, we let $f(u_{i,1}) = 7k - i$ for $i \in [1, k]$. Since all labels are greater than $k - 1$, $6k - 1$ must be a vertex or an edge label in the first component, i.e., (i) $f(u_{1,2}) = 6k - 1$ or (ii) $f(u_{1,1}u_{1,2}) = 6k - 1$.

If (i) holds, then $f(u_{1,1}u_{1,2}) = k$. It follows that $f(u_{1,1}u_{1,3}) = 6k - 2$. By Lemma 4.1 $f(u_{1,2}u_{1,3}) = 5k - 2$ and hence $f(u_{1,3}) = k + 1$. Now $6k - 3$ is neither a vertex nor an edge label, a contradiction.

If (ii) holds, then $f(u_{1,2}) = k$. It follows that $f(u_{1,3}) = 6k - 2$ and $f(u_{1,1}u_{1,3}) = k + 1$. Consequently, $f(u_{1,2}u_{1,3}) = 5k - 2$ and $6k - 3$ is neither a vertex nor an edge label, a contradiction.

By Corollary 2.2, we have if nC_3 is k -super graceful, then $1 \leq k \leq n$.

Corollary 4.3: For all $k \geq 1$, $2C_3$ is not k -super graceful.

Proof: By Theorem 4.2, $2C_3$ is not 2-super graceful. By Lemma 4.1, we can show that $2C_3$ is not super graceful.

We now know $P_n, n \geq 2$, is super graceful with all even edge labels. Moreover, nK_2 is super graceful with all even edge labels for $n = 4, 5, 8, 9$.

Problem 4.1: For which $k \geq 2$ is there a k -super graceful graph with all even edge labels?

Note that for $n \geq 3$, $m \geq 1$, $C_n + mK_1$ is super graceful. Particularly, $C_n + K_1$ is a super graceful $(n+1, n)$ -graph. However, one can easily show that $C_3 + P_2$, $C_3 + P_3$ and $C_4 + P_2$ are not super graceful with all odd vertex-label set. We now give two families of super graceful $(n+1, n)$ -graphs with all odd vertex labels.

Theorem 4.4: The graph $C_n + P_m$ is super graceful with all odd vertex labels if

- (i) $n = 3$, $m \geq 4$,
- (ii) $n = 4$, $m \geq 3$.

Proof: Let $P_m = u_1v_1u_2v_2 \cdots u_{m/2}v_{m/2}$ for even m , and $P_m = u_1v_1u_2v_2 \cdots u_{(m-1)/2}v_{(m-1)/2}u_{(m+1)/2}$ for odd m .

Suppose $n = 3$. We shall define a super-graceful labeling $f: V(C_3 + P_{4r+s}) \cup E(C_3 + P_{4r+s}) \rightarrow [1, 8r + 2s + 5]$ for each case, where $r \geq 1$.

(A) $s = 0$. Label the vertices of C_3 by $4r+1, 4r+3, 4r+7$ and the corresponding edges by $2, 4, 6$.

- (1) $f(u_{2i-1}) = 4i - 1$ and $f(u_{2i}) = 4i - 3$, $1 \leq i \leq r$.
- (2) $f(v_1) = 8r + 5$; $f(v_{2i-1}) = 8r + 11 - 4i$, $2 \leq i \leq r$; and $f(v_{2i}) = 8r + 5 - 4i$, $1 \leq i \leq r$.
- (3) $f(u_1v_1) = 8r + 2$; $f(u_jv_j) = 8r + 8 - 4j$, $2 \leq j \leq 2r$.
- (4) $f(v_1u_2) = 8r + 4$; $f(v_{2i-1}u_{2i}) = 8r + 14 - 8i$, $2 \leq i \leq r$ and $f(v_{2i}u_{2i+1}) = 8r + 2 - 8i$, $1 \leq i \leq r - 1$.

(B) $s = 1$. Label the vertices of C_3 by $4r+1, 4r+5, 4r+7$ and the corresponding edges by $4, 2, 6$.

- (1) $f(u_{2r+1}) = 4r + 3$, $f(u_{2i-1}) = 4i - 1$ and $f(u_{2i}) = 4i - 3$, $1 \leq i \leq r$.
- (2) $f(v_{2i-1}) = 8r + 9 - 4i$ and $f(v_{2i}) = 8r + 11 - 4i$, $1 \leq i \leq r$.
- (3) $f(v_ju_{j+1}) = 8r + 8 - 4j$ for $1 \leq j \leq 2r$.
- (4) $f(u_{2i-1}v_{2i-1}) = 8r + 10 - 8i$ and $f(u_{2i}v_{2i}) = 8r + 14 - 8i$, $1 \leq i \leq r$.

- (C) $s = 2$. For $m = 6$, label the vertices of C_3 by 5,7,11 and the corresponding edges by 2,4,6. Label the edges of P_6 by 9,17,1,15,3,13 consecutively and the corresponding edges by 8,16,14,12,10.

Now, we consider $r \geq 2$. Label the vertices of C_3 by $4r+3$, $4r+5$, $4r+9$ and the corresponding edges by 2,4,6.

- (1) $f(u_{2r-1}) = 4r-3$, $f(u_{2r}) = 4r+1$, $f(u_{2r+1}) = 4r-1$,
 $f(v_1) = 8r+9$, $f(v_{2r}) = 4r+11$, $f(v_{2r+1}) = 4r+7$.
 - (2) $f(u_{2i-1}) = 4i-1$ and $f(u_{2i}) = 4i-3$, $1 \leq i \leq r-1$.
 - (3) $f(v_{2i}) = 8r+9-4i$ and $f(v_{2i+1}) = 8r+11-4i$, $1 \leq i \leq r-1$.
 - (4) $f(u_1v_1) = 8r+6$, $f(v_1u_2) = 8r+8$, $f(v_{2r-2}u_{2r-1}) = 16$,
 $f(u_{2r-1}v_{2r-1}) = 18$, $f(v_{2r-1}u_{2r}) = 14$, $f(u_{2r}v_{2r}) = 10$,
 $f(v_{2r}u_{2r+1}) = 12$, $f(u_{2r+1}v_{2r+1}) = 8$.
 - (5) $f(u_jv_j) = 8r+12-4j$ for $2 \leq j \leq 2r-2$.
 - (6) $f(v_{2i-2}u_{2i-1}) = 8r+14-8i$ and $f(v_{2i-1}u_{2i}) = 8r+18-8i$,
 $2 \leq i \leq r-1$.
- (D) $s = 3$. Label the vertices of C_3 by $4r+3$, $4r+7$, $4r+9$ and the corresponding edges by 4,2,6.
- (1) $f(u_1) = 3$, $f(u_2) = 5$, $f(u_3) = 1$, $f(u_4) = 9$, $f(v_1) = 8r+7$, $f(v_2) = 8r+11$,
 $f(v_3) = 8r+9$. Following steps are only for $r \geq 2$.
 - (2) $f(u_{2i}) = 4i+1$ and $f(u_{2i-1}) = 4i-5$, $3 \leq i \leq r+1$.
 - (3) $f(v_{2i}) = 8r+11-4i$ and $f(v_{2i+1}) = 8r+13-4i$, $2 \leq i \leq r$.
 - (4) $f(u_1v_1) = 8r+4$, $f(v_1u_2) = 8r+2$, $f(u_2v_2) = 8r+6$, $f(v_2u_3) =$
 $8r+10$, $f(u_3v_3) = 8r+8$, $f(v_3u_4) = 8r$.
 - (5) $f(v_{j-1}u_j) = 8r+16-4j$ for $5 \leq j \leq 2r+2$.
 - (6) $f(u_{2i}v_{2i}) = 8r+10-8i$ and $f(u_{2i+1}v_{2i+1}) = 8r+14-8i$, $2 \leq i \leq r$.

Suppose $n = 4$. Consider $C_4 + P_3$. Label the vertices of C_4 by 1,11,5,13 consecutively and the corresponding edges by 10,6,8,12. Label the

vertices of P_3 by 3, 7, 9 consecutively and the corresponding edges by 4, 2. Now assume $m = 4r + s$, where $r \geq 1$ and $0 \leq s \leq 3$. We shall define a super-graceful labeling $f: V(C_4 + P_{4r+s}) \cup E(C_4 + P_{4r+s}) \rightarrow [1, 8r + 2s + 7]$ for each case.

(A) $s = 0$. For $r = 1$, label the vertices of C_4 by 1, 13, 5, 15 consecutively and the corresponding edges by 12, 8, 10, 14; and label the vertices of P_4 by 3, 9, 7, 11 consecutively and the corresponding edges by 6, 2, 4. Now suppose $r \geq 2$. Label the vertices of C_4 by 1, $8r + 5$, 5, $8r + 7$ consecutively and the corresponding edges by $8r + 4$, $8r$, $8r + 2$, $8r + 6$.

$$(1) \quad f(u_1) = 3, f(u_2) = 7, f(u_{2i-1}) = 4i + 3 \text{ and } f(u_{2i}) = 4i + 1, \\ 2 \leq i \leq r.$$

$$(2) \quad f(v_{2i-1}) = 8r + 5 - 4i \text{ and } f(v_{2i}) = 8r + 7 - 4i, 1 \leq i \leq r.$$

$$(3) \quad f(u_1 v_1) = 8r - 2, f(v_1 u_2) = 8r - 6, f(u_2 v_2) = 8r - 4.$$

$$(4) \quad f(v_j u_{j+1}) = 8r - 4i \text{ for } 2 \leq j \leq 2r - 1.$$

$$(5) \quad f(u_{2i-1} v_{2i-1}) = 8r + 2 - 8i \text{ and } f(u_{2i} v_{2i}) = 8r + 6 - 8i, 2 \leq i \leq r.$$

(B) $s = 1$. Label the vertices of C_4 by 1, $8r + 5$, 3, $8r + 9$ consecutively and the corresponding edges by $8r + 4$, $8r + 2$, $8r + 6$, $8r + 8$.

$$(1) \quad f(u_{2i-1}) = 8r + 11 - 4i, 1 \leq i \leq r + 1; \text{ and } f(u_{2i}) = 8r + 5 - 4i, \\ 1 \leq i \leq r.$$

$$(2) \quad f(v_{2i-1}) = 3 + 4i \text{ and } f(v_{2i}) = 1 + 4i, 1 \leq i \leq r.$$

$$(3) \quad f(u_j v_j) = 8r + 4 - 4j, 1 \leq j \leq 2r.$$

$$(4) \quad f(v_{2i-1} u_{2i}) = 8r + 2 - 8i, 1 \leq i \leq r.$$

$$(5) \quad f(v_{2i} u_{2i+1}) = 8r + 6 - 8i, 1 \leq i \leq r.$$

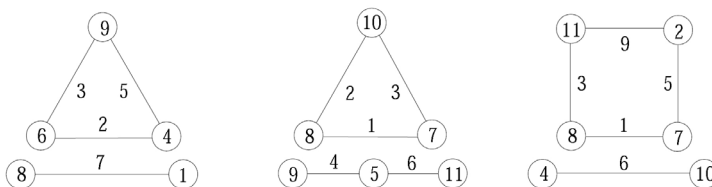
(C) $s = 2$. Label the vertices of C_4 by 1, $8r + 7$, 3, $8r + 11$ consecutively and the corresponding edges by $8r + 6$, $8r + 4$, $8r + 8$, $8r + 10$.

$$(1) \quad f(u_1) = 8r + 9, f(u_2) = 8r + 5, f(u_{2i+1}) = 8r + 5 - 4i, 1 \leq i \leq r; \\ \text{and } f(u_{2i}) = 8r + 11 - 4i, 2 \leq i \leq r.$$

- (2) $f(v_{2i-1}) = 3 + 4i, 1 \leq i \leq r+1$; and $f(v_{2i}) = 1 + 4i, 1 \leq i \leq r$.
- (3) $f(v_1 u_2) = 8r - 2$ and $f(v_{j-1} u_j) = 8r + 8 - 4j$ for $3 \leq j \leq 2r+1$.
- (4) $f(u_1 v_1) = 8r + 2, f(u_2 v_2) = 8r, f(u_{2r+1} v_{2r+1}) = 2, f(u_{2i+1} v_{2i+1}) = 8r - 2 - 8i, 1 \leq i \leq r-1$; and $f(u_{2i} v_{2i}) = 8r + 10 - 8i, 2 \leq i \leq r$.
- (D) $s = 3$. Label the vertices of C_4 by $1, 8r+9, 3, 8r+13$ consecutively and the corresponding edges by $8r+8, 8r+6, 8r+10, 8r+12$.
- (1) $f(u_{2i-1}) = 8r + 15 - 4i$ and $f(u_{2i}) = 8r + 9 - 4i, 1 \leq i \leq r+1$.
- (2) $f(v_{2i-1}) = 3 + 4i, 1 \leq i \leq r+1$; and $f(v_{2i}) = 1 + 4i, 1 \leq i \leq r$.
- (3) $f(u_j v_j) = 8r + 8 - 4j$ for $1 \leq j \leq 2r+1$.
- (4) $f(v_{2r+1} u_{2r+2}) = 2, f(v_{2i-1} u_{2i}) = 8r + 6 - 8i$ and $f(v_{2i} u_{2i+1}) = 8r + 10 - 8i, 1 \leq i \leq r$.

Clearly, each labeling defined above is a super graceful labeling of the respective graph having all odd vertex labels.

Remark 4.1: Here are super graceful labelings for $C_3 + P_2$, $C_3 + P_3$ and $C_4 + P_2$ respectively:



Combining the above remark, we have

Corollary 4.5: The graphs $C_3 + P_m$ and $C_4 + P_m$, are super graceful, where $m \geq 2$.

Theorem 4.6: For $n \geq 2$, the graph $P_{n+r} + P_n$ is 2-super graceful with edge-label set $[2, 2n + r - 1]$ for $r = 2, 3$.

Proof: Suppose $r = 2$. First, define a labeling f for $P_{2n+1} = u_1 u_2 \dots u_{2n+1}$ by $f(u_{2j-1}) = 3n + 3 - j, j \in [1, n + 1]$; $f(u_{2j}) = 3n + 3 + j, j \in [1, n]$; $f(u_{2i+1}) = i + 1, i \in [1, 2n]$. Next, delete the edge $u_n u_{n+1}$ that has label $n + 1$. Finally,

join a new vertex v adjacent to u_{2n+1} by labeling v and vu_{2n+1} by $3n+3$ and $n+1$, respectively.

For $r=3$, begin with P_{2n+2} , we can also get a 2-super graceful labeling for $P_{n+3} + P_n$ in a similar way.

Theorem 4.7: For $n \geq 2$, (i) $P_{n+2} + P_n$ is $(n+1)$ -super graceful with edge-label set $[2n+2, 4n+1]$ and (ii) $P_{n+3} + P_n$ is $(n+1)$ -super graceful with edge-label set $[2n+3, 4n+3]$.

Proof:

- (i) First, define a labeling f for $P_{2n+1} = u_1 u_2 \dots u_{2n+1}$ by $f(u_{2j-1}) = 5n+3-j$, $j \in [1, n+1]$; $f(u_{2j}) = n+j$, $j \in [1, n]$; $f(u_i u_{i+1}) = 4n+2-i$, $1 \leq i \leq 2n$. Finally, delete the edge $u_{n+1} u_{n+2}$ that has label $3n+1$ and join a new vertex u to u_1 by labeling u and uu_1 by $2n+1$ and $3n+1$.
- (ii) Begin with P_{2n+2} , we can get an $(n+1)$ -super graceful labeling for $P_{n+3} + P_n$ in a similar way.

We note that C_{10} admits a 5-super graceful labeling with consecutive vertex labels 24, 6, 20, 12, 21, 10, 23, 7, 22, 5 and corresponding edge labels 18, 14, 8, 9, 11, 13, 16, 15, 17, 19. Moreover, C_{12} is 6-super graceful with consecutive vertex labels 27, 6, 29, 7, 26, 14, 24, 13, 28, 8, 25, 9 and corresponding edge labels 21, 23, 22, 19, 12, 10, 11, 15, 20, 17, 16, 18.

Problem 4.2: For $k \geq 5$, find sufficient conditions such that C_{2k} is k -super graceful.

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