

Constacyclic codes over the non-chain finite commutative ring

$$\frac{\mathbb{Z}_4[u, v]}{\langle u^2 - u, v^2, uv \rangle}$$

Karthick Gowdhaman *

Department of Mathematics

Presidency University

Bangalore

Karnataka

India

T. Aaron Gulliver [†]

Department of Electrical and Computer Engineering

University of Victoria

Victoria

BC

Canada

Cruz Mohan [§]

Department of Mathematics

Bharathidasan University

Tiruchirapalli

Tamil Nadu

India

and

Department of Mathematics

Bishop Heber College

Tiruchirapalli

Tamil Nadu

India

* E-mail: karthygowtham@gmail.com (Corresponding Author)

[†] E-mail: agullive@ece.uvic.ca

[§] E-mail: cruzmohan@gmail.com

Durairajan Chinnappillai [†]

Department of Mathematics

Bharathidasan University

Tiruchirapalli

Tamil Nadu

India

Abstract

This paper considers $(1+2u)$ -constacyclic and $(1+2u)$ -skew constacyclic codes over the ring $R = \mathbb{Z}_4[u,v]/\langle u^2 - u, v^2, uv \rangle$, and some Gray maps over this ring are presented. The Gray images of these codes are shown to be cyclic, quasi-cyclic, and permutation equivalent to quasi-cyclic codes. In addition, $(1+2u)$ constacyclic codes are given with images that are new $\mathbb{Z}_4$ codes according to the online database of linear $\mathbb{Z}_4$ codes [3].

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1. Introduction

Cyclic codes and their algebraic structure have been extensively studied because of their many practical applications. Hammons et al. [8] constructed non-linear binary codes using $\mathbb{Z}_4$ linear codes. This led to tremendous research interest in codes over $\mathbb{Z}_4$. Pless et al. [15] considered $\mathbb{Z}_4$ cyclic codes and obtained idempotent generators for some of them. Yildiz et al. [20] explored codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$. Later Ashraf et al. [2] investigated $(1+u)$ -constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$. Linear and self-dual codes over $\mathbb{Z}_4[x]/\langle x^2 + 2x \rangle$ which is isomorphic to $\mathbb{Z}_4[x]/\langle x^2 - 1 \rangle$ were studied by Martínez-Moro et al. and Yildiz et al. in [12, 19], respectively. Yu et al. Authors have constructed codes over $\mathbb{Z}_4$ in [21], [5], [16], [14], [13] and found new $\mathbb{Z}_4$ codes that are not in the online database of linear $\mathbb{Z}_4$ codes [3].

Recently, Islam et al. [11] and Islam and Prakash [10] studied the $\mathbb{Z}_4$ -images of constacyclic codes over $\mathbb{Z}_4[u]/\langle u^k \rangle$ and $\mathbb{Z}_4[u,v]/\langle u^2, v^2, uv - vu \rangle$, respectively. Boucher et al. [6] studied codes over non-commutative rings and provided generalizations of cyclic codes which are skew cyclic and θ -cyclic codes. Skew cyclic codes were considered by [17], [4,6,7,18].

[†] E-mail: cdurai66@rediffmail.com

Based upon the preceding findings, this paper considers constacyclic codes within the non-chain finite commutative ring $\frac{\mathbb{Z}_4[u,v]}{\langle u^2-u, v^2, uv \rangle}$. Section 2 furnishes the background information, while Section 3 introduces Gray maps for α -constacyclic codes. The structure of α -constacyclic codes and their generating polynomials are studied in Section 4. Section 5 introduces Nechaev's permutation, accompanied by illustrative examples of codes. Sections 6 and 7 sequentially explore (θ, α) -cyclic codes and their corresponding Gray images.

2. Background

Consider $R = \mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$ with $u^2 = u$ and $v^2 = 0$. This is a finite commutative extension of $\mathbb{Z}_4$ and isomorphic to $\frac{\mathbb{Z}_4[u,v]}{\langle u^2-u, v^2, uv \rangle}$. The non-trivial ideals of R are $I_2 = \{2r : r \in R\}$, $I_u = \{ur : r \in R\}$, $I_v = \{vr : r \in R\}$, $I_{2u} = \{2ur : r \in R\}$, $I_{2v} = \{2vr : r \in R\}$, $I_{u,v} = \{ur_1 + vr_2 : (r_1, r_2) \in R\}$, $I_{2u,v} = \{2ur_1 + 2vr_2 : (r_1, r_2) \in R\}$, $I_{u,2v} = \{ur_1 + 2vr_2 : (r_1, r_2) \in R\}$, $I_{2u,2v} = \{2ur_1 + 2vr_2 : (r_1, r_2) \in R\}$, $I_{2,v} = \{2r_1 + vr_2 : (r_1, r_2) \in R\}$, $I_{2,u} = \{2r_1 + ur_2 : (r_1, r_2) \in R\}$, $I_{2,u,v} = \{2r_1 + ur_2 + vr_3 : (r_1, r_2, r_3) \in R\}$. $I_u \not\subseteq I_v$ and vice versa implies R is a non-chain ring. Therefore, R is a local ring with unique maximal ideal $I_{u,v,2}$ and quotient ring $\frac{R}{\langle I_{u,v,2} \rangle}$ which is isomorphic to $\mathbb{Z}_2$. A linear code ζ of length n is defined as an R -submodule of R^n .

Let α be a unit in R . Then an α -constacyclic shift is defined

$$\phi_\alpha(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = (\alpha\varepsilon_{n-1}, \varepsilon_0, \dots, \varepsilon_{n-2}).$$

A code whose codewords satisfy this shift is called an α -constacyclic code. When $\alpha = 1$, the α -constacyclic codes are called cyclic codes, and when $\alpha = -1$ they are called negacyclic codes.

It is convenient to associate each codeword of an α -constacyclic code with a polynomial in $\frac{R[\omega]}{(\omega^n - \alpha)}$ through the following linear map

$$\phi : \zeta \mapsto \frac{R[\omega]}{(\omega^n - \alpha)}, \quad \phi(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = \varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_{n-1}\omega^{n-1}.$$

Then a set of α -constacyclic codewords in R^n can be seen as a set of polynomials over $\frac{R[\omega]}{(\omega^n - \alpha)}$. In addition, an α -cyclic shift in C can be represented by $\omega c(\omega)$ which leads to the following theorem.

Theorem 2.1: A linear code $\zeta \subseteq R^n$ is an α -constacyclic code if and only if it is an ideal of $\frac{R[\omega]}{(\omega^n - \alpha)}$.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \mathbb{Z}_4^{mm}$ where $\varepsilon_i \in \mathbb{Z}_4^n$, $i = \{1, 2, \dots, m\}$. Then we can define the map

$$\nu_m : \mathbb{Z}_4^{mm} \rightarrow \mathbb{Z}_4^{mm}, \nu(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) = (\varphi_1(\varepsilon_1), \varphi_1(\varepsilon_2), \dots, \varphi_1(\varepsilon_m)),$$

where φ_1 is the cyclic shift defined above. A code closed under φ_1 is called a quasi-cyclic code of index m .

A unit in R has the form $\iota^* + u\kappa + v\varsigma$ where ι^* is a unit in $\mathbb{Z}_4$ and $\kappa, \varsigma \in \mathbb{Z}_4$. In the remainder of this paper, we set $\alpha = 1 + 2u$.

3. Gray Maps and its Images

In this section we define three Gray maps and have studied its structure, the Gray maps are given by

1. $\delta_1 : R \mapsto \mathbb{Z}_4^2$ defined by

$$\delta_1(\iota + u\kappa + v\varsigma) = (2\iota + \kappa + \varsigma, 3\iota + \kappa) \quad (1)$$

2. $\delta_2 : R \mapsto \mathbb{Z}_4^3$ defined by

$$\delta_2(\iota + u\kappa + v\varsigma) = (\iota + \varsigma, 2\kappa + 2\iota, \varsigma) \quad (2)$$

3. $\delta_3 : R \mapsto \mathbb{Z}_4^2$ defined by

$$\delta_3(\iota + u\kappa + v\varsigma) = (\iota + 3\kappa + \varsigma, 3\iota + \kappa + \varsigma) \quad (3)$$

These maps can be extended to an arbitrary length n . Lee weight of $\iota \in \mathbb{Z}_4$ is defined as $w_L(\iota) = \min(\iota, 4 - \iota)$. Then, $w_L(0) = 0$, $w_L(1) = w_L(3) = 1$ and $w_L(2) = 2$ respectively. Lee weight of $r = (\iota + u\kappa + v\varsigma) \in R$ defined as $w_L(r) = w_L(\delta_1(r))$. The Lee distance of a code ζ is given by $d_L(\zeta) = \min(w_L(\varepsilon_i - \varepsilon_j))$ where $\varepsilon_i, \varepsilon_j \in \zeta$. Define the Euclidean weight of $\omega \in \mathbb{Z}_4$ as $w_E(\omega) = \min\{\omega^2, (4 - \omega)^2\}$. Then, 0, 1, 2, 3 have Euclidean weights 0, 1, 4, 1, respectively. For an element $\varepsilon \in R$, define the Euclidean weight as $w_E(\varepsilon) = w_E(\delta_1(\varepsilon))$. The Euclidean distance of a code ζ is given by $d_E(\zeta) = w_E(\delta_1(\varepsilon))$.

Lemma 3.1: Consider the the Gray map δ_1 . Then, $\sigma\delta_1 = \delta_1\varphi_\alpha$.

Proof: Let $\varepsilon_i = \iota_i + u\kappa_i + v\varsigma_i \in R$, $0 \leq i \leq n-1$, so then $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$. Therefore

$$\begin{aligned} \sigma\delta_1(\varepsilon) &= \sigma\delta_1(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \sigma(2\iota_0 + \kappa_0 + \varsigma_0, \dots, 2\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_0 + \varsigma_0, \dots, 3\kappa_{n-1} + \varsigma_{n-1}) \\ &= (3\kappa_{n-1} + \varsigma_{n-1}, 2\iota_0 + \kappa_0 + \varsigma_0, \dots, 2\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, \\ &\quad 3\kappa_0 + \varsigma_0, \dots, 3\kappa_{n-2} + \varsigma_{n-2}). \end{aligned}$$

On the RHS

$$\begin{aligned}\delta_1 \varphi_\alpha(\varepsilon) &= \delta_1 \varphi_\alpha(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \psi_1(\alpha \varepsilon_{n-1}, \varepsilon_0, \dots, \varepsilon_{n-2}) \\ &= (3\kappa_{n-1} + \varsigma_{n-1}, 2\iota_0 + \kappa_0 + \varsigma_0, \dots, 2\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_0 \\ &\quad + \varsigma_0, \dots, 3\kappa_{n-2} + \varsigma_{n-2}).\end{aligned}$$

Hence, $\sigma \delta_1 = \delta_1 \varphi_\alpha$.

Theorem 3.2: $\delta_1(\zeta)$ is a cyclic code over $\mathbb{Z}_4^{2n}$, whenever ζ is a α -constacyclic code over R^n .

Proof: $\varphi_\alpha(\zeta) = \zeta$, as ζ is a α -constacyclic code. Therefore by Lemma 3.1,

$$\delta_1 \varphi_\alpha(\zeta) = \delta_1(\zeta) = \sigma \delta_1(\zeta).$$

Hence, the result.

Lemma 3.3: Consider the Gray map δ_2 . Then, $\nu_3 \delta_2 = \delta_2 \varphi_\alpha \varphi_\alpha$.

Proof: Let $z_i = \iota_i + u\kappa_i + v\varsigma_i \in R$, $0 \leq i \leq n-1$, so then $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$. Therefore

$$\begin{aligned}\nu_3 \delta_2(\varepsilon) &= \nu_3 \delta_2(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \nu_3(\iota_0 + \varsigma_0, \dots, \iota_{n-1} + \varsigma_{n-1}, 2\iota_0 + 2\kappa_0, \dots, 2\iota_{n-1} + 2\kappa_{n-1}, \varsigma_0, \dots, \varsigma_{n-1}) \\ &= (\iota_{n-1} + \varsigma_{n-1}, \iota_0 + \varsigma_0, \dots, \iota_{n-2} + \varsigma_{n-2}, 2\iota_{n-1} + 2\kappa_{n-1}, \\ &\quad 2\iota_0 + 2\kappa_0, \dots, 2\iota_{n-2} + 2\kappa_{n-2}, \varsigma_{n-1}, \varsigma_0, \dots, \varsigma_{n-2}).\end{aligned}$$

On the RHS

$$\begin{aligned}\delta_2 \varphi_\alpha(\varepsilon) &= \delta_2 \varphi_\alpha(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \psi_2(\alpha \varepsilon_{n-1}, \varepsilon_0, \dots, \varepsilon_{n-2}) \\ &= (\iota_{n-1} + \varsigma_{n-1}, \iota_0 + \varsigma_0, \dots, \iota_{n-2} + \varsigma_{n-2}, 2\iota_{n-1} + 2\kappa_{n-1}, \\ &\quad 2\iota_0 + 2\kappa_0, \dots, 2\iota_{n-2} + 2\kappa_{n-2}, \varsigma_{n-1}, \varsigma_0, \dots, \varsigma_{n-2}).\end{aligned}$$

Hence, $\nu_3 \delta_2 = \delta_2 \varphi_\alpha$.

Theorem 3.4: $\delta_2(\zeta)$ is a quasi-cyclic code over $\mathbb{Z}_4^{3n}$ with index 3. If ζ is an α -constacyclic code over R^n .

Proof: ζ is a α -constacyclic, $\Rightarrow \varphi_\alpha(\zeta) = \zeta$. By Lemma 3.3, we have

$$\delta_2 \varphi_\alpha(\zeta) = \delta_2(\zeta) = \nu_3 \delta_2(\zeta).$$

This shows the result.

Lemma 3.5: Consider the Gray map δ_3 . Then, $\varepsilon v_2 \delta_3 = \delta_3 \varphi_\alpha$ where φ_α is the α -constacyclic shift, v_2 is the quasi-cyclic shift operator and ε a permutation on $\mathbb{Z}_4^{2n}$ defined by $\varepsilon(\omega_1, \omega_2, \dots, \omega_{2n}) = (\omega_{\varepsilon(1)}, \omega_{\varepsilon(2)}, \dots, \omega_{\varepsilon(2n)})$ with the condition $\varepsilon = (1, n+1)$.

Proof: Let $\varepsilon_i = \iota_i + u\kappa_i + v\varsigma_i \in R$, $0 \leq i \leq n-1$. Then, $z = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$ and

$$\begin{aligned} \varepsilon v_2 \delta_3(\varepsilon) &= \varepsilon v_2 \delta_3(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \varepsilon v_2(\iota_0 + 3\kappa_0 + \varsigma_0, \dots, \iota_{n-1} + 3\kappa_{n-1} + \varsigma_{n-1}, 3\iota_0 + \kappa_0 + \varsigma_0, \dots, 3\iota_{n-1} \\ &\quad + \kappa_{n-1} + \varsigma_{n-1}) \\ &= \varepsilon(\iota_{n-1} + 3\kappa_{n-1} + \varsigma_{n-1}, \iota_0 + 3\kappa_0 + \varsigma_0, \dots, \iota_{n-2} + 3\kappa_{n-2} + \varsigma_{n-2}, 3\iota_{n-1} \\ &\quad + \kappa_{n-1} + \varsigma_{n-1}, 3\iota_0 + \kappa_0 + \varsigma_0, \dots, 3\iota_{n-2} + \kappa_{n-2} + \varsigma_{n-2}) \\ &= (3\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, \iota_0 + 3\kappa_0 + \varsigma_0, \dots, \iota_{n-2} + 3\kappa_{n-2} + \varsigma_{n-2}, \iota_{n-1} \\ &\quad + 3\kappa_{n-1} + \varsigma_{n-1}, \iota_0 + 3\kappa_0 + \varsigma_0, \dots, \iota_{n-2} + 3\kappa_{n-2} + \varsigma_{n-2}). \end{aligned}$$

On the RHS

$$\begin{aligned} \delta_3 \varphi_\alpha(\varepsilon) &= \psi_3 \varphi_\alpha(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \delta_3(\alpha \varepsilon_{n-1}, \varepsilon_0, \dots, \varepsilon_{n-2}) \\ &= (3\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, \iota_0 + 3\kappa_0 + \varsigma_0, \dots, \iota_{n-2} + 3\kappa_{n-2} + \varsigma_{n-2}, \iota_{n-1} + 3\kappa_{n-1} \\ &\quad + \varsigma_{n-1}, \iota_0 + 3\kappa_0 + \varsigma_0, \dots, \iota_{n-2} + 3\kappa_{n-2} + \varsigma_{n-2}). \end{aligned}$$

Hence, $\varepsilon v_2 \delta_3 = \delta_3 \varphi_\alpha$.

Theorem 3.6: Let n be a positive integer, If $\zeta \subseteq R^n$ is an α -constacyclic code, then $\delta_3(\zeta) \subseteq \mathbb{Z}_4^{2n}$ is permutation equivalent to a quasi-cyclic code with index 2 over R .

Proof: If $\zeta \subseteq R^n$ is α -constacyclic code, $\Rightarrow \varphi_\alpha(\zeta) = \zeta$, and by Lemma 3.5, we have

$$\delta_3 \varphi_\alpha(\zeta) = \delta_3(\zeta) = \varepsilon v_2 \delta_3(\zeta).$$

Hence the result.

Definition 3.1: Let $\zeta \subseteq \mathbb{Z}_4^n$, then ζ is called r -cyclic code if $\sigma^r(\zeta) = \zeta$ where σ is the cyclic shift operator. Note that for $r \geq 2$, every cyclic code is r -cyclic but not the converse.

We now define the permutations of the Gray maps $\delta_i, i = 1, 2, 3$.

Definition 3.2: Let $\varepsilon_i = t_i + u\kappa_i + v\varsigma_i \in R$, $0 \leq i \leq n-1$. Then, we have

1. $(\delta_1)_\rho : R^n \rightarrow \mathbb{Z}_4^{2n}$ defined by

$$(\delta_1)_\rho(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = (2t_0 + \kappa_0 + \varsigma_0, 3\kappa_0 + \varsigma_0, \dots, 2t_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_{n-1} + \varsigma_{n-1}) \quad (4)$$

2. $(\delta_2)_\rho : R^n \rightarrow \mathbb{Z}_4^{3n}$ defined by

$$(\delta_2)_\rho(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = (t_0 + \varsigma_0, 2t_0 + 2\kappa_0 + \varsigma_0, \dots, t_{n-1} + \varsigma_{n-1}, 2t_{n-1} + 2\kappa_{n-1} + \varsigma_{n-1}) \quad (5)$$

3. $(\delta_3)_\rho : R^n \rightarrow \mathbb{Z}_4^{2n}$ defined by

$$(\delta_3)_\rho(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = (t_0 + 3\kappa_0 + \varsigma_0, 3t_0 + \kappa_0 + \varsigma_0, \dots, t_{n-1} + 3\kappa_{n-1} + \varsigma_{n-1}, 3t_{n-1} + \kappa_{n-1} + \varsigma_{n-1}). \quad (6)$$

Lemma 3.7: Let $(\delta_1)_\rho$ be the Gray map given in (4). Then, $(\delta_1)_\rho \sigma = \sigma^2(\delta_1)_\rho$.

Proof: Let $\varepsilon_i = t_i + u\kappa_i + v\varsigma_i \in R$, $0 \leq i \leq n-1$. Then, $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$ and

$$\begin{aligned} (\delta_1)_\rho \sigma(\varepsilon) &= (\delta_1)_\rho \sigma(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= (\delta_1)_\rho(\varepsilon_{n-1}, \varepsilon_0, \dots, \varepsilon_{n-2}) \\ &= (2t_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_{n-1} + \varsigma_{n-1}, 2t_0 + \kappa_0 + \varsigma_0, 3\kappa_0 + \varsigma_0, \dots, 2t_{n-2} \\ &\quad + \kappa_{n-2} + \varsigma_{n-2}, 3\kappa_{n-2} + \varsigma_{n-2}) \end{aligned}$$

and

$$\begin{aligned} \sigma^2(\delta_1)_\rho(\varepsilon) &= \sigma^2(\delta_1)_\rho(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \sigma^2(2t_0 + \kappa_0 + \varsigma_0, 3\kappa_0 + \varsigma_0, \dots, 2t_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_{n-1} + \varsigma_{n-1}) \\ &= (2t_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_{n-1} + \varsigma_{n-1}, 2t_0 + \kappa_0 + \varsigma_0, 3\kappa_0 + \varsigma_0, \dots, 2t_{n-2} \\ &\quad + \kappa_{n-2} + \varsigma_{n-2}, 3\kappa_{n-2} + \varsigma_{n-2}) \end{aligned}$$

so that $(\delta_1)_\rho \sigma = \sigma^2(\delta_1)_\rho$.

Theorem 3.8: $(\delta_1)_\rho(\zeta)$ is a 2-cyclic code over $\mathbb{Z}_4^{2n}$ for a cyclic code ζ over R^n , then

Proof: If $\zeta \subseteq R^n$ is cyclic code, $\Rightarrow \sigma(\zeta) = \zeta$. By Lemma 3.7, we have

$$(\delta_1)_\rho \sigma(\zeta) = (\delta_1)_\rho(\zeta) = \sigma^2((\delta_1)_\rho(\zeta)),$$

Therefore, $(\delta_1)_\rho(\zeta) \subseteq \mathbb{Z}_4^{2n}$ is a 2-cyclic code.

Lemma 3.9: Let σ be a cyclic shift operator and $(\delta_2)_\rho$ and $(\delta_3)_\rho$ is Gray map given in (5) and (6), then

- (i) $(\delta_2)_\rho \sigma = \sigma^3(\delta_2)_\rho$
- (ii) $(\delta_3)_\rho \sigma = \sigma^2(\delta_3)_\rho$.

Proof: The proof is omitted as it is similar to that for Lemma 3.7.

By applying the proof of Lemma 3.9 and Theorem 3.8. We have the following

Theorem 3.10: If ζ is a cyclic code of length n over R , then

- (i) $(\delta_2)_\rho(\zeta)$ is a 3-cyclic code of length $3n$ over $\mathbb{Z}_4$
- (ii) $(\delta_3)_\rho(\zeta)$ is a 2-cyclic code of length $2n$ over $\mathbb{Z}_4$.

4. Generators of α -constacyclic Codes over R

We now consider the generators of odd length cyclic codes over R . The relationship with α -constacyclic codes is considered using an isomorphism from $R[\omega]/\langle\omega^n - 1\rangle$ to $R[\omega]/\langle\omega^n - \alpha\rangle$.

Lemma 4.1: [1] Let $\zeta \subseteq \mathbb{Z}_4^n$ be a cyclic code. If n is odd then $\mathbb{Z}_4[\omega]/(\omega^n - 1)$ is a principal ideal ring and $\zeta = (f(\omega), 2g(\omega)) = (f(\omega) + 2g(\omega))$ where $f(\omega)$ and $g(\omega)$ generate cyclic codes with $g(\omega) \mid f(\omega) \mid (x^n - 1) \bmod 4$.

Let ζ be a cyclic code over R . Define a map ϕ from R to $\mathbb{Z}_4 + u\mathbb{Z}_4$ where $u^2 = u$ by

$$\begin{aligned} \phi: R &\mapsto \mathbb{Z}_4 + u\mathbb{Z}_4 \\ \phi(\iota + u\kappa + v\zeta) &= a + ub \end{aligned}$$

It is clear that ϕ is a homomorphism which can be extended as follows

$$\begin{aligned} \phi: R[\omega]/(\omega^n - 1) &\mapsto \mathbb{Z}_4 + u\mathbb{Z}_4[\omega]/(\omega^n - 1) \\ \phi(\varepsilon_0 + \varepsilon_1\omega + \cdots + \varepsilon_{n-1}\omega^{n-1}) &= \phi(\varepsilon_0) + \phi(\varepsilon_1)\omega + \cdots + \phi(\varepsilon_{n-1})\omega^{n-1} \end{aligned}$$

Now, restrict this map to cyclic codes $\zeta \subseteq R[\omega]/(\omega^n - 1)$. Then, $\ker \phi = \{va(\omega) : a(\omega) \in \mathbb{Z}_4[\omega]\} = \langle v(f_3(\omega) + 2g_3(\omega)) \rangle$ where $g_3(\omega) \mid f_3(\omega) \mid (\omega^n - 1) \bmod 4$. Further, $\phi(C)$ is an ideal over $\mathbb{Z}_4 + u\mathbb{Z}_4$. Then from [7, Theorem 9],

$\phi(\zeta) = \langle uJ_1, (1+3u)J_2 \rangle$ where J_1 and J_2 are ideals over $\mathbb{Z}_4$. Since it is assumed that n is odd we have $\phi(\zeta) = \langle v(f_1(\omega) + 2g_1(\omega)), (1+3u)((f_2(\omega) + 2g_2(\omega))) \rangle$ where $g_i(\omega) | f_i(\omega) | (\omega^n - 1) \bmod 4$. Therefore

$$\zeta = \langle u(f_1(\omega) + 2g_1(\omega)) + vp_1(\omega), (1+3u)((f_1(\omega) + 2g_1(\omega))) + vp_2(\omega), v(f_3(\omega) + 2g_3(\omega)) \rangle,$$

and the result follows.

Theorem 4.2: *If ζ is a cyclic code of odd length n over R , then*

$$\zeta = \langle u(f_1(\omega) + 2g_1(\omega)) + vp_1(\omega), (1+3u)((f_1(\omega) + 2g_1(\omega))) + vp_2(\omega), v(f_3(\omega) + 2g_3(\omega)) \rangle,$$

where $g_i(\omega) | f_i(\omega) | \omega^n - 1 \bmod 4$ and $p_i(\omega)$ is a polynomial over R .

Theorem 4.3: *For n an odd integer, the map $\tau : R[\omega] / \langle \omega^n - 1 \rangle \rightarrow R[\omega] / \langle \omega^n - \alpha \rangle$ defined by $\tau(f(\omega)) = f(\alpha\omega)$ is a ring isomorphism.*

Proof: Let $f(\omega) = g(\omega)$ in $R[\omega] / \langle \omega^n - 1 \rangle$, so then $f(\omega) \equiv g(\omega) \bmod (\omega^n - 1)$. Replacing ω by $\alpha\omega$ on both sides gives $f(\alpha\omega) - g(\alpha\omega) \equiv 0 \bmod (\omega^n \alpha^n - 1)$, which implies that $f(\alpha\omega) - g(\alpha\omega) \equiv 0 \bmod \alpha^n(\omega^n - \alpha)$ since $\alpha^n = \alpha$ for an odd integer n . Thus, $f(\alpha\omega) = g(\alpha\omega)$ in $R[\omega] / \langle \omega^n - \alpha \rangle$ so τ is an injective and well-defined map. Moreover, since $R[\omega] / \langle \omega^n - 1 \rangle$ and $R[\omega] / \langle \omega^n - \alpha \rangle$ are finite rings with the same number of elements and τ is injective, then τ is surjective. Further, it is easy to check that τ is a ring homomorphism. Hence τ is a ring isomorphism.

Corollary 4.1: *If ζ is a linear code of odd length n over R , then it is a cyclic code if and only if $\tau(\zeta)$ is an α -constacyclic code over R .*

Proof: Since τ is a ring isomorphism between $R[\omega] / \langle \omega^n - 1 \rangle$ and $R[\omega] / \langle \omega^n - \alpha \rangle$, then ζ is an ideal of $R[\omega] / \langle \omega^n - 1 \rangle$ if and only if $\tau(\zeta)$ is an ideal of $R[\omega] / \langle \omega^n - \alpha \rangle$, and the result follows.

Corollary 4.2: *If ζ is a linear code of odd length n over R , then it is a cyclic code if and only if $\phi(\zeta)$ is an α -constacyclic code where $\phi : R^n \rightarrow R^n$ defined by $\phi(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) = (\varepsilon_0, \alpha\varepsilon_1, \dots, \alpha^{n-2}\varepsilon_{n-2}, \alpha^{n-1}\varepsilon_{n-1})$.*

5. Nechaev's Permutation

We now study Nechaev's permutation and use the Gray maps given previously to obtain cyclic and quasi-cyclic codes over $\mathbb{Z}_4$.

Definition 5.1: [14] If m is an odd positive integer and $\xi = (1, m+1) (3, m+3) \dots (2i+1, m+2i+1) \dots (m-2, 2m-2)$ a permutation of $\{0, 1, \dots, 2m-1\}$. Then, Nechaev's permutation is defined as $\pi(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{2m-1}) = (\varepsilon_{\xi(0)}, \varepsilon_{\xi(1)}, \dots, \varepsilon_{\xi(2m-1)})$.

Lemma 5.1: For the Gray map δ_1 given previously, $\delta_1\varphi = \pi\delta_1$.

Proof: Let $\varepsilon_i = \iota_i + u\kappa_i + v\varsigma_i \in R$, $0 \leq i \leq n-1$. Then, $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$ and

$$\begin{aligned} \delta_1\varphi(\varepsilon) &= \delta_1\varphi(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \delta_1(\varepsilon_0, \alpha\varepsilon_1, \dots, \alpha^{n-1}\varepsilon_{n-1}) \\ &= (2\iota_0 + \kappa_0 + \varsigma_0, 3\kappa_1 + \varsigma_1, \dots, 2\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_0 + \varsigma_0, \\ &\quad 2\iota_1 + \kappa_1 + \varsigma_1, \dots, 3\kappa_{n-1} + \varsigma_{n-1}) \end{aligned}$$

Further

$$\begin{aligned} \pi\delta_1(\varepsilon) &= \pi\psi_1(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \pi(2\iota_0 + \kappa_0 + \varsigma_0, \dots, 2\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_0 + \varsigma_0, \dots, 3\kappa_{n-1} + \varsigma_{n-1}) \\ &= (2\iota_0 + \kappa_0 + \varsigma_0, 3\kappa_1 + \varsigma_1, \dots, 2\iota_{n-1} + \kappa_{n-1} + \varsigma_{n-1}, 3\kappa_0 + \varsigma_0, \\ &\quad 2\iota_1 + \kappa_1 + \varsigma_1, \dots, 3\kappa_{n-1} + \varsigma_{n-1}) \end{aligned}$$

and therefore $\psi_1\varphi = \pi\delta_1$.

Theorem 5.2: Let $\zeta \subseteq R^n$ be cyclic of odd length, let $T = \delta_1(\zeta)$. Then, $\pi(T)$ is cyclic over $\mathbb{Z}_4^{2n}$.

Proof: $\zeta \subseteq R^n$ is cyclic code, $\Rightarrow T = \delta_1(\zeta)$. By Lemma 5.1,

$$\pi\delta_1(\zeta) = \pi(T) = \psi_1\varphi(\zeta).$$

From Corollary 4.2 and Theorem 3.2, $\delta_1\varphi(\zeta) \subseteq \mathbb{Z}_4^{2n}$ is cyclic. Thus, $\pi(T)$ is cyclic.

Lemma 5.3: For the Gray map δ_2 given in (2), $\delta_2\varphi = \pi\delta_2$.

Proof: Since the proof is similar to Lemma 5.1 and hence it is omitted.

By applying the proof of Lemma 5.3 and Theorem 5.2, we have the following

Theorem 5.4: For a cyclic code ζ of odd length n over R , let $T = \delta_2(\zeta)$. Then, $\pi(T)$ is a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.

Lemma 5.5: For the Gray map δ_3 given in (3), $\delta_3\varphi = \pi\delta_3$.

Proof: The proof is omitted as it is similar to that for Lemma 5.1.

By applying Lemma 5.5 and Theorem 5.2 we obtain

Theorem 5.6: For a cyclic code ζ of odd length n over R , let $T = \delta_3(\zeta)$. Then, $\pi(T)$ is permutation equivalent to a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.

Example 5.1: Let $n=9$ and $p_i(\omega)=0, i=1,2,3$. Then by Theorem 4.2, we have an ideal over R generated by

NOTE:

- $d_L(\zeta)$ - Lee distance of C
- $d_E(\zeta)$ - Euclidean distance of ζ

$$\zeta = \langle u(f_1(\omega) + 2g_1(\omega)) + vp_1(\omega), (1+3u)((f_1(\omega) + 2g_1(\omega))) + vp_2(\omega), v(f_3(\omega) + 2g_3(\omega)) \rangle$$

1. Let $f_i(\omega) + 2g_i(\omega) = (\omega - 1)(\omega^6 + \omega^3 + 1) + 2(\omega - 1)$. Then, $\delta_1(\zeta)$ has parameter $[18, 4^4 2^{12}, 2]$, $d_L(\delta_1(\zeta)) = 4$, $d_E(\delta_1(\zeta)) = 6$ and is linear. By Theorem 5.2, $\pi(T_1)$ is a cyclic code over $\mathbb{Z}_4$. Further, $T_2 = \delta_2(\zeta)$ has parameter $[27, 4^6 2^{12}, 2]$ and is linear. By Theorem 5.3, $\pi(T_1)$ is a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].
2. Let $f_i(\omega) + 2g_i(\omega) = (\omega^6 + \omega^3 + 1)(\omega^2 + \omega + 1) + 2(\omega^2 + \omega + 1)$. Then, $\delta_2(\zeta)$ has parameter $[27, 4^3 2^{12}, 2]$, $d_L(\delta_2(\zeta)) = 4$, $d_E(\delta_2(\zeta)) = 8$ and is linear. By Theorem 5.4, $\pi(T)$ is quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].
3. Let $f_i(\omega) + 2g_i(\omega) = (\omega^6 + \omega^3 + 1)(\omega^2 + \omega + 1) + 2(\omega^6 + \omega^3 + 1)$. Then, $\delta_3(\zeta)$ has parameter $[18, 4^2 2^4, 3]$, $d_L(\delta_2(\zeta)) = 6$, $d_E(\delta_2(\zeta)) = 9$ and is linear. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ and $\delta_2(\zeta)$ has a parameter $[27, 4^3 2^6, 3]$ and is linear. By Theorem 5.4, $\pi(T_2)$ is a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].

Example 5.2: Let $n=15$. Then by Theorem 4.2, we have an ideal over R generated by

$$\zeta = \langle u(f_1(\omega) + 2g_1(\omega)) + vp_1(\omega), (1 + 3u)((f_1(\omega) + 2g_1(\omega)) + vp_2(\omega), v(f_3(\omega) + 2g_3(\omega))) \rangle$$

1. Let $f_i(\omega) + 2g_i(\omega) = 3\omega^4 + 2\omega^2 + \omega + 3$, and $p_i(\omega) = 0$, $i = 1, 2, 3$. Then, $T_3 = \delta_3(\zeta)$ has parameter $[30, 4^{22}2^0]$ and $d_H(\zeta) = d_L(\zeta) = 3$. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].
2. Let $f_i(\omega) + 2g_i(\omega) = \omega^5 + \omega^4 + 2\omega^3 + \omega^2 + 1$, and $p_i(\omega) = 0$, $i = 1, 2, 3$. Then, $T_3 = \delta_3(\zeta)$ has parameter $[30, 4^{20}2^2]$, $d_H(\zeta) = 3$, $d_L(\zeta) = 4$ and is linear. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].
3. Let $f_i(\omega) + 2g_i(\omega) = \omega^8 + 3\omega^7 + \omega^5 + \omega^4 + \omega^3 + \omega + 3$, $i = 1, 2$, $f_3(\omega) + 2g_3(\omega) = 3\omega + 1$, and $p_i(\omega) = 0$, $i = 1, 2, 3$. Then, $T_3 = \delta_3(\zeta)$ has parameter $[30, 4^{21}2^4]$, $d_H(\zeta) = 2$. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].
4. Let $f_i(\omega) + 2g_i(\omega) = \omega^8 + 3\omega^7 + \omega^5 + \omega^4 + \omega^3 + \omega + 3$, $i = 1, 2$, $f_3(\omega) + 2g_3(\omega) = \omega^6 + \omega^5 + \omega^4 + 3\omega^3 + 3\omega^2 + 3$, and $p_i(\omega) = 0$, $i = 1, 2, 3$. Then, $T_3 = \delta_3(\zeta)$ has parameter $[30, 4^{17}2^8]$, $d_H(\zeta) = 2$ and is linear. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].
5. Let $f_i(\omega) + 2g_i(\omega) = \omega^8 + 3\omega^7 + \omega^5 + \omega^4 + \omega^3 + \omega + 3$, $i = 1, 2$, $f_3(\omega) + 2g_3(\omega) = \omega^{10} + \omega^9 + 3\omega^8 + \omega^6 + \omega^4 + 3\omega^2 + 3\omega + 3$, and $p_i(\omega) = 0$, $i = 1, 2, 3$. Then, $T_3 = \delta_3(\zeta)$ has parameter $[30, 4^{13}2^{12}]$, $d_H(\zeta) = 2$ and is linear. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ that is not in the database [3].

Example 5.3: Let $n = 17$. Then by Theorem 4.2, we have an ideal over R generated by

$$\zeta = \langle u(f_1(\omega) + 2g_1(\omega)) + vp_1(\omega), (1 + 3u)((f_1(\omega) + 2g_1(\omega)) + vp_2(\omega), v(f_3(\omega) + 2g_3(\omega))) \rangle$$

Let $f_i(\omega) + 2g_i(\omega) = \omega^9 + \omega^8 + 2\omega^7 + \omega^6 + \omega^3 + 2\omega^2 + \omega + 1$, $p_i(\omega) = 0$, $i = 1, 2, 3$. Then, $T_3 = \delta_3(\zeta)$ has parameter $[34, 4^{16}2^2]$, $d_H(\zeta) = 5$. By Theorem 5.6, $\pi(T_3)$ is permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$ which is not in the database [3].

6. Theta Cyclic Codes

We now study skew cyclic codes over R using a non-trivial automorphism. Further, (θ, α) -cyclic codes of odd length over R are considered which are α -constacyclic codes over R .

Let θ be a non-trivial automorphism over R defined by $\theta(\iota + u\kappa + v\zeta) = \iota + u\kappa - v\zeta$. It is evident that θ is an automorphism on R . We recall the multiplication of skew polynomials defined by the distribution rule along with the condition $(\iota\omega^i)(\kappa\omega^j) = \iota\theta^i(\kappa)\omega^{i+j}$, $\iota, \kappa \in \mathbb{Z}_4$. Then, $R[\omega; \theta] = \{f(\omega) \in R[\omega]\}$ is a non-commutative ring (called a skew polynomial ring), under the above multiplication and usual addition of polynomials. Note that the automorphism has order 2. Therefore, when n is an even integer, $\langle \omega^n - \alpha \rangle$ is a two-sided ideal in $R[\omega; \theta]$, so then $R_{n, \alpha, \theta} = R[\omega; \theta] / \langle \omega^n - \alpha \rangle$ is a non-commutative ring. However, for an integer n (not necessarily even), $R_{n, \alpha, \theta}$ is a left $R[\omega; \theta]$ -module where the left multiplication is defined as $f(\omega)(g(\omega) + \langle \omega^n - \alpha \rangle) = f(\omega)g(\omega) + \langle \omega^n - \alpha \rangle$ for $f(\omega), g(\omega) \in R[\omega; \theta]$.

Definition 6.1: [6] An R -submodule ζ of R^n is called a (θ, α) -cyclic (or skew α -constacyclic) code of length n if for any $\varsigma = (\varsigma_0, \varsigma_1, \dots, \varsigma_{n-1}) \in \zeta$ we have $\Gamma_{\theta, \alpha}(\varsigma) = (\alpha\theta(\varsigma_{n-1}), \theta(\varsigma_0), \dots, \theta(\varsigma_{n-2})) \in \zeta$. The map $\Gamma_{\theta, \alpha}$ is known as the (θ, α) shift operator. For $\alpha = 1$, ζ is called a θ -cyclic (skew cyclic) code.

Definition 6.2: [9] A linear code ζ of length $n = st$ over R is called an α -quasi-twisted code of length n and index t if for any $(\varepsilon_{00}, \varepsilon_{01}, \dots, \varepsilon_{0t-1}, \dots, \varepsilon_{s-10}, \varepsilon_{s-11}, \dots, \varepsilon_{s-1t-1}) \in \zeta$ we have $(\alpha\varepsilon_{s-10}, \alpha\varepsilon_{s-11}, \dots, \alpha\varepsilon_{s-1t-1}, \varepsilon_{00}, \varepsilon_{01}, \dots, \varepsilon_{0t-1}, \dots, \varepsilon_{s-20}, \varepsilon_{s-21}, \dots, \varepsilon_{s-2t-1}) \in \zeta$. Note that an α -quasi-twisted code of length n and index 1 is an α -constacyclic code.

Theorem 6.1: If $\varepsilon(\omega) = \varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_{n-1}\omega^{n-1} \in R_{n, \alpha, \theta}$ is the polynomial corresponding to the codeword $c = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in \zeta$, then ζ is a (θ, α) -cyclic code if and only if ζ is a left $R[\omega; \theta]$ -submodule of $R_{n, \alpha, \theta}$.

Proof: Let ζ be a (θ, α) -cyclic code of length n over R , $\iota(\omega) \in R[\omega; \theta]$, and $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in \zeta$ which can be represented as $\varepsilon = \varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_{n-1}\omega^{n-1}$. Then

$$\begin{aligned} \omega\varepsilon(\omega) &= \omega\varepsilon_0 + \omega\varepsilon_1\omega + \dots + \omega\varepsilon_{n-1}\omega^{n-1} \\ &= \theta(\varepsilon_0)\omega + \theta(\varepsilon_1)\omega^2 + \dots + \theta(\varepsilon_{n-2})\omega^{n-1} + \theta(\varepsilon_{n-1})\omega^n \end{aligned}$$

$$\begin{aligned}
&= \alpha\theta(\varepsilon_{n-1}) + \theta(\varepsilon_0)\omega + \theta(\varepsilon_1)\omega^2 + \dots + \theta(\varepsilon_{n-2})\omega^{n-1}, (\text{since } \omega^n = \alpha) \\
&= (\alpha\theta(\varepsilon_{n-1}), \theta(\varepsilon_0), \theta(\varepsilon_1), \dots, \theta(\varepsilon_{n-2})) = \Gamma_{\theta, \alpha}(\varepsilon) \in C.
\end{aligned}$$

Analogously, for an integer $i \geq 2$, $\omega^i \varepsilon(\omega) \in \zeta$. Since ζ is an R -submodule, $\iota(\omega)\varepsilon(\omega) \in \zeta$, and consequently ζ is a left $R[\omega; \theta]$ -submodule of $R_{n, \alpha, \theta}$. Conversely, let ζ be a left $R[\omega; \theta]$ -submodule of $R_{n, \alpha, \theta}$ and $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in \zeta$ which can be represented as $\varepsilon = \varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_{n-1}\omega^{n-1}$. Then, $\omega\varepsilon(\omega) = \Gamma_{\theta, \alpha}(\varepsilon) \in \zeta$, and hence ζ is a (θ, α) -cyclic code of length n over R .

Theorem 6.2: α -constacyclic code over R^n . If ζ is a (θ, α) -cyclic code of odd length n

Proof: Let ζ be a (θ, α) -cyclic code of odd length n over R . We know that the automorphism θ has order 2. Since $\gcd(n, 2) = 1$, there exist integers s, t such that $2s = 1 + nt$. Let $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in \zeta$ which can be represented as $\varepsilon(\omega) = \varepsilon_0 + \varepsilon_1\omega + \varepsilon_2\omega^2 + \dots + \varepsilon_{n-1}\omega^{n-1}$. As ζ is a (θ, α) -cyclic code, $\Gamma_{\theta, \alpha}(\varepsilon) = \omega\varepsilon(\omega) \in \zeta$. Then, $\omega^{2s}\varepsilon(\omega) \in \zeta$ where

$$\begin{aligned}
\omega^{2s}\varepsilon(\omega) &= \omega^{2s}\varepsilon_0 + \omega^{2s}\varepsilon_1\omega + \omega^{2s}\varepsilon_2\omega^2 + \dots + \omega^{2s}\varepsilon_{n-1}\omega^{n-1} \\
&= \theta^{2s}(\varepsilon_0)\omega^{2s} + \theta^{2s}(\varepsilon_1)\omega^{2s+1} + \dots + \theta^{2s}(\varepsilon_{n-2})\omega^{2s+n-2} + \theta^{2s}(\varepsilon_{n-1})\omega^{2s+n-1} \\
&= \alpha^t\varepsilon_0\omega + \alpha^t\varepsilon_1\omega^2 + \dots + \alpha^t\varepsilon_{n-2}\omega^{n-1} + \alpha^{t+1}\varepsilon_{n-1}, (\text{since } \theta^2 \equiv I, \text{ and} \\
&\quad \omega^{2s} = \omega^{1+nt} = \alpha^t\omega) \\
&= \alpha^t(\varepsilon_0\omega + \varepsilon_1\omega^2 + \dots + \varepsilon_{n-2}\omega^{n-1} + \alpha\varepsilon_{n-1}).
\end{aligned}$$

This implies that $\alpha^t\omega^{2s}\varepsilon(\omega) = \alpha^{2t}(\varepsilon_0\omega + \varepsilon_1\omega^2 + \dots + \varepsilon_{n-2}\omega^{n-1} + \alpha\varepsilon_{n-1}) = (\alpha\varepsilon_{n-1} + \varepsilon_0\omega + \varepsilon_1\omega^2 + \dots + \varepsilon_{n-2}\omega^{n-1}) = \Gamma_{\alpha}(\varepsilon) \in \zeta$. Therefore the result.

Theorem 6.3: Let n be an even integer. If $\zeta \subseteq R^n$ is a (θ, α) -cyclic, then it is an α -quasi-twisted code of length n and index 2 over R .

Proof: Let $\zeta \subseteq R^n$ be a (θ, α) -cyclic code, $n = 2k$ and

$$\varepsilon = (\varepsilon_{00}, \varepsilon_{01}, \dots, \varepsilon_{k-10}, \varepsilon_{k-11}) \in \zeta.$$

Then

$$\begin{aligned}
\Gamma_{\theta, \alpha}^2(\varepsilon) &= (\alpha\theta^2(\varepsilon_{k-10}), \alpha\theta^2(\varepsilon_{k-11}), \theta^2(\varepsilon_{00}), \theta^2(\varepsilon_{01}), \dots, \theta^2(\varepsilon_{k-20}), \theta^2(\varepsilon_{k-21})) \in \zeta \\
&= (\alpha\varepsilon_{k-10}, \alpha\varepsilon_{k-11}, \varepsilon_{00}, \varepsilon_{01}, \dots, \varepsilon_{k-20}, \varepsilon_{k-21}) \in \zeta \text{ (since } \theta^2 \equiv I).
\end{aligned}$$

Therefore, ζ is an α -quasi-twisted code of length n and index 2 over R .

7. Gray Images of (θ, α) -constacyclic Codes

We now define three new Gray maps and discuss the $\mathbb{Z}_4$ -images of (θ, α) -cyclic codes which are cyclic, quasi-cyclic, and permutation equivalent to quasi-cyclic codes.

For $\iota, \kappa, \varsigma \in \mathbb{Z}_4$, we define the following Gray maps.

1. $\Phi_1 : R^n \rightarrow \mathbb{Z}_4^2$ defined by

$$\phi_1(\iota + u\kappa + v\varsigma) = (a\iota + \kappa + 2\varsigma, 3\iota + 3\kappa + 2\varsigma) \quad (7)$$

2. $\Phi_2 : R \rightarrow \mathbb{Z}_4^2$ defined by

$$\phi_2(\iota + u\kappa + v\varsigma) = (2\iota + 2\kappa + 2\varsigma, 2\kappa + 2\varsigma) \quad (8)$$

3. $\Phi_3 : R \rightarrow \mathbb{Z}_4^2$ defined by

$$\phi_3(\iota + u\kappa + v\varsigma) = (2\iota + 3\kappa + 2\varsigma, \kappa + 2\varsigma) \quad (9)$$

Lemma 7.1: Consider the Gray map Φ_1 . Then, $\Phi_1 \Gamma_{\theta, \alpha} = \sigma \Phi_1$.

Proof: Let $z = \iota_i + u\kappa_i + v\varsigma_i$, $0 \leq i \leq n-1$, so then $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$ and $\alpha(\iota_i + u\kappa_i + v\varsigma_i) = \iota_i + u(2\iota_i + 3\kappa_i) + v(\varsigma_i)$. Therefore $\theta(\alpha(\iota_i + u\kappa_i + v\varsigma_i)) = \iota_i + u(2\iota_i + 3\kappa_i) + 3v(\varsigma_i)$ and hence

$$\begin{aligned} \Phi_1 \Gamma_{\theta, \alpha}(z) &= \Phi_1 \Gamma_{\theta, \alpha}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \Phi_1(\theta(\alpha \varepsilon_{n-1}), \theta(\varepsilon_0), \dots, \theta(\varepsilon_{n-2})) \\ &= (3\iota_{n-1} + 3\kappa_{n-1} + 2\varsigma_{n-1}, \iota_0 + \kappa_0 + 2\varsigma_0, \dots, \iota_{n-2} + \kappa_{n-2} + 2\varsigma_{n-2}, \iota_{n-1} \\ &\quad + \varsigma_{n-1} + 2\varsigma_{n-1}, 3\iota_0 + 3\kappa_0 + 2\varsigma_0, \dots, 3\iota_{n-2} + 3\kappa_{n-2} + 2\varsigma_{n-2}). \end{aligned}$$

On the RHS

$$\begin{aligned} \sigma \phi_1(\varepsilon) &= \sigma \phi_1(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= (3\iota_{n-1} + 3\kappa_{n-1} + 2\varsigma_{n-1}, \iota_0 + \kappa_0 + 2\varsigma_0, \dots, \iota_{n-2} + \kappa_{n-2} + 2\varsigma_{n-2}, \iota_{n-1} + \varsigma_{n-1} \\ &\quad + 2\varsigma_{n-1}, 3\iota_0 + 3\kappa_0 + 2\varsigma_0, \dots, 3\iota_{n-2} + 3\kappa_{n-2} + 2\varsigma_{n-2}). \end{aligned}$$

Hence $\phi_1 \Gamma_{\theta, \alpha} = \sigma \phi_1$.

Theorem 7.2: $\Phi_1(\zeta) \subseteq \mathbb{Z}_4^{2n}$ is a cyclic code. If $\zeta \subseteq R^n$ is a (θ, α) -cyclic code.

Proof: ζ is (θ, α) -cyclic code $\Rightarrow \Gamma_{\theta, \alpha}(\zeta) = \zeta$ and by Lemma 7.1,

$$\Phi_1 \Gamma_{\theta, \alpha}(\zeta) = \Phi_1(\zeta) = \sigma \Phi_1(\zeta)$$

Hence $\Phi_1(\zeta)$ is a cyclic code.

Lemma 7.3: Consider the Gray map Φ_2 . Then, $\nu_2 \Phi_2 = \Phi_2 \Gamma_{\theta, \alpha}$.

Proof: Let $\varepsilon = \iota_i + u\kappa_i + v\varsigma_i$ for $0 \leq i \leq n-1$. Then, $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$ and

$$\begin{aligned} \nu_2 \Phi_2(\varepsilon) &= \nu_3 \Phi_2(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \nu_2(2\iota_0 + 2\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-1} + 2\kappa_{n-1} + 2\varsigma_{n-1}, 2\kappa_0 + 2\varsigma_0, \dots, 2\kappa_{n-1} + 2\varsigma_{n-1}) \\ &= (2\iota_{n-1} + 2\kappa_{n-1} + 2\varsigma_{n-1}, 2\iota_0 + 2\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-2} + 2\kappa_{n-2} + 2\varsigma_{n-2}, \\ &\quad 2\kappa_{n-1} + 2\varsigma_{n-1}, 2\kappa_0 + 2\varsigma_0, \dots, 2\kappa_{n-2} + 2\varsigma_{n-2}). \end{aligned}$$

Further

$$\begin{aligned} \Phi_2 \Gamma_{\theta, \alpha}(\varepsilon) &= \Phi_2 \Gamma_{\theta, \alpha}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \Phi_2(\theta(\alpha \varepsilon_{n-1}), \theta(\varepsilon_0), \dots, \theta(\varepsilon_{n-2})) \\ &= (2\iota_{n-1} + 2\kappa_{n-1} + 2\varsigma_{n-1}, 2\iota_0 + 2\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-2} + 2\kappa_{n-2} + 2\varsigma_{n-2}, \\ &\quad 2\kappa_{n-1} + 2\varsigma_{n-1}, 2\kappa_0 + 2\varsigma_0, \dots, 2\kappa_{n-2} + 2\varsigma_{n-2}), \end{aligned}$$

and hence $\nu_2 \Phi_2 = \Phi_2 \Gamma_{\theta, \alpha}$.

Theorem 7.4: $\Phi_2(\zeta) \subseteq \mathbb{Z}_4^{2n}$ is a quasi-cyclic with index 2. If $\zeta \subseteq R^n$ is (θ, α) -cyclic.

Proof: ζ is (θ, α) -cyclic, $\Rightarrow \Gamma_{\theta, \alpha}(\zeta) = \zeta$ by Lemma 7.3,

$$\Phi_2 \Gamma_{\theta, \alpha}(\zeta) = \phi_2(\zeta) = \nu_2 \phi_2(\zeta).$$

Hence $\Phi_2(\zeta)$ is a quasi-cyclic over $\mathbb{Z}_4$.

Lemma 7.5: Consider the Gray map Φ_3 . Then, $\varepsilon \nu_2 \Phi_3 = \Phi_3 \Gamma_{\theta, \alpha}$.

Proof: Let $\varepsilon = \iota_i + u\kappa_i + v\varsigma_i$ for $0 \leq i \leq n-1$. Then, $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \in R^n$ and so

$$\begin{aligned} \varepsilon \nu_2 \Phi_3(\varepsilon) &= \varepsilon \nu_2 \Phi_3(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\ &= \varepsilon \nu_2(2\iota_0 + 3\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-1} + 3\kappa_{n-1} + 2\varsigma_{n-1}, \kappa_0 + 2\varsigma_0, \dots, \kappa_{n-1} + 2\varsigma_{n-1}) \\ &= \varepsilon(2\iota_{n-1} + 3\kappa_{n-1} + 2\varsigma_{n-1}, 2\iota_0 + 3\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-2} + 3\kappa_{n-2} + 2\varsigma_{n-2}, \kappa_{n-1} \end{aligned}$$

$$\begin{aligned}
& +2\varsigma_{n-1}, \kappa_0 + 2\varsigma_0, \dots, \kappa_{n-2} + 2\varsigma_{n-2}) \\
& = (\kappa_{n-1} + 2\varsigma_{n-1}, 2\iota_0 + 3\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-2} + 3\kappa_{n-2} + 2\varsigma_{n-2}, 2\iota_{n-1} + 3\kappa_{n-1} \\
& + 2\varsigma_{n-1}, \kappa_0 + 2\varsigma_0, \dots, \kappa_{n-2} + 2\varsigma_{n-2}).
\end{aligned}$$

On the RHS

$$\begin{aligned}
\Phi_3 \Gamma_{\theta, \alpha}(\varepsilon) &= \Phi_3 \Gamma_{\theta, \alpha}(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}) \\
&= \Phi_3(\theta(\Lambda \varepsilon_{n-1}), \theta(\varepsilon_0), \dots, \theta(\varepsilon_{n-2})) \\
&= (\kappa_{n-1} + 2\varsigma_{n-1}, 2\iota_0 + 3\kappa_0 + 2\varsigma_0, \dots, 2\iota_{n-2} + 3\kappa_{n-2} + 2\varsigma_{n-2}, 2\iota_{n-1} + 3\kappa_{n-1} \\
&+ 2\varsigma_{n-1}, \kappa_0 + 2\varsigma_0, \dots, \kappa_{n-2} + 2\varsigma_{n-2}).
\end{aligned}$$

$$\text{Hence, } \varepsilon \nu_2 \Phi_3 = \Phi_3 \Gamma_{\theta, \alpha}.$$

Theorem 7.6: $\Phi_3(\zeta) \subseteq \mathbb{Z}_4^{2n}$ is permutation equivalent to a quasi-cyclic code. If $\zeta \subseteq R^n$ is (θ, α) -cyclic.

Proof: ζ be a (θ, α) -cyclic, $\Gamma_{\theta, \alpha}(\zeta) = \zeta$ by Lemma 7.5,

$$\Phi_3 \Gamma_{\theta, \alpha}(\zeta) = \Phi_3(\zeta) = \varepsilon \nu_2 \Phi_3(\zeta).$$

Hence, $\Phi_3(\zeta)$ is permutation equivalent to a quasi-cyclic code.

8. Conclusion

This paper considered $(1+2u)$ -constacyclic and $(1+2u)$ -skew constacyclic codes over $R = \frac{\mathbb{Z}_4[u, v]}{\langle u^2 - u, v^2, uv \rangle}$. Some Gray maps over this ring were defined. It was shown that the Gray images of these constacyclic codes are cyclic, quasi-cyclic, and permutation equivalent to QC codes. Several $\mathbb{Z}_4$ codes were presented which are not in the online database of linear $\mathbb{Z}_4$ codes [3].

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