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The construction of fuzzy subgroups of groups with units modulo 20 and 21

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Abstract

The number of fuzzy subgroups (NFSGs) of the groups of units U_{20} and U_{21} are determining in this study. The (NFSGs) of U_{20} was found based on diagram and using the

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lemma which state that the (NFSGs) of G is equal to the number of chains (NC) on the subgroups lattice of G. Moreover, we explain that there are only two fuzzy subgroups for a prime order group. Lagrange's theorem is more important in this work, from Lagrange's theorem, there is no nontrivial subgroup of U_{21} since the order of the group is prime. We prove that, if $P_1(\mu) = U_{21}$, then we get $\mu_1(x) = \theta_1$, $\forall x \in U_{21}$. Thus we have one fuzzy subgroup of $P_1(\mu) = U_{21}$. If $P_1(\mu) = \{1\}$, then we obtain $\mu_2(x) = \theta_1$, $\forall x \in \{1\}$ and $\mu_2(x) = \theta_2, \forall x \in U_{21} \setminus \{1\}$. Thus we have 2 fuzzy subgroups of U_{21}. We came up with the numbers 21 and 2 respectively.

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1. Introduction

Zadeh [1] proposed the notions of fuzzy numbers and fuzzy arithmetic. Since then, other authors have looked into the features of fuzzy numbers and fuzzy algebra and potential applications [2]. Fuzzy sets are like permutation sets in symmetric groups [3]. Fuzzy equations are difficult to solve because operations on fuzzy integers, such as addition and multiplication, are not invertible [4]. As a result, the class of all fuzzy numbers is not a group that can be added to or multiplied. Let $n \geq 2$, an element a $a \in Z_n$ is said to be a unit if there is an element $b \in Z_n$ such that $ab = 1$. Here the product is multiplication modulo n. We denote the set of all units in Z_n by U_n [5]. Sulaiman [6] in 2011, found the (NFSGs). Next, in 2012 [7] he determined all the 30 subgroups of S_4 and drew the lattice subgroups and then he [8] established (NFSGs) of S_4 and found out to be 220. Humera and Reza in 2014 [9] determined (NFSGs) of finite abelian group. Onasanya and Ilori in 2014 [10], Explored the characteristics of an anti-fuzzy subgroup within a group, specifically concerning pseudo cosets. In this work, the (NFSGs) of the groups of units U_{20} and U_{21} is shown. We came up with the numbers 21 and 20 respectively.

2. Basic Definitions and Results

The notions of subgroup lattice and fuzzy subgroup are reviewed in this section.

Definition 2.1: [7] Let $A = \{U_\alpha\}$ be a family of all subgroups of W with relation lattice. We say A is a subgroup lattice (SGL) of W, where The lattice of subgroups (SGs) of W is a partially ordered set whose components are (SGs) of W, with set inclusion standing as the partial order relation.

Theorem 2.2: [11] *If W has order (i.e $|W| = n$ for some integer $n > 0$) and H is a subgroup of W, then $|H|$ such that $|H| \backslash |W|$*

Theorem 2.3: [12] *If W has order and $a \in W$, with a smallest positive integer n such that $a^n = 1$, where 1 is the identity of G, then $n \backslash |W|$.*

Definition 2.4: [12, 13] Assume that D has order with $D = p^t k$ where $0 < t \in Z$, p represents a prime number, and $(p,k) = 1$. The subgroup H of D with $|H| = p^t$ Normally known as Sylow p −subgroup of D.

Theorem 2.5: (First Sylow Theorem) [12] *Assume that D has order and $|D| = p^t k$ for $0 < t \in Z$, with p is a prime and $(p,k) = 1$. Then there is W subgroup of D with $|W| = p^r$, $\forall 1 \le r \le t$.*

Theorem 2.6: (Third Sylow Theorem) [12] *Assume that the number of sylow p-subgroups of D is equal d, then $d \backslash m$ and $d \equiv 1 \bmod p$. We say $N \subseteq D$ is a normal subgroup of D, if $\forall g \in D$ & $n \in N$, $gng^{-1} \in N$.*

Definitions 2.7: [8] *Assume $h: D \longrightarrow [0,1]$ is a map, where D is a group. We say h is a fuzzy subgroup (FSG) of D if $h(sr) \ge min\{h(s), h(r)\}$, $\forall s, r \in D$ and $h(s^{-1}) \ge h(s)$, $\forall s \in D$.*

Theorem 2. 8: [8] *A map $h: D \longrightarrow [0,1]$ is a (FSG), if there is a chain $b_1 \le b \le \cdots \le b_n = D$ in (SGL) of D with h can be presented by $h(s) = \begin{cases} \theta_1, & s \in b_1 \\ \theta_2, & s \in b_2 \backslash b_1 \\ \vdots & \\ \theta_n, & s \in b_n \backslash b_{n-1} \end{cases}$, where '$b_i \backslash b_{i-1}$' stands for complement of b_i in b_{i-1}.*

Definitions 2.9: [8] Assume that each one of μ and γ is (FSG) of G with $\mu(x) = \begin{cases} \theta_1, & s \in P_1 \\ \theta_2, & s \in P_2 \backslash P_1 \\ \vdots & \\ \theta_t, & s \in P_t \backslash P_{t-1} \end{cases}$, and $\gamma(x) = \begin{cases} \delta_1, & s \in T_1 \\ \delta_2, & s \in T_2 \backslash T_1 \\ \vdots & \\ \delta_k, & s \in T_k \backslash T_{k-1} \end{cases}$. So, μ and γ are said to be equivalent and symbolized by $\mu \sim \gamma$, if (a) $t = k$, and (b) $P_i = T_i, \forall i \in \{1,2,\dots,t\}$. We can then show that this relation is an equivalence. Also, if μ and γ are not equivalent, then they are different.

Example 2.10: Let $G = \mathbb{Z}_{18}$ and $\mu, \gamma, \alpha, \beta$ be four functions from $\mathbb{Z}_{18}$ to $[0,1]$ such that $\mu(u) = \begin{cases} 1, u \in \{0,6,12\} \\ \frac{1}{2}, u \in \{3,9,15\} \\ \frac{1}{3}, u \in \{1,2,4,5,7,8,10,11,14,16,17\} \end{cases}$. Then $P_1(\mu) = \{0,6,12\}$, $P_2(\mu) = \{0,3,6,9,12,15\}$, $P_3(\mu) = \mathbb{Z}_{18}$, and

$$\gamma(u)=\begin{cases}1, u\in\{0,1,2,3,4,5\}\\ \frac{1}{2}, u\in\{6,7,8,9,10,11\}\\ \frac{1}{3}, u\in\{12,13,14,15,16,17\}\end{cases}, \alpha(u)=\begin{cases}1, & u\in\{0,2,4,6,8,10,12,14,16\}\\ 1/2, & u\in\{1,3,5,7,9,11,13,15,17\}\end{cases},$$

$$\beta(u)=\begin{cases}1, & u\in\{0,2,4,6,8,10,12,14,16\}\\ 0, & u\in\{1,3,5,7,9,11,13,15,17\}\end{cases}.$$

Now since $P_1(\mu)=\{0,6,12\}, P_2(\mu)=\{0,3,6,9,12,15\}$, and $P_3(\mu)=\mathbb{Z}_{18}$ are subgroups of $\mathbb{Z}_{18}$, from [Theorem (2.8)], μ is a (FSG) of $\mathbb{Z}_{18}$. Also, α and β are fuzzy subgroups (FSGs) of $\mathbb{Z}_{18}$ but γ is not a (FSG) of $\mathbb{Z}_{18}$ since $P_1(\gamma)$ is not a subgroup of $\mathbb{Z}_{18}$. Also since $|\mu|\neq|\alpha|$ and $|\mu|\neq|\beta|$, we get μ is different with α and β. In other word, $|\alpha|=|\beta|$ and it is clearly we get $P_i(\alpha)=P_i(\beta)\ \forall i\in\{1,2\}$. Thus $\alpha\sim\beta$.

Lemma 2.11: [8] *The (NFSGs) of D is equal to the (NC) on the (SGL) of D.*

3. Main Results

Some results concerning the number of fuzzy subgroups of group of units modulo 20 and 21 are shown here.

Subgroups of U_{20}. 3.1: Take $U_{20}=\{1,3,7,9,11,13,17,19\}$ with $|U_{20}|=8$. From the Lagrange's theorem, non-trivials subgroups of U_{20} must have orders 2 or 4, see table (1). The subgroup of order 1 is $H_1=\{1\}$. Now, we determining the remain subgroups of U_{20}.

Table 1
The orders of subgroups of U_{20}

Order of subgroups	Element
1	1
2	9,11,19
4	3,7,13,17

Consider subgroups have order 2, assume K is any subgroup of U_{20} have order 2, we get K is a cyclic subgroup as 2 is a prime number. Thus we get the following possibilities: $K_2=\{1,9\}$, $K_3=\{1,11\}$, $K_4=\{1,19\}$. If M is a subgroup of U_20, then it has an order of 4. According to Theorem 2.3, the elements of M can only be of certain orders 1,2 or 4. Where M consists of components of the order 4, then $M_5=\langle 3\rangle=\langle 7\rangle=\{1,3,7,9\}$, $M_6=\langle 13\rangle=\{1,13,9,17\}$. If M has elements of order 2, then $M_7=\{1,9,11,19\}$. Therefore, we consider fig. 1. for (SGL) of U_{20}.

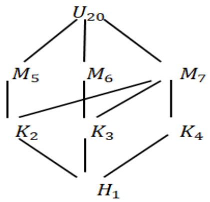

Figure 1
(SGL) of group U_{20} with non-prime order

Subgroups of $\boldsymbol{U_{21}}$. 3.2: We know that U_{21} is a group of prime order. Then from the Lagrange's theorem, we have there is no nontrivial subgroup of U_{21}. Hence the (SGL) as fig. 2.

$$\begin{array}{c} U_{21} \\ | \\ \{1\} \end{array}$$

Figure 2
(SGL) of group $\boldsymbol{U_{21}}$ with prime order

Constructing fuzzy subgroups of $\boldsymbol{U_{20}}$. 3.3: We will find the (NFSGs) of U_{20} based on the diagram and by using the lemma 2.11. Let μ be a (FSG) of U_{20}. Any subgroup of U_{20} could be selected to become $P_1(\mu)$. If $P_1(\mu) = U_{20}$, then we have: $\mu_1(x) = \theta_1, \forall x \in U_{20}$.

Thus we have one fuzzy subgroup of U_{20}. If $P_1(\mu) = M_5$, then we obtain: $\mu_2(x) = \begin{cases} \theta_1, & x \in M_5 \\ \theta_2, & x \in U_{20} \setminus M_5 \end{cases}$, $\mu_3(x) = \begin{cases} \theta_1, & x \in M_6 \\ \theta_2, & x \in U_{20} \setminus M_6 \end{cases}$, $\mu_4(x) = \begin{cases} \theta_1, & x \in M_7 \\ \theta_2, & x \in U_{20} \setminus M_7 \end{cases}$. If $P_1(\mu) = K_2$, then we get $\mu_5(x) = \begin{cases} \theta_1, & x \in K_2 \\ \theta_2, & x \in U_{20} \setminus K_2 \end{cases}$, $\mu_6(x) = \begin{cases} \theta_1, & x \in K_2 \\ \theta_2, & x \in M_5 \setminus K_2 \\ \theta_3, & x \in U_{20} \setminus M_5 \end{cases}$, $\mu_7(x) = \begin{cases} \theta_1, & x \in K_2 \\ \theta_2, & x \in M_7 \setminus K_2 \\ \theta_3, & x \in U_{20} \setminus M_7 \end{cases}$. If $P_1(\mu) = K_3$, then we have $\mu_8(x) = \begin{cases} \theta_1, & x \in K_3 \\ \theta_2, & x \in U_{20} \setminus K_3 \end{cases}$, $\mu_9(x) = \begin{cases} \theta_1, & x \in K_3 \\ \theta_2, & x \in M_6 \setminus K_3 \\ \theta_3, & x \in U_{20} \setminus M_6 \end{cases}$, $\mu_{10}(x) = \begin{cases} \theta_1, & x \in K_3 \\ \theta_2, & x \in M_7 \setminus K_3 \\ \theta_3, & x \in U_{20} \setminus M_7 \end{cases}$. If $P_1(\mu) = K_4$, then we get : $\mu_{11}(x) = \begin{cases} \theta_1, & x \in K_4 \\ \theta_2, & x \in U_{20} \setminus K_4 \end{cases}$, $\mu_{12}(x) = \begin{cases} \theta_1, & x \in K_3 \\ \theta_2, & x \in M_7 \setminus K_4 \\ \theta_3, & x \in U_{20} \setminus M_7 \end{cases}$. If $P_1(\mu) = \{1\}$, then we consider that $\mu_{13}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in U_{20} \setminus \{1\} \end{cases}$, $\mu_{14}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in M_5 \setminus \{1\} \\ \theta_3, & x \in U_{20} \setminus M_5 \end{cases}$, $\mu_{15}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in M_6 \setminus \{1\} \\ \theta_3, & x \in U_{20} \setminus M_6 \end{cases}$, $\mu_{16}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in M_7 \setminus \{1\} \\ \theta_3, & x \in U_{20} \setminus M_7 \end{cases}$, $\mu_{17}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in K_2 \setminus \{1\} \\ \theta_3, & x \in M_5 \setminus K_2 \\ \theta_4, & x \in U_{20} \setminus M_5 \end{cases}$, $\mu_{18}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in K_2 \setminus \{1\} \\ \theta_3, & x \in M_7 \setminus K_2 \\ \theta_4, & x \in U_{20} \setminus M_7 \end{cases}$, $\mu_{19}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in K_3 \setminus \{1\} \\ \theta_3, & x \in M_6 \setminus K_3 \\ \theta_4, & x \in U_{20} \setminus M_6 \end{cases}$, $\mu_{20}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in K_3 \setminus \{1\} \\ \theta_3, & x \in M_7 \setminus K_3 \\ \theta_4, & x \in U_{20} \setminus M_7 \end{cases}$, $\mu_{21}(x) = \begin{cases} \theta_1, & x \in \{1\} \\ \theta_2, & x \in K_4 \setminus \{1\} \\ \theta_3, & x \in M_7 \setminus K_4 \\ \theta_4, & x \in U_{20} \setminus M_7 \end{cases}$. Thus we have 21 fuzzy subgroup of U_{20}.

Constructing fuzzy subgroup U_{21}. 3.4: Let $P_1(\mu) = U_{21}$, then we consider that $\mu_1(x) = \theta_1, \forall x \in U_{21}$. Thus we get one (FSG) of $P_1(\mu) = U_{21}$. Also, if $P_1(\mu) = \{1\}$, then we obtain: $\mu_2(x) = \begin{cases} \theta_1, & \forall x \in \{1\} \\ \theta_2, & \forall x \in U_{21} \backslash \{1\} \end{cases}$. Thus we have 2 fuzzy subgroup of U_{21}.

Proposition: 3.5: There are only two fuzzy subgroups for a group of prime order.

Proof: Any group of prime order has no nontrivial subgroup by the Lagrange theorem, so there are only the trivial groups and so two fuzzy subgroups.

4. Conclusion

The number of (SGL) of a group D that is equal to the (NC) in the (SGL) aids in counting (NFSGs) for U_{20} and U_{21} by first identifying their respective subgroups. We discovered the numbers to be 21 and 2 respectively. In future study, we will use several types of fuzzy subgroups.

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