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Study on-purely small sub-modules

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Abstract

D is a unitary left and *R* is an associative ring with identity. Module *R*. We call a sub-module *C* of an *R*-module *D* called purely-small (for short Pu-small), denoted by ($C<<_{Pu}$ D) if $C + B \neq D$ For every proper pure sub-module B of D. When D is finitely generated multiplication and faithful, we study A purely-small in D in this paper. Conditions when[A:D] purely-small in R are given. This paper determines to investigate some new results on the notion of purely small sub-modules introduced by Ahmad, Dhari, and Atiya various properties of this kind of sub-modules considered.

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1. Introduction

In this paper, D is a unitary left R-module, R is a commutative ring with identity. It is a sub-module C of D. It is called a pure if $ID \cap C = IC$ for every ideal τ of R [1, P.31]. Nauom, A. and Al-Bahranny B. introduced the concept of Pure Sum Property (PSP), the sum of any of two pure sub-modules is again pure, [2]. The proper sub-module C of D is called small ($C \ll D$), if for any sub-module B of D such that $C + H = D$ implies that B = D, [3]. Define a generalisation of tiny sub-m in [4]. In [4], a generalisation of small sub-modules,

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which are referred to as pure small (Pr-small), is defined as follows: a sub-module C of an R-module D is referred to as a Pr-small sub-module of D and is defined as $\mathrm{C} \ll_{\mathrm{Pr}} \mathrm{D}$ if $C + \mathrm{H} = D$ implies that B is a pure sub-module of D. Ahmed, Dhari, and Atiya introduced the concept of purely small, which is a generalisation of a small sub-module. A sub-module C of D is referred to as purely small ($\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$), if $\mathrm{C} + \mathrm{V} = \mathrm{D}$ for every proper pure sub-module V of D. In the same way, if $\mathrm{C} + \mathrm{V} \neq \mathrm{D}$ for every proper pure ideal ι of R, where V is a pure submodule of D, then ($\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$) and V=D. An ideal τ of R is referred to as purely small if $\tau + \iota = R$, [5].odules, which are referred to as pure small (Pr-small) a sub-module C of an R-module D is referred to as a Pr-small sub-module of D if $C + \mathrm{H} = D$ implies that B is a pure sub-module of D. This can be represented as ($\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$). Ahmed, Dhari, and Atiya introduced the concept of purely small, which is a generalisation of a small sub-module. A sub-module C of D is referred to as purely small ($\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$)if C+V=D for every proper pure sub-module V of D. In the same way, if $\mathrm{C} + \mathrm{V} \neq \mathrm{D}$ for every proper pure ideal ι of R, where V is a pure submodule of D, then ($\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$)and $V = D$. An ideal τ of R is referred to as purely small if $\tau + \iota = R$, [5]. Majid, M, and David in [6] and Elbast, Z and Smith in [7] present the concept of a multiplication module, where an R - module D is a multiplication module if and only if $C = [C\colon D]D$, for every sub-module C of D. Purely small sub-modules led Ahmed, Dhari, and Atiya to introduce purely hollow modules. In this paper we rewrite the meaning of purely small sub-modules and purely hollow modules which are given in [5,8,9] we give some new results for this concept and give many other properties of Pu-small sub-module and purely hollow modules also study the behavior of Pu-small sub-modules in a certain class of modules.

2. Purely Small Sub-modules

Now in this section, we present the Pu-small sub-module as a concept and we illustrate it with examples. We also give some simple properties of this class of sub-modules. This section commences with a definition:

Definition 2.1: An R-module C submodule is referred to as purely-small (for short Pu-small), denoted by ($\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$) if $\mathrm{C} + \mathrm{B} \neq \mathrm{D}$. For each proper pure submodule B of D.Equally:- $\mathrm{C} \ll_{\mathrm{Pu}} \mathrm{D}$ is a pure submodule of D in case and only if $C + \mathrm{H} = D$, which implies that $\mathrm{H} = D$. For each proper pure ideal ι of a ring R. If $\tau + \iota \neq \mathrm{R}$, the ideal I is referred to as Pu-small [5].

Proposition 2.3: Let ω be an R -module and $\beta, \gamma \leq \mathrm{D}$ such that $\beta \leq \gamma \leq \omega$, if $\gamma \ll_{\mathrm{Pu}} \mathrm{M}$, then $\beta \ll_{\mathrm{Pu}} \omega$

Proof: Let κ be a pure sub-module of an R -module ω such that $\beta + \kappa = \omega$. Since $\beta \leq \gamma$ implies $\gamma + \kappa = \omega$, But $\gamma \ll_{\mathrm{Pu}} \omega$, then $\kappa = \omega$,hence $\beta \ll_{\mathrm{Pu}} \omega$.

Corollary 2.4: *If β, γ are sub-modules of an R -module ω . If $\beta + \gamma \ll_{Pu} \omega$, then $\beta \ll_{Pu} \omega$ and $\gamma \ll_{Pu} \omega$.*

Proof: Suppose $\beta + \gamma \ll_{Pu} \omega$, since $\beta \leq \beta + \gamma \leq \omega$, then by Proposition (2.3) $\beta \ll_{Pu} \omega$. Similarly, We can prove $\gamma \ll_{Pu} \omega$.

Corollary 2.5: *A_i are Pu-small in ω , ∀ i = 1, 2, ... , n if $\sum_{i=1}^{n} A_i$ is Pu-small in D.*

Proof: Suppose $\sum_{i=1}^{n} A_i \ll_{Pu} \omega$, since $A_i \leq \sum_{i=1}^{n} A_i \leq \omega$, then by Proposition (2.3) $A_i \ll_{Pu} \omega$.

The following two propositions give more properties of Pu-small sub-module. Recall that If $\beta \leq \gamma$ and γ is a direct summand of ω with $\ll \omega$, then $\beta \ll \gamma$ .Recall that If the sum of any of two pure sub-modules is again pure, it is said an R -module has the pure sum property (PSP), [2].

Proposition 2.6: Let ω be an R –module, with PSP property and $\kappa \leq \beta \leq \omega$ such that $\kappa \ll_{Pu} \omega$ also A direct summand of D , then $C \ll_{Pu}$ A.

Proof: Let E be the pure sub-module of A such that $C + E = A$, since A is a direct summand of ω, then $\omega = A \oplus F$, for some $F \leq M$, hence $\omega = (C + E) \oplus F = C + (E \oplus T)$. Since $A \leq_{\oplus} \omega$, then A is a pure sub-module of ω. [1, P.31]. Now $E \leq_p A \leq_p \omega$ implies $E \leq_p D\omega$, also $F \leq_{\oplus} \omega$, then $F \leq_P D$. Since ω has the PSP property, then $E + T$ is pure in ω. But $C \ll_{Pu} \omega$, hence $E + T = \omega$.Then $A + T = E + T$, so that E =A ,which implies that $C \ll_{Pu}$ A. Now, we give the following remark [12-14].

Proposition 2.8: With a finitely produced multiplication faithful R-module D, suppose that $\beta \leq \omega$ Then $A \ll_{Pu} \omega$ only in the event $[\beta: \omega] \ll_{Pu} R$.

Proof: Assume that $\beta \ll_{Pu} \omega$ .To show that $[\beta: \omega] \ll_{Pu} R$. Assume Γ be a pure ideal of R, such that $[\beta: D] + \Gamma = R$. Then $[\beta: \omega]\omega + I\omega = R\omega = \omega$, hence $A + I\omega = \omega$. Since ω is a finitely generated multiplication module and IΓ R, then $IM \leq_P \omega$ by [6, Proposition (1.4)]. But $A \ll_{Pu} \omega$, then $\Gamma\omega = \omega$, so that $\Gamma = R$ [7]. Thus $[A: \omega] \ll_{Pu} R$. For the converse, let $A + B = \omega$, for some pure sub-module B of ω. Since ω is multiplication then $A = [A: \omega]\, \omega$, $B = [B: \omega]\, \omega$. Thus $[\beta: \omega]\omega + \Gamma\omega = R\omega = \omega$. But ω is a finitely generated multiplication faithful R-module, then $[A: \omega] + [B: \omega] = R, [B: \omega]$ is pure in R [6]. Since $[A: \omega] \ll_{Pu} R$ and $[B: \omega]$ are pure in R, hence $[B: \omega] = R$. Thus $[B: \omega]\omega = R\omega$, implies $B = \omega$, Therefore $A \ll_{Pu} \omega$.

3. Purely Hollow Modules

In this section, we continue the work of [5] and we give some properties of this class work of modules. Recall that an R- module D is called hollow if every proper sub-module of D is small, [8].

Definition 3.1: [5] An R-module $D \neq 0$ defined purely hollow for Brief Pu-hollow, if every proper sub-module of D is Pu-small. If all of R's ideals are Pu-hollow, then a ring R is said to be Pu-hollow.

1- clearly every hollow module is Pu-hollow, in general. But the converse is not true. Consider Q isn't a hollow Z -module. But it is Pu-hollow since (0), just the only proper pure sub-module of Q, then every proper sub-module of Q, is a Pu-small sub-module.

2- In Z_6 $(\bar{2})$,$(\bar{3})$ are not Pu-small, then Z_6 is not Pu-hollow as Z –module.

3- Every simple module is Pu-hollow. Every pure simple module is Pu-hollow.

Now, we can introduce the following proposition:

Proposition 3.3: Assume that v_1 and v_2 be R-modules such that v_1 is isomorphic to v_2. Then v_1 is Pu - hollow if and only if D_2 is Pu -hollow.

Proof: Assume that v_1 is Pu-hollow R-module , since v_1 is isomorphic to v_2, then there exists f: $v_1 \rightarrow v_2$ be isomorphism, to prove v_2 is Pu-hollow. Let A be a proper sub-module of v_2 such that $A + B = v_2$, where B is a pure sub-module of v_2 , since f is an isomorphism, then by [9]. At that time $f^{-1}(B)$ is pure in v_1 . Now $f^{-1}(A) + f^{-1}(B) = f^{-1}(v_2) = v_1$.But D_1 is Pu-hollow, so $f^{-1}(A)$ is Pu small sub-module of M_1, implies $f^{-1}(B) = n_1$, later $f(f^{-1}(B)) = f(v_1) = v_2$, since f is isomorphism $f\left(f^{-1}(B)\right) = B$.Thus $B = v_2$, therefore v_2 is Pu -hollow R-module.

Remark 3.4: If D is a Pu -hollow R -module and $C \leq D$, then $\frac{D}{C}$ not necessarily Pu –hollow, as the following example shows.

We will give a necessary condition under which the Remark (3.4) is true.

Proposition 3.5: Let$D \neq 0$ be a finitely generated faithful multiplication R-module. Then the R –module D is Pu-hollow in case and only if R is Pu-hollow.

Proof: Follows using Proposition (2.8)

Remark 3.6: Let D be an R -module and $A \leq D$. If $\frac{D}{A}$ is a Pu -hollow R–module, then D not necessarily to be Pu –hollow. Now for example: -

The following remark shows that not necessarily a purely hollow module has a direct sum of purely hollow modules.

Remark 3.7: Two direct sums of Pu -hollow R -module is not necessary Pu -hollow as the following example shows:

Example 3.8: Each of Z_2 and Z_3 as Z-module are Pu -hollow, but $Z_2 \oplus Z_3 \simeq Z_6$ is not Pu -hollow as Z -module (See Remark and Example (3.2), 2).

Now, we give the condition below which is a direct sum of a purely hollow module, also purely hollow.

Proposition 3.9: Let D and C be a Pu-hollow R- module with $\mathrm{ann}_R\, D + \mathrm{ann}_R\, C = R$, then $D \oplus C$ is Pu -hollow.

Proof: Let B be a proper sub-module of $D \oplus C$ Since $\mathrm{ann}_R D + \mathrm{ann}_R C = R$, then $B = B_1 \oplus B_2$, with $B_1 \leq D$, $B_2 \leq C$, [10].

To show B is Pu -small sub-module of $D \oplus C$. Let L be a pure sub-module of $D \oplus C$ such that $B + L = D \oplus C$. Since $\mathrm{ann} D + \mathrm{ann} C = R$, then [11]. $L = L_1 \oplus L_2$, where $L_1 \leq D$ and $L_2 \leq C$. But L_1, direct summand of L, also L_2 direct summand of L, Then by [1, p31] L_1 and L_2 are pure in L, but L is pure in $D \oplus C$ by hypothesis, then L_1 is pure is D and L_2 is pure in C. On the other hand $B + L = D \oplus C$. $(B_1 \oplus B_2) + (L_1 \oplus L_2) = D \oplus C$. Then $(B_1 + L_1) \oplus (B_2 + L_2) = D \oplus C$. Then $(B_1 + L_1) = D$ and $B_2 + L_2 = C$. Since D and C are Pu -hollow, then B_1 and B_2 are Pu -small in D and C respectively, implies $L_1 = D$ and $L_2 = C$ i.e $L_1 \oplus L_2 = D \oplus C$, Which implies that $D \oplus C$ is Pu -hollow.

Proposition 3.11: Let D be an R- module $D \neq 0$ with (PSP) Property. D is Pu - hollow module in case and only if there exists a pure sub- module A of D such that $\beta \ll_{Pu} D$, and $\frac{D}{\beta}$ is Pu-hollow.

Proof: $\Longrightarrow$)It is followed directly by taking $A = 0$.

($\Longleftarrow$ To prove D is Pu-hollow . Let B be a proper sub-module of D and L be a pure sub-module of D such that $L + B = D$ then $\frac{D}{\beta} = \frac{L+\beta}{\beta} + \frac{B+\beta}{\beta}$ implies that $D \neq L + \beta$ since $\beta \ll_{Pu} D$. If $D = L + \beta$ with β is Pu-small of D, so $D = L$ which is a contradiction. Thus $\frac{D}{\beta} \neq \frac{L+\beta}{\beta}$ Now, $L \leq_p D$ and M have the (PSP) Property, then $L + \beta \leq_P D$, [2] hence by [1, P.31] $\frac{L+\beta}{\beta} \leq_P \frac{D}{\beta}$, since $\frac{D}{\beta}$ is Pu-hollow, then $\frac{L+\beta}{\beta} = \frac{D}{\beta}$ and $D = L + \beta$, but $\beta s \ll_{Pu} D$, then $L = D$. Thus D is Pu-hollow.

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