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Strongly extending semimodules over a semiring

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Abstract

In this work, the concept of strongly extending modules has been introduced for semimodules category. We call a semimodule C is strongly extending if for each subsemimodule A in C there is a stable direct summand B of C such that A is essential in B. Various characterizations of strongly extending semimodules are obtained. It is discussed direct sum property for strongly extending semimodules and gives more than results which this property is valid. Also, we provide many results concerned with the hereditary property for extending semimodules. A useful sufficient condition is given for the equivalence between extending and strongly extending semimodules. Further, we get a partial relation between extending semimodules and subtractive semimodules.

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1. Introduction

The study of the concept of the semiring is due to Dedekind in his paper [1] and then later many authors studied and developed this concept extensively. A semiring is an algebraic system S = (S,+,.) where (S,+) and (S,.) semigroups are commutative, such that s(t + r)=st + sr for all s,t,r ∈ S, and there exists 0 ∈ S such that s +0=s and s.0=0.s = 0 for all s ∈ R. Many authors studied and developed this concept extensively and obtained various applications of semirings in mathematics and branches computer science (see[2]). Moreover,

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in 1966 the concept of inverse semimodule over a semiring appeared in Yusuf's paper [3] and he gave some similar to results in modules over a ring for inverse semimodules. Later many authors studied and developed this concept extensively. A (left) Si- semimodule C over the semiring $Si$is a commutative monoid $(C, +, 0_C)$ together with a scalar multiplication $(r, m) \to irm$ from $S \times C \to C$ which satisfies identities $(rt)c = r(tc), r(c + d) = rc + rd, (r + t)c = rc + tc, 1c = c \; and \; r0_c = 0 = 0_S c$ for all $r, t \in S$ and $c, d \in C$ [2].

In the last decades the injective object and their generalizations in various categories played very important role in different areas such commutative algebra, ring theory, homology theory and cohomology theory and others (see [4]). Regarding the modules over the rings category, the concept of extending modules and its generalizations studied by numerous authors (for more details see [5]). A module M becomes extending module if each of its submodules be essential in constructing a direct summand of iM. Equivalently, a module iM is called extending module if and only if each of its closed submodules be a direct summand of iM. Following [6], a stronger concept of extending modules are introduced. A moduleiM is named strongly extending module if each of its submodules is essential in a stable (fully invariant) direct summand of M. Recently, extending semimodules over semiring was introduced and studied. An S-semimodule C be called extending semimodule if each of its subsemimodules is essential in a direct summand of C [7].

In the present paper, the strong concept of extending semimodules overia semiring introduced and considered.

Let C be a left S-semimodule. A nonempty subset T of C is said to be subsemimodulei of C if iT is closed under addition and scalar multiplication. A non-zero subsemimoduleI iT of C is called an essential of C, if T ∩ H ≠ (0) for every nonzero subsemimodule H of C. Let $C \; and \; N$ be two semimodules over a semiring S. A subsemimodule T of a semimodule C is called closed if it has not any proper essential extension in C [8]. A subsemimoduleI T of CI is called to be closure of a subsemimodule K in C if T is closed subsemimodule and D essential in T[8]. A subsemimodule $T \; of \; C$ is said to be a complement of a subsemimodule K if $T \cap K = 0$ and T is a maximal with this property [8]**.**. A subsemimodule T of C is called subtractive (= Mi -subsemimodule) if for all $c_1, c \in C$ and all $r \in S$, if $c_1, c_1 + c_2 \in T$, then c_2 is also in T [9]. An iS -semimodule iC is called$I$ subtractive if all submodules of C are subtractive. Also, C is called closed- subtractive (= kc-semimodule) if every closed subsemimoduleof C is subtractive [7]. An S-semimodule C is called direct summand of subsemimodules T_1,T_2 of C, if each element c in C can be uniquely written as $c = c_1 + c_2$ where $c_1 \in T_1$ and $c_2 \in T_2$ and denoted by $C=T_1 \oplus T_2$. Also, T_1 is called direct summand of C if there exists a subsemimodule T_2 of M such that $C=T_1 \oplus T_2$ [2]. For short, we will use summand instead of direct summand. A homomorphism from $C \; to \; N$ is a function $f: C \to N$ such that for all $r \in S$ and $c_1, c_2 \in C$, the following equations hold, $f(c_1 + c_2) = f(c_1) + f(c_2)$; $f(rc) = rf(c)$ [9].

AnIS -semimoduleIS is called cancellative if for each $c, c_1, c_2 \in C, c + c_1 = c + c_2$ this implies that $c_1 = c_2$ [2]. A non-zero semimoduleI C is said to be indecomposable if the only direct summands of C are {0}1 and C itself [8]. Also, a non-zero semimodule C is called simple if C has no proper subsemimodules [10]. An S- semimodule C is called to be semisimple if it is a direct sum of its simple subsemimodule [11]. An Si- semimodule C is said uniform if C $\neq$ 0 and for each non-zero subsemimodule of C is essential [7]. A semimodule C is called injective if, for any S-semimodules A and B, any S-monomorphism α : Ai → B, and any S-homomorphism ϕ : A → C, there is an IS -homomorphism ϕ : B → C such that ϕi = ϕ ∘ α. If E is an injective S- semimodule, and it is essential extension of a semimodule C, then E is called to be an injective hull of C denoted by $E(C)$ [12]. A subsememodule T of an S-semimodule C is called stable if, $f(T) \subseteq T$ for each R- homomorphism $f: T \rightarrow C$. An S-semimodule C is called fully stable if each subsemimodule of C is stable [13]. Also, a subsemimodule T of C be called fully invariant if $if(T) \subseteq T$ for each endomorphism f of C. C is called to be Duo if each subsemimodulei is fully invariant [13].

2. Strongly extending semimodules

Consideration by the same concept in module theory as well as a strong concept of extending semimodules over a semiring, we introduce the next concept:

Definition 2.1: An S- semimodule C be called strongly extending if, each subsemimodule of C is essential in a stable summand of C. A semiring S be called left (resp. right) strongly extending if IS is left (resp. right) strongly extending S-semimodule. A semiring S is strongly extending if it be both left and right strongly extending.

Form [13] each stable subsemimodule be fully invariant and the converse be not true. The following lemma Hence, we get the following result.

Lemma 2.2: *Full invariant summands are stable.*

Proof: A assume that N is a fully invariant summand is an S- semimodule H. Suppose that $f: N \rightarrow C$ is any S- homomorphism. Since N be a summand in C thus there is the projection mapping $\varphi: C = N \oplus T \rightarrow N$ where T a subsemimodule of C. Hence $(f\ o\ \varphi): C \rightarrow C$ and since iN be fully invariant of iC, then we have $(f\ o\ \varphi)\ (N) \subseteq N$ and so $f(iN) = (f\ o\ \varphi)\ (N)i \subseteq N$. Hence , N is a stable subsemimodule of iC.

The following corollaries is directly from Lemma (2.2).

Corollary 2.3: *Every duo semi-simple semimodule is full stable.*

Note that the condition of semi-simple semimodule in above corollary is necessary. one can easily check that the semiring of integers numbers $\mathbb{Z}$ is not semi-simple, not fully stable, but it is a duo semiring.

we have a useful equivalent statement for strongly extending property. (compare with the definition of strongly extending modules in [6])

Corollary 2.4: *Let C a strongly extending semimodule if each subsemimodule in C is Iessential in fully Iinvariant summand in C.*

Furthermore, we can rewrite the definition of strongly extending semiring as the following. The proof is routine.

Corollary 2.5: *A semiringIS is left i(resp. right) strongly extending if and only every left (resp. right) ideal of IS is essential in two-sided summand of S.*

Corollary 2.6: *A duo (and so fully stable) extending semimodule is strongly extending.*

Since every commutative semring is duo, so we have:

Corollary 2.7: *A commutative semiring be strongly extending semimodule if and only if is extending.*

Remarks 2.8

1. It is obvious that strongly extending semimodule is extending, while the reverse is not valid. Consider $C = \mathbb{R}\oplus\mathbb{R}$ as $\mathbb{R}^+$- semimodule . Thus C is extending $\mathbb{R}^+$ - semimodule (since it is semi-simple), while it not strongly extending since the subsemimodule $0\oplus\mathbb{R}$ is non-trivial summand of C which is not stable.
2. It is clear that every uniform semimodule is strongly extending. While the converse is not true , for example, the semimodule $(\mathbb{Z}_6, +_6)$ over $(Z_0^+, +, .)$ is extending and it is not uniform.
3. Unlike extending semimodule, a semisimple semimodule need not be strongly extending (see the example in (1)).
4. An S- semimodule C is uniform if and only if C is indecomposable and strongly extending semimodule.

Proof: For the sufficient direction, let N be a subsemimodule of C . By strongly extending of C , N is essential in a stable summand of C, since C is indecomposable, then (0) and C are the only summand of C. Thus N is essential subsemimodule of C . Hence C is uniform S- semimodule. For the necessary direction. From (2), C is strongly extending. Let D be a proper summand of C. Thus $C = D \oplus K$, where K is a subsemimodule of C and so $D \cap K = (0)$. But K is essential subsemimodule of C (since C is uinform), hence D=(0), Therefore, C is indecomposable.

5. It is easy to verify that an indecomposable semimodule is strongly extending if and only if it is extending.

There is no direct relation between subtractive semimodules and strongly extending semimodules. Here, we get a partial relation between (strongly) extending semimodules and subtractive semimodules.

Lemma 2.9: *Every extending (and so strongly extending) seminodule is closed-subtractive.*

Proof: Suppose that K is a closed subsemimodule ofof an S-semimodule C. By extending property, ([7], proposition 2.3) , then iK be a summand of C. By ([14], corollary 3.2), then K is a subtractive subsemimodule. Thus C is closed-subtractive semimodule.

The next result gives an equivalent characterization of strongly extending semimodules.

Proposition 2.10: AnIS -semimodule C be strongly extending if and only if for each closed subsemimodule of C is stable summand.

Proof: For necessary direction, assume that K is a closed subsemimodule in C . As C is strongly extending, hence there is stable summand B in C where K is essential in B . However, K is closed subsemimodule in C, Thus, $K = B$ (i.e.) K represents a stable summand in C. For the sufficient direction, let A be a subsemimodule of C . By Lemma (2.9), C is closed-subtractive semimodule. Hence (by ([7], remark 2.9)), there exists a closed subsemimodule $K\ of\ C$ such that A is essential in K. Thus, according to the given hypothesis, iK is a stable summand of C, (i.e.) A is essential in stable summand of K. Then C is strongly extending S- semimodule.

By using lemma (2.2), one can restate proposition (2.10) as follows. The proof is routine.

Corollary 2.11: *An S-semimodule C be strongly extending if and only if for each closed subsemimodule of C is fully invariant summand.*

Theorem 2.12: *Let C be a S- semimodule with injective hull. Then C is strongly extending if and only if $A \cap C$ is a stable direct summand of C, for every direct summand A of $E(C)$.*

Proof: For the first direction, assume that C is strongly extending, let $E(C) = A \oplus B$, where B is a subsemimodule of $E(C)$. Suppose that $A \cap C$ is an essential in K, where K is a subsemimodule of C. Let $k \in K$, where $k = a + b$, where $a \in A$ and b∈ B. Suppose that $k \notin A$, then b $\neq$ (0). But C is essential in $E(C)$

and $(0) \neq b \in B \subseteq E(C)$, therefore there exists $r \in R$ such that $(0) \neq rb \in C$. Now $rk = ra + rb \in A \cap C \subseteq K$. Since A is a direct summand, corollary 3.2), thus A is subtractive. Then $ra, ra + rb \in A$, hence $rb \in A$. Thus $rb \in K \cap B$, but $A \cap C$ is an essential in K, so $(0) = (A \cap C) \cap B$ is essential in $K \cap B = (0)$. Then $rb = (0)$ which is a contradiction. Thus $A \cap C$ is closed subsemimodule of C, from (proposition 2.10), since C is strongly extending ssemimodule, then $A \cap C$ is as$i$table direc$i$tsummand of C. For the reverse direction, suppose that Ai be a subsemimodule of iC. Let Bi be a$i$relative complement$i$ of A, then $A \oplus B$ is$i$ essential of C by ([16] , lemma 2.20). Also, ([16], proposition 2.25) states that C is$i$ essential $i$in $E(M)$. Thus, $A \oplus B$ is essential in $E(C)$ proposition 3.3 and so $E(A) \oplus E(B) = E(A \oplus B) = E(C)$ by ([13], lemma 2.21). Since $E(A)$ is a direct summand of C, then by hypothesisi E(A)$\cap C$ is$i$a stable$i$direct summand$i$of C But A is$i$essential in $E(A)$ and C is essential$i$in C, then $A = A \cap C$ is essential in $E(A) \cap C$ by, proposition 3.3). Therefore, C is strongly$i$extending.

The next result achieves another characterization of strongly extending semimodules.

Theorem 2.13: *AnIS - semimodule C is called to be strongly extending semimodule if each subsemimodule A of iC, there exists a direct decomposition ($Ci = C_1 \oplus C_2$) where, $A \subseteq C_1$ and iC_1 is a stable subsemimodule of C. Besides, $A + C_2$ must be essential subsemimodule in iC.*

Proof: Let C semimodule be strongly extending, and suppose that A is a subsemimodule in C. Then, A is necessarily essential in a stable a summand $C_1 \, of \, C$. Put $C = C_1 \oplus C_2$, where C_2 be a subsemimodule of C . Since A and C_2 are essential subsemimodules in C_1 and C_2 respectively, then $A + C_2$ becomes essential in $C_1 \oplus C_2 = C$ by ([8], lemma 2.1.9). Conversely, if A is a subsemimodule of . According to the given hypothesis, there exists a direct decomposition $C = C_1 \oplus C_2$, where $A \subseteq C_1$ knowing that C_1 is a stable subsemimodule in C. Besides $A + C_2$ be essential in C. Suppose that K is a non-zero subsemimodule in C_1, and since K is subsemimodule of C, then $(A + C_2) \cap K$ is a non-zero value (as $A + C_2$ is essential in C). Suppose that $k = a + m_2 \neq (0)$, such that $k \in K, a \in A$ and $m_2 \in C_2$. Since C_1 is summand of C, then (from ([14] ,corollary 3.2)), C_1 is subtractive subsemimodule. But $, a + m_2 \in C_1$, hence $m_2 \in C_1$, which means $m_2 \in C_1 \cap C_2 = (0)$, thus $k = a \in K \cap A$ and so $K \cap A \neq (0)$. Thus, A becomes essential in C_1. Therefore, C semimodule is strongly extending.

We do not know whether the hereditary property of strongly extending semimodules is valid. The following results verify this property for some kinds of subsemimodules.

Proposition 2.14: A subsemimodule in strongly extending semimodulei is said to be strongly extending subsemimodulei if it is a closed.

Proof: Suppose that N is a closed subsemimodule in strongly extending S-semimodule C. Assume that K is a closed subsemimodule in N, then K is said to be closed subsemimodule in C by ([7], Lemma 2.8) and as C semimodule is strongly extending, K is stable summand in C. But $K \subseteq N$, hence K is summand in N by ([14] , corollary 3.3). If we assume $f: K \to N$ is an S- homomorphism and considering the following sequence $K \xrightarrow{f} N \xrightarrow{i} C$, where i refers to the inclusion mapping. Then $(i\ of): K \to C$ and as K is stable in C , then $(i\ o\ f)$i(K)i $\subseteq K$. Thus, $f(K) \subseteq K$. Then, K represents a stable summand in N. Thus, N becomes strongly extending.

Corollary 2.15: *A summand of strongly extending semimodule is strongly extending semimodule.*

Proof: A assume that N is any summand in a strongly extending semimodule C. By Lemma 2.9 C is closed-subtractive and following Remark (1.1.21) in [7] N is closed subsemimodule of C. Directly, according to proposition (2.14), N becomes strongly extending semimodule.

Proposition 2.16: Every subsemimodule N of a strongly extending S – semimodule *C* that intersects with a stable summand in C is a stable summand in N, is strongly extending.

Proof: Suppose that A is a subsemimodule in N . As C is strongly and A is a subsemimodule in C, hence there exists stable summand K in C where A is essential in K . However $A \subseteq K \cap N \subseteq K$, hence by ([15], Lemma 2.2.3) A is essential in $K \cap N$ and considering the given hypothesis , $K \cap N$ becomes a stable summand in N . Thus, N is a stable extending.

Proposition 2.17: Let $C = C_1 \oplus C_2$ be an IS – isemimodule. Hence C is strongly extending if and only if iC is closed-subtractive semimodule and every closed subsemimodule K in C where $K \cap C_1 = 0$ or $K \cap C_2 = 0$ is a stable summand.

Proof: The main part is clear from proposition (2.10). For the sufficient part, suppose that every closed subsemimodule iK of C with $K \cap C_1 = 0$ or $iK \cap C_2 = 0$ is stable summand. Let L be any closed subsemimodule in C. There is a closed $H\ in\ L$ where $L \cap C_2$ is also essential in H and as L subsemimodule is closed in C , Thus, H is a closed subsemimodule in C by ([7], Lemma 2.2.13). Clearly, since $(L \cap C_2) \cap C_1$ is also essential in $H \cap C_1$, thus $H \cap C = 0$. Then, according to the given hypothesis, $C = H \oplus H_1$ for a subsemimodule H_1 in $C\ and\ H$ represents a stable summand in C, then L is subtractive. Now, $Li = L \cap C = Li \cap (H \oplus H_1) = H \cap (Li \cap H_1)$. Therefore, (L$\cap H_1$) is also closed in C (as (L$\cap H_1$) is closed in L). Also, (L$\cap H_1$)$\cap C_2$=0. Using the hypothesis, $(L \cap H_1)$

represents a stable summand in C , and concequently of H_1 (since $L \cap H_1 \subseteq H_1$). Thus, $H_1 = (L \cap H_1) \oplus H$, where K is a subsemimodule of H_1. Now $C = H \oplus H_1 = H \oplus ((L \cap H_1) \oplus K)i = L \oplus K$. Hence, L is a summand in C . Suppose that $g: L \to C$ is any S- homomorphism , $i_1: H \to iH \oplus (L \cap H_1 i), i_2: (L \cap H_1 i) \to iH \oplus (L \cap H_1)$ be the inclusion mappings and $\pi_1: C \to C_1$, $\pi_2: C \to C_2$ be the projection mappings . Put $f_1 = \pi_1 \, o \, g \, o \, i_1$, $f_2 = \pi_2 \, o \, g \, o \, i_2$. Hence $g = g_1 \oplus g_2$, $g(L) = g(Hi \oplus (L \cap H_1)) = g_1(H) \oplus g_2(Li \cap H_1) \subseteq H \oplus (Li \cap H_1)$ and $g(L) \subseteq L$. Thus, L is a stable summand in C .Hence, C becomes strongly extending.

As strong form of closed-subtractive semimodules introduces in the next concept:

Definition 2.18: A semimodule is called strongly closed-subtractive (shortly, sc-subtractive) if every closed subsemimodule is fully invariant subtractive semimodule.

It is clear from the definitions that we have the next implications:

Strongly extending semimodule ⟹sc-subtractive semimodule ⟹ closed-subtractive semimodule

Proposition 2.19: An *S*-semimodulei Ci is sc-subtractive semimodulei if and only if every subsemimodule of C is essential in a subtractive stable subsemimodule.

Proof: For the first direction, let N subsemimodule of C, since C is kcs-semimodule. Then C is closed-subtractive semimodule by([7], remark (1.1.15)), then N essential subsemimodule K such that K is closure. Then N is closed, N is a direct summand. By ([14], corollary 3.2), N is subtractive semimodule. By hypothesis N is stable. Then N is essential in a subtractive stable subsemimodule. Conversely, suppose that iN is a closed subsemimodule of C, by hypothesis iN is essential in subtractive stable subsemimodule T. Since N is closed subsemimodule, then $N = T$. Thus N is subtractive stable subsemimodule. Then C is sc-subtractive semimodule.

In the same above approach, the following result is concluded.

Proposition 2.20: A *S*-semimodule C is closed-subtractive semimodule if and only if every subsemimodule of *C* is essential in a subtractive subsemimodule of *C*.

Following [7], if C is an S-semimodule such that $T \leq K \leq C$ and T is subtractive in *C* , then *T* is subtractive in *K*.

It is easy to check that a subsemimodule of subtractive semimodule is subtractive. But this fact need not be true for closed-subtractive semiomodules. In the next result, we investigate some subsemimodules that hold the hereditary property of closed-subtractive semimodules.

Proposition 2.21: A closed subsemimodule of closed-subtractive is closed-subtractive.

Proof: Let IM be a closed-subtractive semimodule and T be a closed subsemimodule of M. Let iK be a closed subsemimodulei of T, then K be a closed subsemimodule of iM [7]. Since iM is closed-subtractive, hence Ki is subtractive in Mi and so K is subtractive in T. Thus, T is closed-subtractive.

Corollary 2.22: *A summand in closed-subtractive semimodule is closed-subtractive.*

The next theorem gives when a summand property in strongly extending semimodules is valid.

Theorem 2.23: *Suppose that $C = \bigoplus_{i\in I} C_i$ is an S- semimodule, such that C_i is subsemimodule in C for every $i \in I$. Hence, following statements are considered equivalent:*

(1) *C is a strongly extending semimodule.*
(2) *Every C_i be a strongly extending semimodule and every closed subsemimodule in C is a stable subtractive.*
(3) *Every C_i is extending and every closed subsemimodule of C is stable subtractive.*

Proof:

$(\mathbf{1}) \Rightarrow (\mathbf{2})$. Considering (Proposition 2.14) and (Proposition 2.10).

$(\mathbf{2}) \Rightarrow (\mathbf{3})$. It is obvious.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$. Let K be a closed subsemimodule of C. By hypothesis, iK is fully invariant subsemimodule of C. So from [13], $K = \bigoplus_{i\in I} (K \cap C_i)$. By lemma (2.9), C is closed-subtractive semimodule and so by Proposition (2.21), K is closed-subtractive. Hence, $K \cap C_i$ becomes a summand in K , thus $K \cap C_i$ is closed in K by ([7], Lemma 2.2.13). But K be closed in C , thus $K \cap C_i$ be closed in C by ([7], Lemma 2.2.13). Since $IK \cap C_i \subseteq C_i \subseteq C$, then $iK \cap C_i$ is closed in C_i .Using the extending property of C_i , $K \cap C_i$ represents a summand in C_i . Hence, $iK = \bigoplus_{i\in I} (K \cap iC_i)$ be a summand of $C = i \bigoplus_{i\in I} C_i$. Therefore, K will be fully invariant t summand in C . Thus, using proposition (2.4), C is a strongly extending semimodule.

The following result obtains another characterization of strongly extending semimodules by using sc-subtractive semimodules.

Proposition 2.24: An S-semimodule C be strongly extending semimodule if and only if C is extending semimodule and sc- subtractive semimodule.

Proof: Let C be a strongly extending semimodule. Directly, C is extending semimodule. Now, let N is a closed subsemimodule of C, by (proposition 2.5). Thus N is stable summand of C. By ([14] , corollary 3.2) then N is subtractive subsemimodule.Then N is sc- subtractive semimodule. Conversely, let N a subsemimodule lof C, since C is extending semimodule, hence, there is a summand D of C such that, $N \leq^{e} D$. C is sc- subtractive so it is closed-subtractive thus by ([7] , remake (1.1.21)), then D is a closed subsemimodule. By sc-subtractive property, D is stable subtractive in C. So, we have D is stable summand and by proposition (2.10), C be strongly extending semimodule.

Following a module with direct summands are stable submodules is called SS-module. Also, from [16], a module in which each direct summand is fully invariant submodule is called weak duo. In fact, these concepts are equivalent. Here, we introduce version these concepts, we

Definition 2.25: An S- semimodule C is said SS-semimodule if each summand of C is stable subsemimodule.

By lemma (2.2), the property of SS-semimodule is equivalent to each summand be fully invariant subsemimodule.

The next useful result gives a relation between sc-subtractive semimodules and SS-semimodule.

Lemma 2.26: *Every sc-subtrective semimodule is ss-semimodule.*

Proof: Suppose that H is a summand of iC. Since C is sc-subtractive semimodule, so by ([7], Remark (1.1.21)), then D is a closed subsemimodule. By sc-subtractive property, D is stable subsemimodule. Then C is a ss-semimodule.

Proposition 2.27: Let C be strongly extending semimodule if and only if C is extending semimodule and M is ss-semimodule.

Proof: For the necessary part, let C be strongly extending semimodule. Directly, C is extending semimodule. Now, let D be a summand of C. Since C is strongly extending semimodule by (lemma 2.9), then C is closed-subtractive semimodule. By ([7], remark (1.1.21)), D is closed subsemimodule. By (proposition 2.5), then D is stable subsemimodule. Then C is ss-semimodule. For the sufficient part, let N be an subsemimodule of iC, since C is extending semimodule then there exist is a summand D of C such that N essential in D. But C is ss-semimodule, then D is stable. Hence N be essential in a stable summand D of C and so C is strongly extending semimodule.

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