

Supplement t-extending modules

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Abstract

It is known that the topic of extending module has been addressed by many researchers. In this work, we will focus on studying and presenting a new class of modules namely, supplement t-extending which is weaker than t-extending and supplement extending modules, respectively. Throughout this study, many results, examples, properties and

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characterizations about this concept are given. In addition, many relationships with other related concepts are discussed.

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1. Introduction

All modules in this work will be unitary left R -modules, where the ring R is associative with unity.

A submodule N of W is named essential (briefly $N \leq^e W$), if $\forall C \leq W$ with $N \cap C = 0$ then must be $C=0$ [5]. While, $B \leq W$ is named closed (briefly $B \leq^c W$) if, when $B \leq^e N \not\leq W$ then $B=N$ [5]. W is named extending (or CS-module) If, $\forall U \leq W$, then $U \leq^e D \leq \oplus W$, this equivalent to say that W is CS-module if and only if, $\forall B \leq^c W$ implies $B \leq \oplus W$ see [4]. The sub module $C \leq W$ is named t -essential (briefly $C \leq^{tes} W$) if $\forall D \leq W$ with $C \cap D \leq Z_2(W)$ then $D \leq Z_2(W)$ where $Z_2(W)$ is the Goldie torsion submodule of W (a module W is said to be Z_2 -torsion if $Z_2(W)=W$). Every essential is t -essential submodule of W [1]. A non-zero R -module W is named uniform if each $(0) \neq C \leq W$, then $C \leq^e W$ [4]. While, A non-zero R -module W is named t -uniform if $\forall (0) \neq C \leq W$, then $C \leq^{tes} W$, clearly every uniform is t -uniform [6]. The sub-module $C \leq W$ is named t -closed (briefly $C \leq^{tc} W$) if, when $C \leq^{tes} N \not\leq W$ then $C=N$, so clear all t -closed is closed but under non-singularity of W they are equivalents [1]. W is called t -extending (shortly t -CS-module), if all $A \leq W$ implies $A \leq^{tes} D \leq \oplus W$, it is clear that each CS-module is t -CS-module [1]. $H \leq W$ is small (briefly $H \leq^s W$) if, each $D \leq W$, when $H + D = W$ must be $D = W$ [7]. Let $S \leq W$ and $V \leq W$ the n S is supplement of V in W (briefly $S \leq^{supp} W$ iff $S + V = W$ and $S \cap V \leq^s S$) [7]. The module W is named supplement - CS-module (shortly, S -CS-module), if every $U \leq W$ then $U \leq^e D \leq^{supp} W$, see [8]. According [2], we have, W is t -semi simple if, every $C \leq W$ there exists $D \leq \oplus W$ such that $D \leq^{tes} C$ also, every semi-simple is t -semi simple and each t -semi simple is t -CS-module. W is named lifting if each $H \leq W$ then there exists $D \leq W$ such that $W = D \oplus C$ and $H \cap C \leq^s C$. There are many researchers who have worked on the subject of the class of CS-module and injectivity and their generalizations, as some of them presented a class of concepts that are stronger than the class of extending modules while others prevented concepts that are weaker than extending modules, see [1], [8]-[11], [13]- [15].

2. Preliminaries

Proposition 2.1[1]: The statements that follow are equivalent for $D \leq W$.

1. $D \leq^{tes} W$

2. $(D+Z_2(W))/Z_2(W) \leq^e W/Z_2(W)$
3. $D+Z_2(W) \leq^e W$
4. W/D is Z_2 -torsion.

Proposition 2.2 [1]: For $H \leq W$. H is t -closed in W if and only if W/H is non-singular.

Corollary 2.3 [1]: Let W be a module.

1. $Z_2(W) \leq^{tc} W$
2. If ϕ is an endomorphism of W , $C \leq^{tc} W$, then $\phi^{-1}(C) \leq^{tc} W$.

Corollary 2.4 [1]: Let $D \leq W$. If $D \leq^{tc} A$ and $A \leq^{tc} W$ then $D \leq^{tc} W$.

Proposition 2.5 [10] The lifting module W is considered to be a lifting module if and only if it is an adequately supplemented module, and each supplement sub module in W is summated and.

3. Supplement t-extending Modules

Definition 3.1: An R -module W is named a supplement t -extending module (shortly, S - t -CS-module) if, for each $C \leq W$ then $C \leq^{tes} V$, where V is supplement in W . Equivalent: The module W is S - t -CS-module iff all t -closed is supplement in W .

Remarks and Examples 3.2:

1. Each S -CS-module is S - t -CS-module, but the converse is not hold. $W = Q \oplus \mathbb{Z}_2$ as $\mathbb{Z}$ -module.
2. All CS-module is S - t -CS-module, but the opposite is uncorrected. For example $W = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$ -module is S - t -CS-module since it's S -CS-module but not CS-module by [8].
3. Each t -semi-simple (and so simple) is S - t -CS-module, but the opposite is't hold. Where, $\mathbb{Z}_2 \oplus \mathbb{Z}$ as $\mathbb{Z}$ -module is S - t -CS-module since it is t -CS-module but not t -semi simple.
4. If W is Z_2 -torsion then W is S - t -CS-module, while the opposite is not hold. Where $\mathbb{Z}_2 \oplus \mathbb{Z}$ as $\mathbb{Z}$ -module it is S - t -CS-module but not Z_2 -torsion since it is not t -semi simple by [6]. Also every singular is S - t -CS-module, while the converse is not hold, where $W = \mathbb{Z}_4$ as $\mathbb{Z}_4$ -module is S - t -CS-module but W is not singular. In particular, $\mathbb{Z}_n$ as $\mathbb{Z}$ -module is S - t -CS-module $(\forall n \in \mathbb{Z})_+$.
5. If the $\mathbb{Z}$ -module W is finitely generated then W is S - t -CS-module, (in fact by [1], W is t -CS-module).
6. All t -uniform (and hence, uniform) is S - t -CS-module. Since W is t -uniform $(0) \leq^{tes} W$. Hence $W/(0) Z_2$ -torsion (proposition (2.1(4))), that is W is Z_2 -torsion then by (4) W S - t -CS-module.

7. If W is t -CS-module, then $W/Z_2(W)$ is S-CS-module, the converse isn't true. $W = \prod_p Z_p$ as $\mathbb{Z}$ -module and $Z_2(W) = \bigoplus_p Z_p$, so $W/Z_2(W)$ is S-t-CS-module. since it is CS-module but not t -CS-module see[1,Example 2.16].
8. If all $D \leq^{\text{supp}} W$, then W is semi simple, and hence by (3), W is S-t-CS-module[14] as shown in Figure 1.

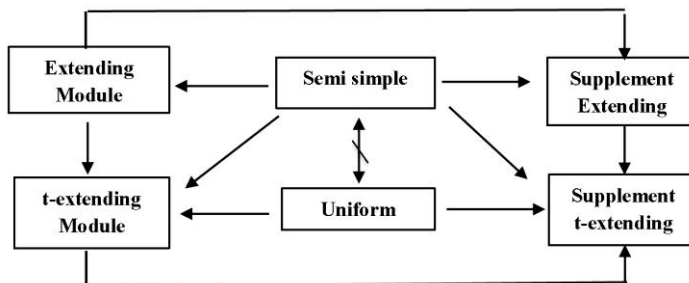


Figure 1

A diagram showing the relations between CS-module and some of its generalizations and other concepts.

Proposition 3.3: If W is t -CS-module then W is S-t-CS-module.

Proof: Assume that $N \leq^{tc} W$, and W is t -CS-module then $N \leq^{\oplus} W$ but (by[7], we have every direct summand is supplement), so N is supplement in W . Hence W is S-t-CS-module.

If W is lifting module then they are equivalent.

Remark 3.4: These two concepts (S-CS-module and t -CS-modules, are different). For example,

1. $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$ -module is S-CS-module but not t -CS-module.
2. $\mathbb{Q} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$ -module is t -CS-module but isn't S-CS-module.

Proposition 3.5: If W is S-t-CS-module, then for every $D \leq W$, $D_2 \leq^{\text{supp}} W$, where $D_2 / D = Z_2(W/D)$.

Proof: The factor module W/D_2 is nonsingular (since isomorphic to $(W/D) / Z_2(W/D)$)

Since W/D_2 is nonsingular, then D_2 is t -closed (by proposition(2.2)) and W is S-t-CS-module. So, $D_2 \leq^{\text{supp}} W$.

Proposition 3.6: Let W be a module, if for every $D \leq^{tc} W$, There's a decomposition $W/D = N/D \oplus B/D$ such that N is supplement in W and $B \leq^{tes} W$ iff W is S-t-CS-module.

Proof: $\Rightarrow$ Let $D \leq {}^{tc} W$, so there exists decomposition $W/D = N/D \oplus B/D$ such that N is supplement in W and $B \leq {}^{tes} W$.

Therefore, by proposition 1(2.1(4)) $D \leq {}^{tes} N$. Since $D \leq {}^{tc} W$. So, $D = N$ (supplement). Hence, W S-t-CS-module

$\Leftarrow$ Let $D \leq W$, and there exists decomposition $W = N \oplus L$ such that $D \leq {}^{tes} N$.

Then $W/D = N/D \oplus (L+D)/D$. Since $W/(L+D) \cong [W/D] / [(L+D)/D] \cong N/D$ and N/D is Z_2 -torsion (since $D \leq {}^{tes} N$), so $L+D \leq {}^{tes} D$.

In [12], W is named generalized-extending if for all $D \leq W$, there exists $N \leq_\oplus W$ such that N/D is a singular.

Proposition 3.7: Every generalized extending module is a S-t-CS-module.

Proof: Let W is a generalized extending, and $C \leq {}^{tes} W$ then, there exists $N \leq_\oplus W$ such that N/D is singular. So, by [5], $C \leq {}^{ess} N$ then by [1], $C \leq {}^{tes} N$ but, $N \leq_\oplus W$. Hence N is supplement in W . Therefore, W is S-t-CS-module.

Lemma 3.8 [3]: If $C \leq S \leq W$ such that $C \leq {}^{supp} W$, then $C \leq {}^{supp} S$.

Proposition 3.9: Let W is S-t-CS-module. Then all t-closed in W is S-t-CS-module.

Proof: Assume that $C \leq {}^{tc} W$ and $N \leq {}^{tc} C$, then by corollary (2.4), $N \leq {}^{tc} W$ but W is S-t-CS-module, so $N \leq {}^{supp} W$ and by lemma(3.7) $N \leq {}^{supp} C$. Hence, C is S-t-CS-module.

We can now proceed to introduce the next theorem.

Theorem 3.10: Let W is nonsingular lifting module, the following statements are equivalents.

- 1) W is C S-module
- 2) W is S-C S-module.
- 3) W is S-t-CS-module.
- 4) W is t-CS-module.

Proof: $1 \Rightarrow 2$ by [8], $2 \Rightarrow 3$ by [Remark (3.2(1))],
 $3 \Rightarrow 4$ by [Proposition (3.3)] and $4 \Rightarrow 1$ by [1]

Proposition 3.11: Let W be S-t-CS-module., then, the endomorphism image of W is S-t-CS-module.

Proof: Let $f: W \rightarrow W$ be an endomorphism map and let $C \leq {}^{tc} N = f(W)$. Then by corollary (2.3)

$$D = f^{-1}(C) \leq {}^{tc} W.$$

But W supp-t-ext , then $D \leq^{\text{supp}} W$. So, there exists $S \leq W$ such that $D + S = W$ and $D \cap S \leq^s D$.

Now, we have $N = f(W) = f(D+S) = f(D) + f(S) = C + f(S)$. Also, by [7], $f(D \cap S) = f(D) \cap f(S)$. Then by [7], $f(D \cap S) \leq^s f(D) \Rightarrow C \cap f(S) \leq^s C$. Hence, N is S - t -CS-module.

Let W be a R -module. A submodule N of W is called relative complement of C in W if $N \cap C = 0$ and N is a maximal (w.r.t) this property, see [7].

Proposition 3.12: If N is relative-complement for some submodule C of W , then N is t -closed sub module of W .

Proof: Suppose that $N \leq^{\text{tes}} F \leq W$, try to prove $N = F$

Since $(F \cap C) \cap N = C \cap N = 0$, then $F \cap C = 0$ and N is a maximal, implies that $F = N$

Hence N is t -closed sub module of W .

Theorem 3.13: Define W as an R -module. and $N \leq W$. If $C \leq W$ is a relative-complement of N , then $N \oplus C \leq^{\text{tes}} W$.

Proof: Let $D \leq W$ such that $D \cap (N \oplus C) \leq Z_2(W)$. try to prove $D \leq Z_2(W)$

Assume $D \not\leq Z_2(W)$. Now, since $D \cap (N \oplus C) \leq Z_2(W)$ then clearly $N \cap (D \oplus C) \leq Z_2(W)$

But C is relative complement of N in W !

Then $D \leq Z_2(W)$, thus $N \oplus C \leq^{\text{tes}} W$.

Proposition 3.14: Let W is S - t -CS-module and $C \leq^{\text{tc}} W$ such that $N \leq C$ then W/N is S - t -CS-module.

Proof: Let $C/N \leq^{\text{tc}} W/N$. to prove that, $C/N \leq^{\text{supp}} W/N$

Since $C \leq^{\text{tc}} W$ and it is S - t -CS-module, then $C \leq^{\text{supp}} W$.

Thus, there exists $S \leq W$, such that $W = C + S$ and $C \cap S \leq^s C$

But $N \leq C \Rightarrow W/N = C/N + (S+N)/N$. Now, we should prove that, $C/N \cap (S+N)/N \leq^s C/N$

Then, by (modular law)[7], $(C \cap (S+N))/N \leq^s C/N$ that is $(N + (S \cap C))/N \leq^s C/N$

Now, let $U/N = (N + (S \cap C))/N = (U + N + (S \cap C))/N = C/N$. Then $U + N + (S \cap C) = C$ but $C \cap S \leq^s C$

So, $U + N = C$, but $N \leq U$. Therefore $U = C$ and $U/N = C/N$. So, $(N + (S \cap C))/N \leq^s C/N$

Hence, $C/N \leq^{\text{supp}} W/N$. So, it is S - t -CS-module

W is named a supplement-simple R -module if the only supplement submodule in W are W and 0 , [8].

Proposition 3.15: W is supplement simple, if W is S-t-CS-module, then W is t-uniform.

Proof: Let $0 \neq N \leq W$ and C be a relative-complement of N in W , by theorem (3.13), $N \oplus C \leq^{\text{tes}} W$. But by proposition (3.12), C is t-closed in W and W is S-t-CS-module then, $C \leq^{\text{supp}} W$. And W is a supplement simple, $C \neq W$, then, $C = 0$. Hence, $N \leq^{\text{tes}} W$, implies W is t-uniform.

Proposition 3.16: If each sub-module in W which contain $Z_2(W)$ is essential in supplement of W then, each submodule of W is S-t-CS-module.

Proof: Suppose $C \leq W$. there exists $N \leq^{\text{supp}} W$ such that $C + Z_2(W) \leq^e N$
Hence by (proposition(2.1)), $C \leq^{\text{tes}} N$. So, W is S-t-CS-module.

4. Discussion and Conclusion

Through this work, it was concluded that all t-extending modules are supplement t-extending modules, and the opposite is generally only achieved under a certain condition, which is the lifting module condition. It was also concluded that there is no relationship between the concept of supplement extending and t-extending, and examples of that were given. We have reached many relationships between our concept and many other concepts, such as generalized extending, finitely generated, and other concepts. A comprehensive theorem was also given to verify the equivalence of the following concepts extending, supplement extending, supplement t-extending, and t-extending under the conditions of nonsingular and lifting module.

References

- [1] S h. Asgari and A. Haghany, t-extending modules and t-baer modules. Com. Al gebra Vol. 39: pp.1605 – 1623, (2011).
- [2] S h. Asgari, A. Haghany and Y.Tolooei, t-semi simple modules and t-semi simple rings. Comm. Algebra Vol. 41: pp.1882 – 1902, (2013).
- [3] C. Nebiyev and N.Sokmez, Modules which Lie a supplement module, *Int. J. of Computational Cognition*, Vol. 8, No.2, (2010).
- [4] N. V. Dung, D.V. Huynh, P. F. Smith and R. Wisbauer, Extending Modules. Pitman Res. Notes in Math. 313. Harlow: Long man, (1994).
- [5] K. R. Goodearl Ring Theory, Non Singular Rings and Module, Mar. Dekker, Inc. New York and Basel, (1976).

- [6] Hadi. M.A.I and Shyaa, D. F, Strongly t-semisimple modules and Strongly t-semisimple Rings. *International Journal of Pure and Applied Mathematics*, 15(1): 27-41, (2017).
- [7] F. Kasch, Modules and Rings, Acad. Press, London, (1982).
- [8] M. M. Tawfeeq, Supplement Extending-Modules, thesis, Collage of Science, University of Baghdad, (2015).
- [9] Marrwa Abdallah Salih, Taghreed Hur Majeed and Mahdi Saleh Nayef, On duality of injective topological module, *Journal of Interdisciplinary Mathematics*, Vol. 25, pp:1409-1414, (2022).
- [10] S. H. Mohamed and B. J. Muller, Continuous and Discrete Modules, London Math. Soc. Lecture Note Series 14, Cambridge Univ. press, (1990).
- [11] Sepril Top, Totally Weak Supplemented Modules, thesis, Izmer Institute of Technology, (2007).
- [12] Zeng, Q. Y, On generalized extending modules. J. Zhejiang Univ. Sci. A 8:939–945, (2007).
- [13] Zahraa M. Abd Al-Majeed and Mahdi Saleh Nayef, On Supplement Rationally-Extending Modules (ICAPAST (2021), published Online in AIP Conf. Proc.2398, pp. 060073-1-060073-8 (2022); <https://doi.org/10.1063/5.0093715>.
- [14] Zahraa M. Abd Al-Majeed and Mahdi Saleh Nayef, Tr-Rationally closed submodules and RCLS-modules, *IOP Journal of Physics*, (2021). doi:10.1088/1742-6596/1963/1/012024.
- [15] S.A. Abdul-Kareem, Small quasi principally injective S-acts, *Journal of Interdisciplinary Mathematics*, 25(2), 259-272 (2022). <https://doi.org/10.1080/09720502.2021.1881221>.

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