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Some types of homogeneous systems of difference equations of dimension four

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Abstract

The suggested study's main objective is to create a new meaning for the word "homogeneous system." With m being a positive integer, this phrase describes a P-self-semi-homogenous system of difference equations on the order of magnitude m. Furthermore, we introduced theorems for scenarios where P equals one and those where it takes any number. Every situation has a value of m, of which some are greater than one and some are equal to one. We provide examples to support each of these values.

Subject Classification: *Primary 39Axx, Secondary 39A06.*

Keywords: *Homogeneous system, Self-semi-homogeneous, P-self-semi-homogeneous, Difference equation.*

1. Introduction

The history of difference equations, including recursion relations, dates back to Mesapotamia's second millennium BCE. Recursion relations were used to calculate approximations to the square root of 2, with Heron of Alexandria

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providing a precursor to difference equations. Fibonacci's Liber Abaci in 1202 explicitly presents recursion in the growth of an idealised rabbit population. The general difference equation

$$x(n + 1) = f(x(n)) \tag{1}$$

is an example of a discrete dynamical system. It generates a sequence of iterations, with as the initial point. These systems are equivalent, embodying an identical physical paradigm. The difference equation is referred to as autonomous or time-invariant and is often discussed in analytic theory and discrete dynamical systems. Consider the following homogeneous system: [4], [6], [7], [10], [11], [14]

$$F(X(n)) = \Xi X(n) \tag{2}$$

The system is a P-semi-homogeneous system of order m, provided that a matrix is located, H is non-zero and non-identity, and the other criteria are satisfied.

$$F(\mathrm{H}x(n)) = P^k \mathrm{H}^m F(x(n)) \tag{3}$$

where p, k, and m are integer numbers [1], [2], [12].

Our research introduces a new idea about the homogeneous system (2) is called p-self-semi-homogeneous system of order m if it satisfies condition (3) with matrices H and ψ that are equal. The following is a possible formulation of Equation (3).

$$F(\mathrm{H}x(n)) = P^k \mathrm{H}^{m+1}(x(n)) \tag{4}$$

Several particular circumstances, in addition to the general case, are investigated in this work. In addition to providing examples and definitional characteristics, the paper establishes related theorems.

2. Specific details pertaining to Homogeneity in Difference Equation systems

The notion of homogeneity, which is established in difference equations, is similarly present in difference equations, resulting in the subsequent system:

$$X(n + 1) = \Xi(n)X(n) + g(n) \tag{5}$$

Where $\Xi(n) = (\xi_{ij}(n))$ is a $k \times k$ nonsingular matrix function, and $g(n), x(n) = (x_1(n), x_2(n), \ldots, x_n(n))^T \in R^k$. Here, T indicates the transpose of a vector. This is a non-homogeneous linear difference system that is not autonomous or time-variable. The system (5) is a homogeneous linear difference system if g(n) = 0.

In this study, matrix H will be considered real non-singular; that is, system (5) will be considered autonomous, or time invariant. [3], [5], [8],[9],[13]

Thus, you may express equation (2) as follows:

$$\begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(z) \\ f_4(w) \end{pmatrix} = \begin{pmatrix} x(n+1) \\ \mathrm{y}(n+1) \\ \mathrm{z}(n+1) \\ \mathrm{w}(n+1) \end{pmatrix} = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix} \begin{pmatrix} x(n) \\ \mathrm{y}(n) \\ \mathrm{z}(n) \\ \mathrm{w}(n) \end{pmatrix}$$

$$\begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(z) \\ f_4(w) \end{pmatrix} = \begin{pmatrix} \xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w \\ \xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w \\ \xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w \\ \xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w \end{pmatrix} \tag{6}$$

3. Self-Semi-Homogenous of order m

In this part, self-semi-homogenous groups of order m were examined when m = 1 and when m > 1.

Case 1: When m=1, where the matrix

$$\mathrm{H} = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}, \Xi = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix}$$

Theorem 3-1**:** *The homogeneous system of difference equations (2) is self-semi-homogenous of order one if and only if the matrix Ξ is a solution of the following algebraic system:*

$$\left.\begin{aligned}
&\eta_{11}b_{11} = \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41} \\
&\eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\xi_{12} + \eta_{12}\eta_{22} + \eta_{13}\eta_{32} + \eta_{14}\eta_{42} \\
&\eta_{11}\eta_{13} + \eta_{13} = \eta_{11}\xi_{13} + \eta_{12}\eta_{23} + \eta_{13}\eta_{33} + \eta_{14}\eta_{43} \\
&\eta_{11}\eta_{14} + \eta_{14} = \eta_{11}\xi_{14} + \eta_{12}\eta_{24} + \eta_{13}\eta_{34} + \eta_{14}\eta_{44} \\
&\eta_{21} + \eta_{22}\eta_{21} = \eta_{21}\eta_{11} + \eta_{22}\xi_{21} + \eta_{23}\eta_{31} + \eta_{24}\eta_{41} \\
&\eta_{22}\xi_{22} = \eta_{21}\eta_{12} + \eta_{22}^2 + \eta_{23}\eta_{32} + \eta_{24}\eta_{42} \\
&\eta_{22}\eta_{23} + \eta_{23} = \eta_{21}\eta_{13} + \eta_{22}\xi_{23} + \eta_{23}\eta_{33} + \eta_{24}\eta_{43} \\
&\eta_{22}\eta_{24} + \eta_{24} = \eta_{21}\eta_{14} + \eta_{22}\xi_{24} + \eta_{23}\eta_{34} + \eta_{24}\eta_{44} \\
&\eta_{31} + \eta_{33}\eta_{31} = \eta_{31}\eta_{11} + \eta_{32}\eta_{21} + \eta_{33}\xi_{31} + \eta_{34}\eta_{41} \\
&\eta_{32} + \eta_{33}\eta_{32} = \eta_{31}\eta_{12} + \eta_{32}\eta_{22} + \eta_{33}\xi_{32} + \eta_{34}\eta_{42} \\
&\eta_{33}\xi_{33} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43} \\
&\eta_{34} + \eta_{33}\eta_{34} = \eta_{31}\eta_{14} + \eta_{32}\eta_{24} + \eta_{33}\xi_{34} + \eta_{34}\eta_{44} \\
&\eta_{41} + \eta_{44}\eta_{41} = \eta_{41}\eta_{11} + \eta_{42}\eta_{21} + \eta_{43}\eta_{31} + \eta_{44}\xi_{41} \\
&\eta_{42} + \eta_{44}\eta_{42} = \eta_{41}\eta_{12} + \eta_{42}\eta_{22} + \eta_{43}\eta_{32} + \eta_{44}\xi_{42} \\
&\eta_{43} + \eta_{44}\eta_{43} = \eta_{41}\eta_{13} + \eta_{42}\eta_{23} + \eta_{43}\eta_{33} + \eta_{44}\xi_{43} \\
&\eta_{44}\xi_{44} = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2
\end{aligned}\right\} \tag{7}$$

where H and Ξ are similar matrices.

Proof: $\Longrightarrow$

Assume (2) is a self -semi -homogeneous of order m=1, and we will prove that (7) is held.

$$F\big(\mathrm{H}x(n)\big) = \mathrm{H}F\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\left(\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}\begin{pmatrix} \mathrm{f}_1(\mathrm{x}) \\ \mathrm{f}_2(\mathrm{y}) \\ \mathrm{f}_3(\mathrm{z}) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} f_1(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) \\ f_2(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) \\ f_3(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) \\ f_3(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix}$$

$$= \begin{pmatrix} \eta_{11}f_1(x) + \eta_{12}f_2(y) + \eta_{13}f_3(y) + \eta_{14}f_4(w) \\ \eta_{21}f_1(x) + \eta_{22}f_2(y) + \eta_{23}f_3(y) + \eta_{24}f_4(w) \\ \eta_{31}f_1(x) + \eta_{32}f_2(y) + \eta_{33}f_3(y) + \eta_{34}f_4(w) \\ \eta_{41}f_1(x) + \eta_{42}f_2(y) + \eta_{43}f_3(y) + \eta_{44}f_4(w) \end{pmatrix}$$

By applying equation (6)

$$\begin{pmatrix} \xi_{11}(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) + \xi_{12}y + \xi_{13}z + \xi_{14}w \\ \xi_{21}x + \xi_{22}(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) + \xi_{23}z + \xi_{24}w \\ \xi_{31}x + \xi_{32}y + \xi_{33}(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) + \xi_{34}w \\ \xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix}$$

$$= \begin{pmatrix} \eta_{11}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{12}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \eta_{13}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{14}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ \eta_{21}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{22}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \eta_{23}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{24}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ \eta_{31}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{32}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \eta_{33}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{34}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ \eta_{41}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{42}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \eta_{43}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{44}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \end{pmatrix}$$

$$\xi_{11}\eta_{11} = \eta_{11}\xi_{11} + \eta_{12}\xi_{21} + \eta_{13}\xi_{31} + \eta_{14}\xi_{41}$$
$$\xi_{11}\eta_{11} + \xi_{12} = \eta_{11}\xi_{12} + \eta_{12}\xi_{22} + \eta_{13}\xi_{32} + \eta_{14}\xi\eta_{42}$$
$$\xi_{11}\eta_{13} + \xi_{13} = \eta_{11}\xi_{13} + \eta_{12}\xi_{23} + \eta_{13}\xi_{33} + \eta_{14}\xi_{43}$$
$$\xi_{11}\eta_{14} + \xi_{14} = \eta_{11}\xi_{14} + \eta_{12}\xi_{24} + \eta_{13}\xi_{34} + \eta_{14}\xi_{44}$$
$$\xi_{22}\eta_{21} + \xi_{21} = \eta_{21}\xi_{11} + \eta_{22}\xi_{21} + \eta_{23}\xi_{31} + \eta_{24}\xi_{41}$$
$$\xi_{22}\eta_{22} = \eta_{21}\xi_{12} + \eta_{22}\xi_{22} + \eta_{23}\xi_{32} + \eta_{24}\xi_{42}$$
$$\xi_{22}\eta_{23} + \xi_{23} = \eta_{21}\xi_{13} + \eta_{22}\xi_{23} + \eta_{23}\xi_{33} + \eta_{24}\xi_{43}$$
$$\xi_{22}\eta_{24} + \xi_{24} = \eta_{21}\xi_{14} + \eta_{22}\xi_{24} + \eta_{23}\xi_{34} + \eta_{24}\xi_{44}$$
$$\xi_{33}\eta_{31} + \xi_{31} = \eta_{31}\xi_{11} + \eta_{32}\xi_{21} + \eta_{33}\xi_{31} + \eta_{34}\xi_{41}$$
$$\xi_{33}\eta_{32} + \xi_{32} = \eta_{31}\xi_{12} + \eta_{32}\xi_{22} + \eta_{33}\xi_{32} + \eta_{34}\xi_{42}$$
$$\xi_{33}\eta_{33} = \eta_{31}\xi_{13} + \eta_{32}\xi_{23} + \eta_{33}\xi_{33} + \eta_{34}\xi_{43}$$
$$\xi_{33}\eta_{34} + \xi_{34} = \eta_{31}\xi_{14} + \eta_{32}\xi_{24} + \eta_{33}\xi_{34} + \eta_{34}\xi_{44}$$
$$\xi_{44}\eta_{41} + \xi_{41} = \eta_{41}\xi_{11} + \eta_{42}\xi_{21} + \eta_{43}\xi_{31} + \eta_{44}\xi_{41}$$

$$\xi_{44}\eta_{42} + \xi_{42} = \eta_{41}\xi_{12} + \eta_{42}\xi_{22} + \eta_{43}\xi_{32} + \eta_{44}\xi_{42}$$
$$\xi_{44}\eta_{43} + \xi_{43} = \eta_{41}\xi_{13} + \eta_{42}\xi_{23} + \eta_{43}\xi_{33} + \eta_{44}\xi_{43}$$
$$\xi_{44}\eta_{44} = \eta_{41}\xi_{14} + \eta_{42}\xi_{24} + \eta_{43}\xi_{34} + \eta_{44}\xi_{44}$$
$$\because \mathrm{H} = \Xi \Leftrightarrow \eta_{11} = \xi_{11}, \eta_{12} = \xi_{12}, \dots, \eta_{44} = \xi_{44}$$

The above system may be expressed as

$$\eta_{11}\xi_{11} = \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}$$
$$\eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\xi_{12} + \eta_{12}\eta_{22} + \eta_{13}\eta_{32} + \eta_{14}\eta_{42}$$
$$\eta_{11}\eta_{13} + \eta_{13} = \eta_{11}\xi_{13} + \eta_{12}\eta_{23} + \eta_{13}\eta_{33} + \eta_{14}\eta_{43}$$
$$\eta_{11}\eta_{14} + \eta_{14} = \eta_{11}\xi_{14} + \eta_{12}\eta_{24} + \eta_{13}\eta_{34} + \eta_{14}\eta_{44}$$
$$\eta_{21} + \eta_{22}\eta_{21} = \eta_{21}\eta_{11} + \eta_{22}\xi_{21} + \eta_{23}\eta_{31} + \eta_{24}\eta_{41}$$
$$\eta_{22}\xi_{22} = \eta_{21}\eta_{12} + \eta_{22}^2 + \eta_{23}\eta_{32} + \eta_{24}\eta_{42}$$
$$\eta_{22}\eta_{23} + \eta_{23} = \eta_{21}\eta_{13} + \eta_{22}\xi_{23} + \eta_{23}\eta_{33} + \eta_{24}\eta_{43}$$
$$\eta_{22}\eta_{24} + \eta_{24} = \eta_{21}\eta_{14} + \eta_{22}\xi_{24} + \eta_{23}\eta_{34} + \eta_{24}\eta_{44}$$
$$\eta_{31} + \eta_{33}\eta_{31} = \eta_{31}\eta_{11} + \eta_{32}\eta_{21} + \eta_{33}\xi_{31} + \eta_{34}\eta_{41}$$
$$\eta_{32} + \eta_{33}\eta_{32} = \eta_{31}\eta_{12} + \eta_{32}\eta_{22} + \eta_{33}\xi_{32} + \eta_{34}\eta_{42}$$
$$\eta_{33}\xi_{33} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43}$$
$$\eta_{34} + \eta_{33}\eta_{34} = \eta_{31}\eta_{14} + \eta_{32}\eta_{24} + \eta_{33}\xi_{34} + \eta_{34}\eta_{44}$$
$$\eta_{41} + \eta_{44}\eta_{41} = \eta_{41}\eta_{11} + \eta_{42}\eta_{21} + \eta_{43}\eta_{31} + \eta_{44}\xi_{41}$$
$$\eta_{42} + \eta_{44}\eta_{42} = \eta_{41}\eta_{12} + \eta_{42}\eta_{22} + \eta_{43}\eta_{32} + \eta_{44}\xi_{42}$$
$$\eta_{43} + \eta_{44}\eta_{43} = \eta_{41}\eta_{13} + \eta_{42}\eta_{23} + \eta_{43}\eta_{33} + \eta_{44}\xi_{43}$$
$$\eta_{44}\xi_{44} = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2$$

Conversly

F is one-order self-semi-homogeneous if matrix H is present such that $F(\mathrm{H}x(n)) = \mathrm{H}F(x(n))$ we claim that this matrix Ξ defined as:

$\psi =$

$$\begin{pmatrix}
\frac{\eta_{11}^2 + \eta_{12}\eta_{12} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}}{\eta_{11}} & \frac{\eta_{11}\eta_{12} + \eta_{12} - \eta_{12}\eta_{22} - \eta_{13}\eta_{32} - \eta_{14}\eta_{42}}{\eta_{11}} & \frac{\eta_{11}\eta_{13} + \eta_{13} - \eta_{12}\eta_{23} - \eta_{13}\eta_{33} - \eta_{14}\eta_{43}}{\eta_{11}} & \frac{\eta_{11}\eta_{14} + \eta_{14} - \eta_{12}\eta_{24} - \eta_{13}\eta_{34} - \eta_{14}\eta_{44}}{\eta_{11}} \\
\frac{\eta_{22}\eta_{21} + \eta_{21} - \eta_{21}\eta_{11} - \eta_{23}\eta_{31} - \eta_{24}\eta_{41}}{\eta_{22}} & \frac{\eta_{21}\eta_{12} + \eta_{22}^2 + \eta_{23}\eta_{32} + \eta_{24}\eta_{42}}{\eta_{22}} & \frac{\eta_{22}\eta_{23} + \eta_{23} - \eta_{21}\eta_{13} - \eta_{23}\eta_{33} - \eta_{24}\eta_{43}}{\eta_{22}} & \frac{\eta_{22}\eta_{24} + \eta_{24} - \eta_{21}\eta_{14} - \eta_{23}\eta_{34} - \eta_{24}\eta_{44}}{\eta_{22}} \\
\frac{\eta_{33}\eta_{31} + \eta_{31} - \eta_{31}\eta_{11} - \eta_{32}\eta_{21} - \eta_{34}\eta_{41}}{\eta_{33}} & \frac{\eta_{33}\eta_{32} + \eta_{32} - \eta_{31}\eta_{12} - \eta_{32}\eta_{22} - \eta_{34}\eta_{42}}{\eta_{33}} & \frac{\eta_{33}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43}}{\eta_{33}} & \frac{\eta_{33}\eta_{34} + \eta_{34} - \eta_{31}\eta_{14} - \eta_{32}\eta_{24} - \eta_{34}\eta_{44}}{\eta_{22}} \\
\frac{\eta_{44}\eta_{41} + \eta_{41} - \eta_{41}\eta_{11} - \eta_{42}\eta_{21} - \eta_{43}\eta_{31}}{\eta_{44}} & \frac{\eta_{44}\eta_{42} + \eta_{42} - \eta_{41}\eta_{12} - \eta_{42}\eta_{22} - \eta_{43}\eta_{32}}{\eta_{44}} & \frac{\eta_{44}\eta_{43} + \eta_{43} - \eta_{41}\eta_{13} - \eta_{42}\eta_{23} - \eta_{43}\eta_{33}}{\eta_{44}} & \frac{\eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2}{\eta_{44}}
\end{pmatrix}$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\left(\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}\begin{pmatrix} \mathrm{f}_1(\mathrm{x}) \\ \mathrm{f}_2(\mathrm{y}) \\ \mathrm{f}_3(\mathrm{z}) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} f_1(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) \\ f_2(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) \\ f_3(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) \\ f_3(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix}$$

$$= \begin{pmatrix} \eta_{11}f_1(x) + \eta_{12}f_2(y) + \eta_{13}f_3(y) + \eta_{14}f_4(w) \\ \eta_{21}f_1(x) + \eta_{22}f_2(y) + \eta_{23}f_3(y) + \eta_{24}f_4(w) \\ \eta_{31}f_1(x) + \eta_{32}f_2(y) + \eta_{33}f_3(y) + \eta_{34}f_4(w) \\ \eta_{41}f_1(x) + \eta_{42}f_2(y) + \eta_{43}f_3(y) + \eta_{44}f_4(w) \end{pmatrix}$$

By implementing equation (6)

$$\begin{pmatrix} \xi_{11}(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) + \xi_{12}y + \xi_{13}z + \xi_{14}w \\ \xi_{21}x + \xi_{22}(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) + \xi_{23}z + \xi_{24}w \\ \xi_{31}x + \xi_{32}y + \xi_{33}(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) + \xi_{34}w \\ \xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix} =$$

$$\left(\begin{matrix} \eta_{11}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{12}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \\ \eta_{21}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{22}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \\ \eta_{31}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{32}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \\ \eta_{41}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + \eta_{42}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + \end{matrix}\right.$$

$$\left.\begin{matrix} \eta_{13}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{14}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ \eta_{23}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{24}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ \eta_{33}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{34}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ \eta_{43}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + \eta_{44}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \end{matrix}\right)$$

Therefore

$$\eta_{11}\xi_{11} = \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41} \Rightarrow \eta_{11}\frac{\eta_{11}^2 + \eta_{12}\eta_{12} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}}{\eta_{11}}$$
$$= \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}$$

$$\therefore \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41} = \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}$$

Also, $\eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\xi_{12} + \eta_{12}\eta_{22} + \eta_{13}\eta_{32} + \eta_{14}\eta_{42}$

$$\eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\frac{\eta_{11}\eta_{12} + \eta_{12} - \eta_{12}\eta_{22} - \eta_{13}\eta_{32} - \eta_{14}\eta_{42}}{\eta_{11}} + \eta_{12}\eta_{22} + \eta_{13}\eta_{32} + \eta_{14}\eta_{42}$$

$$\therefore \eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\eta_{12} + \eta_{12}$$

Also,

$$\eta_{11}\eta_{13} + \eta_{13} = \eta_{11}\xi_{13} + \eta_{12}\eta_{23} + \eta_{13}\eta_{33} + \eta_{14}\eta_{43}$$

$$\eta_{11}\eta_{13}+\eta_{13}=\eta_{11}\frac{\eta_{11}\eta_{13}+\eta_{13}-\eta_{12}\eta_{23}-\eta_{13}\eta_{33}-\eta_{14}\eta_{43}}{\eta_{11}}+\eta_{12}\eta_{23}+\eta_{13}\eta_{33}+\eta_{14}\eta_{43}$$

$$\therefore \eta_{11}\eta_{13}+\eta_{13}=\eta_{11}\eta_{13}+\eta_{13}$$

Also,

$$\eta_{11}\eta_{14}+\eta_{14}=\eta_{11}\xi_{14}+\eta_{12}\eta_{24}+\eta_{13}\eta_{34}+\eta_{14}\eta_{44}$$

$$\eta_{11}\eta_{14}+\eta_{14}=\eta_{11}\frac{\eta_{11}\eta_{14}+\eta_{14}-\eta_{12}\eta_{24}-\eta_{13}\eta_{34}-\eta_{14}\eta_{44}}{\eta_{11}}+\eta_{12}\eta_{24}+\eta_{13}\eta_{34}+\eta_{14}\eta_{44}$$

$$\therefore \eta_{11}\eta_{14}+\eta_{14}=\eta_{11}\eta_{14}+\eta_{14}$$

Also,

$$\eta_{21}+\eta_{22}\eta_{21}=\eta_{21}\eta_{11}+\eta_{22}\xi_{21}+\eta_{23}\eta_{31}+\eta_{24}\eta_{41}$$

$$\eta_{21}+\eta_{22}\eta_{21}=\eta_{21}\eta_{11}+\eta_{22}\frac{\eta_{22}\eta_{21}+\eta_{21}-\eta_{21}\eta_{11}-\eta_{23}\eta_{31}-\eta_{24}\eta_{41}}{\eta_{22}}+\eta_{23}\eta_{31}+\eta_{24}\eta_{41}$$

$$\therefore \eta_{21}+\eta_{22}\eta_{21}=\eta_{21}+\eta_{22}\eta_{21}$$

Also,

$$\eta_{22}\xi_{22}=\eta_{21}\eta_{12}+\eta_{22}^2+\eta_{23}\eta_{32}+\eta_{24}\eta_{42}\Rightarrow \eta_{22}\frac{\eta_{21}\eta_{12}+\eta_{22}^2+\eta_{23}\eta_{32}+\eta_{24}\eta_{42}}{\eta_{22}}=\eta_{21}\eta_{12}+\eta_{22}^2+\eta_{23}\eta_{32}+\eta_{24}\eta_{42}$$

$$\therefore \eta_{21}\eta_{12}+\eta_{22}^2+\eta_{23}\eta_{32}+\eta_{24}\eta_{42}=\eta_{21}\eta_{12}+\eta_{22}^2+\eta_{23}\eta_{32}+\eta_{24}\eta_{42}$$

Also,

$$\eta_{22}\eta_{23}+\eta_{23}=\eta_{21}\eta_{13}+\eta_{22}\xi_{23}+\eta_{23}\eta_{33}+\eta_{24}\eta_{43}$$

$$\eta_{22}\eta_{23}+\eta_{23}=\eta_{21}\eta_{13}+\eta_{22}\frac{\eta_{22}\eta_{23}+\eta_{23}-\eta_{21}\eta_{13}-\eta_{23}\eta_{33}-\eta_{24}\eta_{43}}{\eta_{22}}+\eta_{23}\eta_{33}+\eta_{24}\eta_{43}$$

$$\therefore \eta_{22}\eta_{23}+\eta_{23}=\eta_{22}\eta_{23}+\eta_{23}$$

Also,

$$\eta_{22}\eta_{24}+\eta_{24}=\eta_{21}\eta_{14}+\eta_{22}\xi_{24}+\eta_{23}\eta_{34}+\eta_{24}\eta_{44}$$

$$\eta_{22}\eta_{24}+\eta_{24}=\eta_{21}\eta_{14}+\eta_{22}\frac{\eta_{22}\eta_{24}+\eta_{24}-\eta_{21}\eta_{14}-\eta_{23}\eta_{34}-\eta_{24}\eta_{44}}{\eta_{22}}+\eta_{23}\eta_{34}+\eta_{24}\eta_{44}$$

$$\therefore \eta_{22}\eta_{24}+\eta_{24}=\eta_{22}\eta_{24}+\eta_{24}$$

Also,

$$\eta_{31}+\eta_{33}\eta_{31}=\eta_{31}\eta_{11}+\eta_{32}\eta_{21}+\eta_{33}\xi_{31}+\eta_{34}\eta_{41}$$

$$\eta_{31}+\eta_{33}\eta_{31}=\eta_{31}\eta_{11}+\eta_{32}\eta_{21}+\eta_{33}\frac{\eta_{33}\eta_{31}+\eta_{31}-\eta_{31}\eta_{11}-\eta_{32}\eta_{21}-\eta_{34}\eta_{41}}{\eta_{33}}+\eta_{34}\eta_{41}$$

$$\therefore \eta_{31}+\eta_{33}\eta_{31}=\eta_{31}+\eta_{33}\eta_{31}$$

Also,

$$\eta_{32} + \eta_{33}\eta_{32} = \eta_{31}\eta_{12} + \eta_{32}\eta_{22} + \eta_{33}\xi_{32} + \eta_{34}\eta_{42}$$

$$\eta_{32} + \eta_{33}\eta_{32} = \eta_{31}\eta_{12} + \eta_{32}\eta_{22} + \eta_{33}\frac{\eta_{33}\eta_{32}+\eta_{32}-\eta_{31}\eta_{12}-\eta_{32}\eta_{22}-\eta_{34}\eta_{42}}{\eta_{33}} + \eta_{34}\eta_{42}$$

$$\therefore \eta_{32} + \eta_{33}\eta_{32} = \eta_{32} + \eta_{33}\eta_{32}$$

Also,

$$\eta_{33}\xi_{33} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43} \Rightarrow \eta_{33}\frac{\eta_{33}\eta_{13}+\eta_{32}\eta_{23}+\eta_{33}^2+\eta_{34}\eta_{43}}{\eta_{33}} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43}$$

$$\therefore \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43}$$

Also,

$$\eta_{34} + \eta_{33}\eta_{34} = \eta_{31}\eta_{14} + \eta_{32}\eta_{24} + \eta_{33}\xi_{34} + \eta_{34}\eta_{44}$$

$$\eta_{34} + \eta_{33}\eta_{34} = \eta_{31}\eta_{14} + \eta_{32}\eta_{24} + \eta_{33}\frac{\eta_{33}\eta_{34}+\eta_{34}-\eta_{31}\eta_{14}-\eta_{32}\eta_{24}-\eta_{34}\eta_{44}}{\eta_{22}} + \eta_{34}\eta_{44}$$

$$\therefore \eta_{34} + \eta_{33}\eta_{34} = \eta_{34} + \eta_{33}\eta_{34}$$

Also,

$$\eta_{41} + \eta_{44}\eta_{41} = \eta_{41}\eta_{11} + \eta_{42}\eta_{21} + \eta_{43}\eta_{31} + \eta_{44}\xi_{41}$$

$$\eta_{41} + \eta_{44}\eta_{41} = \eta_{41}\eta_{11} + \eta_{42}\eta_{21} + \eta_{43}\eta_{31} + \eta_{44}\frac{\eta_{44}\eta_{41}+\eta_{41}-\eta_{41}\eta_{11}-\eta_{42}\eta_{21}-\eta_{43}\eta_{31}}{\eta_{44}}$$

$$\therefore \eta_{41} + \eta_{44}\eta_{41} = \eta_{41} + \eta_{44}\eta_{41}$$

Also,

$$\eta_{42} + \eta_{44}\eta_{42} = \eta_{41}\eta_{12} + \eta_{42}\eta_{22} + \eta_{43}\eta_{32} + \eta_{44}\xi_{42}$$

$$\eta_{42} + \eta_{44}\eta_{42} = \eta_{41}\eta_{12} + \eta_{42}\eta_{22} + \eta_{43}\eta_{32} + \eta_{44}\frac{\eta_{44}\eta_{42}+\eta_{42}-\eta_{41}\eta_{12}-\eta_{42}\eta_{22}-\eta_{43}\eta_{32}}{\eta_{44}}$$

$$\therefore \eta_{42} + \eta_{44}\eta_{42} = \eta_{42} + \eta_{44}\eta_{42}$$

Also,

$$\eta_{43} + \eta_{44}\eta_{43} = \eta_{41}\eta_{13} + \eta_{42}\eta_{23} + \eta_{43}\eta_{33} + \eta_{44}\xi_{43}$$

$$\eta_{43} + \eta_{44}\eta_{43} = \eta_{41}\eta_{13} + \eta_{42}\eta_{23} + \eta_{43}\eta_{33} + \eta_{44}\frac{\eta_{44}\eta_{43}+\eta_{43}-\eta_{41}\eta_{13}-\eta_{42}\eta_{23}-\eta_{43}\eta_{33}}{\eta_{44}}$$

$$\therefore \eta_{43} + \eta_{44}\eta_{43} = \eta_{43} + \eta_{44}\eta_{43}$$

Also,

$$\eta_{44}\xi_{44} = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2 \Rightarrow \eta_{44}\frac{\eta_{41}\eta_{14}+\eta_{42}\eta_{24}+\eta_{43}\eta_{34}+\eta_{44}^2}{\eta_{44}} = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2$$

$$\therefore \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2 = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2$$

Consequently, system (2) is self-semi-homogeneous of order 1, as the left and right hands are equivalent.

Corollary 3-2: *A system of standardized distinctions If the following conditions are satisfied, and only then, is Equation (2) self-semi homogeneous of order one:*

$$\left.\begin{aligned}
&\eta_{11}\xi_{11} = \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}\\
&\eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\xi_{12} + \eta_{12}\eta_{22} + \eta_{13}\eta_{32} + \eta_{14}\eta_{42}\\
&\eta_{11}\eta_{13} + \eta_{13} = \eta_{11}\xi_{13} + \eta_{12}\eta_{23} + \eta_{13}\eta_{33} + \eta_{14}\eta_{43}\\
&\eta_{11}\eta_{14} + \eta_{14} = \eta_{11}\xi_{14} + \eta_{12}\eta_{24} + \eta_{13}\eta_{34} + \eta_{14}\eta_{44}\\
&\eta_{21} + \eta_{22}\eta_{21} = \eta_{21}\eta_{11} + \eta_{22}\xi_{21} + \eta_{23}\eta_{31} + \eta_{24}\eta_{41}\\
&\eta_{22}\xi_{22} = \eta_{21}\eta_{12} + \eta_{22}^2 + \eta_{23}\eta_{32} + \eta_{24}\eta_{42}\\
&\eta_{22}\eta_{23} + \eta_{23} = \eta_{21}\eta_{13} + \eta_{22}\xi_{23} + \eta_{23}\eta_{33} + \eta_{24}\eta_{43}\\
&\eta_{22}\eta_{24} + \eta_{24} = \eta_{21}\eta_{14} + \eta_{22}\xi_{24} + \eta_{23}\eta_{34} + \eta_{24}\eta_{44}\\
&\eta_{31} + \eta_{33}\eta_{31} = \eta_{31}\eta_{11} + \eta_{32}\eta_{21} + \eta_{33}\xi_{31} + \eta_{34}\eta_{41}\\
&\eta_{32} + \eta_{33}\eta_{32} = \eta_{31}\eta_{12} + \eta_{32}\eta_{22} + \eta_{33}\xi_{32} + \eta_{34}\eta_{42}\\
&\eta_{33}\xi_{33} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43}\\
&\eta_{34} + \eta_{33}\eta_{34} = \eta_{31}\eta_{14} + \eta_{32}\eta_{24} + \eta_{33}\xi_{34} + \eta_{34}\eta_{44}\\
&\eta_{41} + \eta_{44}\eta_{41} = \eta_{41}\eta_{11} + \eta_{42}\eta_{21} + \eta_{43}\eta_{31} + \eta_{44}\xi_{41}\\
&\eta_{42} + \eta_{44}\eta_{42} = \eta_{41}\eta_{12} + \eta_{42}\eta_{22} + \eta_{43}\eta_{32} + \eta_{44}\xi_{42}\\
&\eta_{43} + \eta_{44}\eta_{43} = \eta_{41}\eta_{13} + \eta_{42}\eta_{23} + \eta_{43}\eta_{33} + \eta_{44}\xi_{43}\\
&\eta_{44}\xi_{44} = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2
\end{aligned}\right\} \tag{8}$$

Example 3.3: Let $F(X(n)) = \Xi X(n)$ be a system of difference equation then its self-semi-homogenous of order one such that

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ there exists } \mathrm{H} = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To show this system is self-semi-homogeneous of order one, Theorem 3.1 will be applied as follows:

$F(X(n)) = \Xi X(n)$

$$\begin{pmatrix} f_1\\ f_2\\ f_3\\ f_4 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} f_1(x_1)\\ f_2(x_2)\\ f_3(x_3)\\ f_4(x_4) \end{pmatrix} = \begin{pmatrix} x_1 + x_4\\ x_2 - x_4\\ x_3\\ x_4 \end{pmatrix}$$

$$F\big(\mathrm{H}x(n)\big) = \mathrm{H}F\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ f_4(x_4) \end{pmatrix}$$

$$\begin{pmatrix} f_1(x_1 + x_4) \\ f_2(x_2 - x_4) \\ f_3(x_3) \\ f_4(x_4) \end{pmatrix} = \begin{pmatrix} f_1(x_1) + f_4(x_4) \\ f_2(x_2) - f_4(x_4) \\ f_3(x_3) \\ f_4(x_4) \end{pmatrix}$$

$$\begin{pmatrix} x_1 + x_4 + x_4 \\ x_2 - x_4 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_4 + x_4 \\ x_2 - x_4 - x_4 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 2x_4 \\ x_2 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_4 \\ x_2 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

$\therefore F\big(\mathrm{H}x(n)\big) = \mathrm{H}F\big(x(n)\big)$

Alternatively, resolve via corollary (3-2).

$\eta_{11}\xi_{11} = \eta_{11}^2 + \eta_{12}\eta_{21} + \eta_{13}\eta_{31} + \eta_{14}\eta_{41}$, that is $1 \cdot 1 = (1)^2 + 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 0 = 1$

$\eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\xi_{12} + \eta_{12}\eta_{22} + \eta_{13}\eta_{32} + \eta_{14}\eta_{42}$, that is $1 \cdot 0 + 0 = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 0$

$\eta_{11}\eta_{13} + \eta_{13} = \eta_{11}\xi_{13} + \eta_{12}\eta_{23} + \eta_{13}\eta_{33} + \eta_{14}\eta_{43}$, that is $1 \cdot 0 + 0 = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0$

$\eta_{11}\eta_{14} + \eta_{14} = \eta_{11}\xi_{14} + \eta_{12}\eta_{24} + \eta_{13}\eta_{34} + \eta_{14}\eta_{44}$, that is $1 \cdot 1 + 1 = 1 \cdot 1 + 0 \cdot -1 + 0 \cdot 1 + 1 \cdot 1 = 2$

$\eta_{21} + \eta_{22}\eta_{21} = \eta_{21}\eta_{11} + \eta_{22}\xi_{21} + \eta_{23}\eta_{31} + \eta_{24}\eta_{41}$, that is $0 + 1 \cdot 0 = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + (-1) \cdot 0 = 0$

$\eta_{22}\xi_{22} = \eta_{21}\eta_{12} + \eta_{22}^2 + \eta_{23}\eta_{32} + \eta_{24}\eta_{42}$, that is $1 \cdot 1 = 0 \cdot 0 + (1)^2 + 0 \cdot 0 + (-1) \cdot 0 = 1$

$\eta_{22}\eta_{23} + \eta_{23} = \eta_{21}\eta_{13} + \eta_{22}\xi_{23} + \eta_{23}\eta_{33} + \eta_{24}\eta_{43}$, that is $1 \cdot 0 + 0 = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + (-1) \cdot 0 = 0$

$\eta_{22}\eta_{24} + \eta_{24} = \eta_{21}\eta_{14} + \eta_{22}\xi_{24} + \eta_{23}\eta_{34} + \eta_{24}\eta_{44}$, that is $1 \cdot -1 + (-1) = 0 \cdot 1 + 1 \cdot (-1) + 0 \cdot 0 + (-1) \cdot 1 = -2$

$\eta_{31} + \eta_{33}\eta_{31} = \eta_{31}\eta_{11} + \eta_{32}\eta_{21} + \eta_{33}\xi_{31} + \eta_{34}\eta_{41}$, that is $0 + 1 \cdot 0 = 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 0 = 0$

$\eta_{32} + \eta_{33}\eta_{32} = \eta_{31}\eta_{12} + \eta_{32}\eta_{22} + \eta_{33}\xi_{32} + \eta_{34}\eta_{42}$, that is $0 + 1 \cdot 0 = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 = 0$

$\eta_{33}\xi_{33} = \eta_{31}\eta_{13} + \eta_{32}\eta_{23} + \eta_{33}^2 + \eta_{34}\eta_{43}$, that is $1 \cdot 1 = 0 \cdot 0 + 0 \cdot 0 + (1)^2 + 0 \cdot 0 = 1$

$\eta_{34} + \eta_{33}\eta_{34} = \eta_{31}\eta_{14} + \eta_{32}\eta_{24} + \eta_{33}\xi_{34} + \eta_{34}\eta_{44}$, that is $0 + 1 \cdot 0 = 0 \cdot 1 + 0 \cdot (-1) + 1 \cdot 0 + 0 \cdot 1 = 0$

$\eta_{41} + \eta_{44}\eta_{41} = \eta_{41}\eta_{11} + \eta_{42}\eta_{21} + \eta_{43}\eta_{31} + \eta_{44}\xi_{41}$, that is $0 + 1 \cdot 0 = 0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 0 = 0$

$\eta_{42} + \eta_{44}\eta_{42} = \eta_{41}\eta_{12} + \eta_{42}\eta_{22} + \eta_{43}\eta_{32} + \eta_{44}\xi_{42}$, that is $0 + 1 \cdot 0 = 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 0$

$\eta_{43} + \eta_{44}\eta_{43} = \eta_{41}\eta_{13} + \eta_{42}\eta_{23} + \eta_{43}\eta_{33} + \eta_{44}\xi_{43}$, that is $0 + 1 \cdot 0 = 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0$

$\eta_{44}\xi_{44} = \eta_{41}\eta_{14} + \eta_{42}\eta_{24} + \eta_{43}\eta_{34} + \eta_{44}^2$, that is $1 \cdot 1 = +0 \cdot 1 + 0 \cdot (-1) + 0 \cdot 0 + (1)^2 = 1$

Remark 3.4: To resolve the aforementioned example, one may utilize Equation (4).

Case 2: When $(m \geq 2)$.

Theorem 3**-5:** *Self-semi-homogeneity of order greater than one is achieved by the homogeneous system of difference equations (2) if and only if the matrix Ξ serves as a solution to the subsequent algebraic system:*

$$\left.\begin{aligned}
\eta_{11}^2 &= A_{11}\xi_{11} + A_{12}\eta_{21} + A_{13}\eta_{31} + A_{14}\eta_{41}\\
\eta_{11}\eta_{12} + \eta_{12} &= A_{11}\xi_{12} + A_{12}\eta_{22} + A_{13}\eta_{32} + A_{14}\eta_{42}\\
\eta_{11}\eta_{13} + \eta_{13} &= A_{11}\xi_{13} + A_{12}\eta_{23} + A_{13}\eta_{33} + A_{14}\eta_{43}\\
\eta_{11}\eta_{14} + \eta_{14} &= A_{11}\xi_{14} + A_{12}\eta_{24} + A_{13}\eta_{34} + A_{14}\eta_{44}\\
\eta_{21} + \eta_{22}\eta_{21} &= A_{21}\eta_{11} + A_{22}\xi_{21} + A_{23}\eta_{31} + A_{24}\eta_{41}\\
\eta_{22}^2 &= A_{21}\eta_{12} + A_{22}\xi_{22} + A_{23}\eta_{32} + A_{24}\eta_{42}\\
\eta_{22}\eta_{23} + \eta_{23} &= A_{21}\eta_{13} + A_{22}\xi_{23} + A_{23}\eta_{33} + A_{24}\eta_{43}\\
\eta_{22}\eta_{24} + \eta_{24} &= A_{21}\eta_{14} + A_{22}\xi_{24} + A_{23}\eta_{34} + A_{24}\eta_{44}\\
\eta_{31} + \eta_{33}\eta_{31} &= A_{31}\eta_{11} + A_{32}\eta_{21} + A_{33}\xi_{31} + A_{34}\eta_{41}\\
\eta_{32} + \eta_{33}\eta_{32} &= A_{31}\eta_{12} + A_{32}\eta_{22} + A_{33}\xi_{32} + A_{34}\eta_{42}\\
\eta_{33}^2 &= A_{31}\eta_{13} + A_{32}\eta_{23} + A_{33}\xi_{33} + A_{34}\eta_{43}\\
\eta_{34} + \eta_{33}\eta_{34} &= A_{31}\eta_{14} + A_{32}\eta_{24} + A_{33}\xi_{34} + A_{34}\eta_{44}\\
\eta_{41} + \eta_{44}\eta_{41} &= A_{41}\eta_{11} + A_{42}\eta_{21} + A_{43}\eta_{31} + A_{44}\xi_{41}\\
\eta_{42} + \eta_{44}\eta_{42} &= A_{41}\eta_{12} + A_{42}\eta_{22} + A_{43}\eta_{32} + A_{44}\xi_{42}\\
\eta_{43} + \eta_{44}\eta_{43} &= A_{41}\eta_{13} + A_{42}\eta_{23} + A_{43}\eta_{33} + A_{44}\xi_{43}\\
\eta_{44}^2 &= A_{41}\xi_{14} + A_{42}\eta_{24} + A_{43}\eta_{34} + A_{44}\xi_{44}
\end{aligned}\right\} \quad (9)$$

where the matrix

$$\mathrm{H} = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}, \Xi = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix} \text{ and}$$

$$\mathrm{H}^m = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

Proof: The identical method utilized in proof theorem (3.1)

Corollary 3.6: *If the following conditions are satisfied, and only then, is Equation (2) self-semi homogeneous of order greater than one:*

$$\left.\begin{aligned}
&\eta_{11}^2 = A_{11}\xi_{11} + A_{12}\eta_{21} + A_{13}\eta_{31} + A_{14}\eta_{41}\\
&\eta_{11}\eta_{12} + \eta_{12} = A_{11}\xi_{12} + A_{12}\eta_{22} + A_{13}\eta_{32} + A_{14}\eta_{42}\\
&\eta_{11}\eta_{13} + \eta_{13} = A_{11}\xi_{13} + A_{12}\eta_{23} + A_{13}\eta_{33} + A_{14}\eta_{43}\\
&\eta_{11}\eta_{14} + \eta_{14} = A_{11}\xi_{14} + A_{12}\eta_{24} + A_{13}\eta_{34} + A_{14}\eta_{44}\\
&\eta_{21} + \eta_{22}\eta_{21} = A_{21}\eta_{11} + A_{22}\xi_{21} + A_{23}\eta_{31} + A_{24}\eta_{41}\\
&\eta_{22}^2 = A_{21}\eta_{12} + A_{22}\xi_{22} + A_{23}\eta_{32} + A_{24}\eta_{42}\\
&\eta_{22}\eta_{23} + \eta_{23} = A_{21}\eta_{13} + A_{22}\xi_{23} + A_{23}\eta_{33} + A_{24}\eta_{43}\\
&\eta_{22}\eta_{24} + \eta_{24} = A_{21}\eta_{14} + A_{22}\xi_{24} + A_{23}\eta_{34} + A_{24}\eta_{44}\\
&\eta_{31} + \eta_{33}\eta_{31} = A_{31}\eta_{11} + A_{32}\eta_{21} + A_{33}\xi_{31} + A_{34}\eta_{41}\\
&\eta_{32} + \eta_{33}\eta_{32} = A_{31}\eta_{12} + A_{32}\eta_{22} + A_{33}\xi_{32} + A_{34}\eta_{42}\\
&\eta_{33}^2 = A_{31}\eta_{13} + A_{32}\eta_{23} + A_{33}\xi_{33} + A_{34}\eta_{43}\\
&\eta_{34} + \eta_{33}\eta_{34} = A_{31}\eta_{14} + A_{32}\eta_{24} + A_{33}\xi_{34} + A_{34}\eta_{44}\\
&\eta_{41} + \eta_{44}\eta_{41} = A_{41}\eta_{11} + A_{42}\eta_{21} + A_{43}\eta_{31} + A_{44}\xi_{41}\\
&\eta_{42} + \eta_{44}\eta_{42} = A_{41}\eta_{12} + A_{42}\eta_{22} + A_{43}\eta_{32} + A_{44}\xi_{42}\\
&\eta_{43} + \eta_{44}\eta_{43} = A_{41}\eta_{13} + A_{42}\eta_{23} + A_{43}\eta_{33} + A_{44}\xi_{43}\\
&\eta_{44}^2 = A_{41}\xi_{14} + A_{42}\eta_{24} + A_{43}\eta_{34} + A_{44}\xi_{44}
\end{aligned}\right\} \tag{10}$$

Example 3.7: Let $F(X(n)) = \Xi X(n)$ be a system of difference equation then its self-semi-homogenous of order five such that

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ there exists } \mathrm{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: By using theorem above

$F(X(n)) = \Xi X(n)$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} \Rightarrow \begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(Z) \\ f_4(Z) \end{pmatrix} = \begin{pmatrix} x \\ 3x - 3z \\ z \\ w \end{pmatrix}$$

$F\big(\mathrm{H}x(n)\big) = \mathrm{H}^5 F\big(x(n)\big)$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(Z) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} f_1(x) \\ f_2(3x-3z) \\ f_3(z) \\ f_4(w) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ 3f_1(x)-3f_3(z) \\ f_3(Z) \\ f_4(w) \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ 3x-3z \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ 3x-3z \\ z \\ w \end{pmatrix}$$

$\therefore F(\mathrm{H}x(n)) = \mathrm{H}F(x(n))$

Instead, the solution can be obtained through corollary (3-2).

4. P-self-semi-homogenous of order m

Case 1: When $m = 1$.

Theorem 4.1: *The homogeneous system of difference equations (2) is p-self-semi-homogenous of order one if and only if the matrix Ξ is a solution of the following algebraic system:*

$$\left.\begin{array}{l}
\eta_{11}\xi_{11} = P\eta_{11}^2 + P\eta_{12}\eta_{21} + P\eta_{13}\eta_{31} + P\eta_{14}\eta_{41} \\
\eta_{11}\eta_{12} + \eta_{12} = P\eta_{11}\xi_{12} + P\eta_{12}\eta_{22} + P\eta_{13}\eta_{32} + \eta_{14}\eta_{42} \\
\eta_{11}\eta_{13} + \eta_{13} = P\eta_{11}\xi_{13} + P\eta_{12}\eta_{23} + P\eta_{13}\eta_{33} + P\eta_{14}\eta_{43} \\
\eta_{11}\eta_{14} + \eta_{14} = P\eta_{11}\xi_{14} + P\eta_{12}\eta_{24} + P\eta_{13}\eta_{34} + P\eta_{14}\eta_{44} \\
\eta_{21} + \eta_{22}\eta_{21} = P\eta_{21}\eta_{11} + P\eta_{22}\xi_{21} + P\eta_{23}\eta_{31} + P\eta_{24}\eta_{41} \\
\eta_{22}\xi_{22} = P\eta_{21}\eta_{12} + P\eta_{22}^2 + P\eta_{23}\eta_{32} + P\eta_{24}\eta_{42} \\
\eta_{22}\eta_{23} + \eta_{23} = P\eta_{21}\eta_{13} + P\eta_{22}\xi_{23} + P\eta_{23}\eta_{33} + P\eta_{24}\eta_{43} \\
\eta_{22}\eta_{24} + \eta_{24} = P\eta_{21}\eta_{14} + P\eta_{22}\xi_{24} + P\eta_{23}\eta_{34} + P\eta_{24}\eta_{44} \\
\eta_{31} + \eta_{33}\eta_{31} = P\eta_{31}\eta_{11} + P\eta_{32}\eta_{21} + P\eta_{33}\xi_{31} + P\eta_{34}\eta_{41} \\
\eta_{32} + \eta_{33}\eta_{32} = P\eta_{31}\eta_{12} + P\eta_{32}\eta_{22} + P\eta_{33}\xi_{32} + P\eta_{34}\eta_{42} \\
\eta_{33}\xi_{33} = P\eta_{31}\eta_{13} + P\eta_{32}\eta_{23} + P\eta_{33}^2 + P\eta_{34}\eta_{43} \\
\eta_{34} + \eta_{33}\eta_{34} = P\eta_{31}\eta_{14} + P\eta_{32}\eta_{24} + P\eta_{33}\xi_{34} + P\eta_{34}\eta_{44} \\
\eta_{41} + \eta_{44}\eta_{41} = P\eta_{41}\eta_{11} + P\eta_{42}\eta_{21} + P\eta_{43}\eta_{31} + P\eta_{44}\xi_{41} \\
\eta_{42} + \eta_{44}\eta_{42} = P\eta_{41}\eta_{12} + P\eta_{42}\eta_{22} + P\eta_{43}\eta_{32} + P\eta_{44}\xi_{42} \\
\eta_{43} + \eta_{44}\eta_{43} = P\eta_{41}\eta_{13} + P\eta_{42}\eta_{23} + P\eta_{43}\eta_{33} + P\eta_{44}\xi_{43} \\
\eta_{44}\xi_{44} = P\eta_{41}\eta_{14} + P\eta_{42}\eta_{24} + P\eta_{43}\eta_{34} + P\eta_{44}^2
\end{array}\right\} \quad (11)$$

In the given matrix H = Ξ

Proof: Employing the same methodology as in proving theorem (3.1).

Corollary 4.2: *A homogenous system of differences Equation (2) is P-self-semi homogenous of order one if and only if the following conditions hold:*

$$\left.\begin{aligned}
&\eta_{11}\xi_{11} = P\eta_{11}^2 + P\eta_{12}\eta_{21} + P\eta_{13}\eta_{31} + P\eta_{14}\eta_{41}\\
&\eta_{11}\eta_{12} + \eta_{12} = P\eta_{11}\xi_{12} + P\eta_{12}\eta_{22} + P\eta_{13}\eta_{32} + \eta_{14}\eta_{42}\\
&\eta_{11}\eta_{13} + \eta_{13} = P\eta_{11}\xi_{13} + P\eta_{12}\eta_{23} + P\eta_{13}\eta_{33} + P\eta_{14}\eta_{43}\\
&\eta_{11}\eta_{14} + \eta_{14} = P\eta_{11}\xi_{14} + P\eta_{12}\eta_{24} + P\eta_{13}\eta_{34} + P\eta_{14}\eta_{44}\\
&\eta_{21} + \eta_{22}\eta_{21} = P\eta_{21}\eta_{11} + P\eta_{22}\xi_{21} + P\eta_{23}\eta_{31} + P\eta_{24}\eta_{41}\\
&\eta_{22}\xi_{22} = P\eta_{21}\eta_{12} + P\eta_{22}^2 + P\eta_{23}\eta_{32} + P\eta_{24}\eta_{42}\\
&\eta_{22}\eta_{23} + \eta_{23} = P\eta_{21}\eta_{13} + P\eta_{22}\xi_{23} + P\eta_{23}\eta_{33} + P\eta_{24}\eta_{43}\\
&\eta_{22}\eta_{24} + \eta_{24} = P\eta_{21}\eta_{14} + P\eta_{22}\xi_{24} + P\eta_{23}\eta_{34} + P\eta_{24}\eta_{44}\\
&\eta_{31} + \eta_{33}\eta_{31} = P\eta_{31}\eta_{11} + P\eta_{32}\eta_{21} + P\eta_{33}\xi_{31} + P\eta_{34}\eta_{41}\\
&\eta_{32} + \eta_{33}\eta_{32} = P\eta_{31}\eta_{12} + P\eta_{32}\eta_{22} + P\eta_{33}\xi_{32} + P\eta_{34}\eta_{42}\\
&\eta_{33}\xi_{33} = P\eta_{31}\eta_{13} + P\eta_{32}\eta_{23} + P\eta_{33}^2 + P\eta_{34}\eta_{43}\\
&\eta_{34} + \eta_{33}\eta_{34} = P\eta_{31}\eta_{14} + P\eta_{32}\eta_{24} + P\eta_{33}\xi_{34} + P\eta_{34}\eta_{44}\\
&\eta_{41} + \eta_{44}\eta_{41} = P\eta_{41}\eta_{11} + P\eta_{42}\eta_{21} + P\eta_{43}\eta_{31} + P\eta_{44}\xi_{41}\\
&\eta_{42} + \eta_{44}\eta_{42} = P\eta_{41}\eta_{12} + P\eta_{42}\eta_{22} + P\eta_{43}\eta_{32} + P\eta_{44}\xi_{42}\\
&\eta_{43} + \eta_{44}\eta_{43} = P\eta_{41}\eta_{13} + P\eta_{42}\eta_{23} + P\eta_{43}\eta_{33} + P\eta_{44}\xi_{43}\\
&\eta_{44}\xi_{44} = P\eta_{41}\eta_{14} + P\eta_{42}\eta_{24} + P\eta_{43}\eta_{34} + P\eta_{44}^2
\end{aligned}\right\} \qquad (12)$$

Example 4.3: Let $F(X(n)) = \Xi X(n)$ be a system of difference equation then its self-semi-homogenous of order three such that $\Xi = \begin{pmatrix} \frac{1}{7} & 0 & \frac{-3}{196} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, there exists $\mathrm{H} = \begin{pmatrix} \frac{1}{7} & 0 & \frac{-3}{196} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: By using theorem above

$F(X(n)) = \Xi X(n)$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & 0 & \frac{-3}{196} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} \Rightarrow \begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(Z) \\ f_4(w) \end{pmatrix} = \begin{pmatrix} \frac{1}{7}x - \frac{3}{196}z \\ 0 \\ x + \frac{1}{7}z \\ 0 \end{pmatrix}$$

$F\big(\mathrm{H}x(n)\big) = P\mathrm{H}F\big(x(n)\big)$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\begin{pmatrix} \frac{1}{7} & 0 & \frac{-3}{196} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} = 4\begin{pmatrix} \frac{1}{7} & 0 & \frac{-3}{196} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(Z) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} \left(\frac{1}{49}x + \frac{-24}{1372}z\right) \\ 0 \\ \left(\frac{8}{7}x + \frac{1}{49}z\right) \\ 0 \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{49}x + \frac{-24}{1372}z\right) \\ 0 \\ \left(\frac{8}{7}x + \frac{1}{49}z\right) \\ 0 \end{pmatrix}$$

$\therefore F\big(\mathrm{H}x(n)\big) = P\mathrm{H}F\big(x(n)\big)$

Alternatively, resolve via corollary (4-2)

Case 2: When $(m \geq 2)$.

Theorem 4.4: *The homogeneous system of difference equations (2) is said to be P-self-semi-homogenous of order greater than one if and only if the matrix Ξ is a solution of the following algebraic system:*

$$\left.\begin{aligned}
&\eta_{11}^2 = PA_{11}\xi_{11} + PA_{12}\eta_{21} + PA_{13}\eta_{31} + PA_{14}\eta_{41} \\
&\eta_{11}\eta_{12} + \eta_{12} = PA_{11}\xi_{12} + PA_{12}\eta_{22} + PA_{13}\eta_{32} + PA_{14}\eta_{42} \\
&\eta_{11}\eta_{13} + \eta_{13} = PA_{11}\xi_{13} + PA_{12}\eta_{23} + PA_{13}\eta_{33} + PA_{14}\eta_{43} \\
&\eta_{11}\eta_{14} + \eta_{14} = PA_{11}\xi_{14} + PA_{12}\eta_{24} + PA_{13}\eta_{34} + PA_{14}\eta_{44} \\
&\eta_{21} + \eta_{22}\eta_{21} = PA_{21}\eta_{11} + PA_{22}\xi_{21} + PA_{23}\eta_{31} + PA_{24}\eta_{41} \\
&\eta_{22}^2 = PA_{21}\eta_{12} + PA_{22}\xi_{22} + PA_{23}\eta_{32} + PA_{24}\eta_{42} \\
&\eta_{22}\eta_{23} + \eta_{23} = PA_{21}\eta_{13} + PA_{22}\xi_{23} + PA_{23}\eta_{33} + PA_{24}\eta_{43} \\
&\eta_{22}\eta_{24} + \eta_{24} = PA_{21}\eta_{14} + PA_{22}\xi_{24} + PA_{23}\eta_{34} + PA_{24}\eta_{44} \\
&\eta_{31} + \eta_{33}\eta_{31} = PA_{31}\eta_{11} + PA_{32}\eta_{21} + PA_{33}\xi_{31} + PA_{34}\eta_{41} \\
&\eta_{32} + \eta_{33}\eta_{32} = PA_{31}\eta_{12} + PA_{32}\eta_{22} + PA_{33}\xi_{32} + PA_{34}\eta_{42} \\
&\eta_{33}^2 = PA_{31}\eta_{13} + PA_{32}\eta_{23} + PA_{33}\xi_{33} + PA_{34}\eta_{43} \\
&\eta_{34} + \eta_{33}\eta_{34} = PA_{31}\eta_{14} + PA_{32}\eta_{24} + PA_{33}\xi_{34} + PA_{34}\eta_{44} \\
&\eta_{41} + \eta_{44}\eta_{41} = PA_{41}\eta_{11} + PA_{42}\eta_{21} + PA_{43}\eta_{31} + PA_{44}\xi_{41} \\
&\eta_{42} + \eta_{44}\eta_{42} = PA_{41}\eta_{12} + PA_{42}\eta_{22} + PA_{43}\eta_{32} + PA_{44}\xi_{42} \\
&\eta_{43} + \eta_{44}\eta_{43} = PA_{41}\eta_{13} + PA_{42}\eta_{23} + PA_{43}\eta_{33} + PA_{44}\xi_{43} \\
&\eta_{44}^2 = PA_{41}\xi_{14} + PA_{42}\eta_{24} + PA_{43}\eta_{34} + PA_{44}\xi_{44}
\end{aligned}\right\} \quad (13)$$

where the matrix

$$\mathrm{H} = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}, \Xi = \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix} \text{ and } \mathrm{H}^m$$

$$= \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

Conversely suppose the homogeneous system of difference equations (2) is P-self-semi-homogenous of order greater than one to prove the equation (13) holds:

$$F\big(\mathrm{H}x(n)\big) = \mathrm{PH}^m F\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\left(\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right)$$

$$= p\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(z) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} f_1(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) \\ f_2(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) \\ f_3(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) \\ f_3(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix}$$

$$= P\begin{pmatrix} A_{11}f_1(x) + A_{12}f_2(y) + A_{13}f_3(y) + A_{14}f_4(w) \\ A_{21}f_1(x) + A_{22}f_2(y) + A_{23}f_3(y) + A_{24}f_4(w) \\ A_{31}f_1(x) + A_{32}f_2(y) + A_{33}f_3(y) + A_{34}f_4(w) \\ A_{41}f_1(x) + A_{42}f_2(y) + A_{43}f_3(y) + A_{44}f_4(w) \end{pmatrix}$$

By implementing equation (6)

$$\begin{pmatrix} \xi_{11}(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) + \xi_{12}y + \xi_{13}z + \xi_{14}w \\ \xi_{21}x + \xi_{22}(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) + \xi_{23}z + \xi_{24}w \\ \xi_{31}x + \xi_{32}y + \xi_{33}(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) + \xi_{34}w \\ \xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix}$$

$$= p\begin{pmatrix} A_{11}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{12}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{13}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{14}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ A_{21}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{22}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{23}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{24}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ A_{31}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{32}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{33}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{34}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ A_{41}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{42}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{43}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{44}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \end{pmatrix}$$

$$\left.\begin{aligned} &\eta_{11}\xi_{11} = pA_{11}\xi_{11} + pA_{12}\xi_{21} + pA_{13}\xi_{31} + pA_{14}\xi_{41} \\ &\xi_{11}\eta_{12} + \xi_{12} = pA_{11}\xi_{12} + pA_{12}\xi_{22} + pA_{13}\xi_{32} + pA_{14}\xi_{42} \\ &\xi_{11}\eta_{13} + \xi_{13} = pA_{11}\xi_{13} + pA_{12}\xi_{23} + pA_{13}\xi_{33} + pA_{14}\xi_{43} \\ &\xi_{11}\eta_{14} + \xi_{14} = pA_{11}\xi_{14} + pA_{12}\xi_{24} + pA_{13}\xi_{34} + pA_{14}\xi_{44} \\ &\xi_{21} + \xi_{22}\eta_{21} = pA_{21}\eta_{11} + pA_{22}\xi_{21} + pA_{23}\xi_{31} + pA_{24}\xi_{41} \\ &\eta_{22}\xi_{22} = pA_{21}\xi_{12} + pA_{22}\xi_{22} + pA_{23}\xi_{32} + pA_{24}\xi_{42} \\ &\xi_{22}\eta_{23} + \xi_{23} = pA_{21}\xi_{13} + pA_{22}\xi_{23} + pA_{23}\xi_{33} + pA_{24}\xi_{43} \\ &\xi_{22}\eta_{24} + \xi_{24} = pA_{21}\xi_{14} + pA_{22}\xi_{24} + pA_{23}\xi_{34} + pA_{24}\xi_{44} \\ &\xi_{31} + \xi_{33}\eta_{31} = pA_{31}\xi_{11} + pA_{32}\xi_{21} + pA_{33}\xi_{31} + pA_{34}\xi_{41} \\ &\xi_{32} + \xi_{33}\eta_{32} = pA_{31}\xi_{12} + pA_{32}\xi_{22} + pA_{33}\xi_{32} + pA_{34}\xi_{42} \\ &\eta_{33}\xi_{33} = pA_{31}\xi_{13} + pA_{32}\xi_{23} + pA_{33}\xi_{33} + pA_{34}\xi_{43} \\ &\xi_{34} + \xi_{33}\eta_{34} = pA_{31}\xi_{14} + pA_{32}\xi_{24} + pA_{33}\xi_{34} + pA_{34}\xi_{44} \\ &\xi_{41} + \xi_{44}\eta_{41} = pA_{41}\xi_{11} + pA_{42}\xi_{21} + pA_{43}\xi_{31} + pA_{44}\xi_{41} \\ &\xi_{42} + \xi_{44}\eta_{42} = pA_{41}\xi_{12} + pA_{42}\xi_{22} + pA_{43}\xi_{32} + pA_{44}\xi_{42} \\ &\xi_{43} + \xi_{44}\eta_{43} = pA_{41}\xi_{13} + pA_{42}\xi_{23} + pA_{43}\xi_{33} + pA_{44}\xi_{43} \\ &\eta_{44}\xi_{44} = pA_{41}\xi_{14} + pA_{42}\xi_{24} + pA_{43}\xi_{34} + pA_{44}\xi_{44} \end{aligned}\right\} \tag{14}$$

Given the equality of matrices H and Ξ, their components are equivalent and can be expressed as follows:

$$\eta_{11} = \xi_{11}, \eta_{12} = \xi_{12}, \eta_{21} = \xi_{21}, \dots, \eta_{44} = \xi_{44}$$

Now, the subsequent may be employed to rephrase clause (14).

$$\eta_{11}^2 = PA_{11}\xi_{11} + PA_{12}\eta_{21} + PA_{13}\eta_{31} + PA_{14}\eta_{41}$$
$$\eta_{11}\eta_{12} + \eta_{12} = PA_{11}\xi_{12} + PA_{12}\eta_{22} + PA_{13}\eta_{32} + PA_{14}\eta_{42}$$
$$\eta_{11}\eta_{13} + \eta_{13} = PA_{11}\xi_{13} + PA_{12}\eta_{23} + PA_{13}\eta_{33} + PA_{14}\eta_{43}$$
$$\eta_{11}\eta_{14} + \eta_{14} = PA_{11}\xi_{14} + PA_{12}\eta_{24} + PA_{13}\eta_{34} + PA_{14}\eta_{44}$$
$$\eta_{21} + \eta_{22}\eta_{21} = PA_{21}\eta_{11} + PA_{22}\xi_{21} + PA_{23}\eta_{31} + PA_{24}\eta_{41}$$
$$\eta_{22}^2 = PA_{21}\eta_{12} + PA_{22}\xi_{22} + PA_{23}\eta_{32} + PA_{24}\eta_{42}$$
$$\eta_{22}\eta_{23} + \eta_{23} = PA_{21}\eta_{13} + PA_{22}\xi_{23} + PA_{23}\eta_{33} + PA_{24}\eta_{43}$$
$$\eta_{22}\eta_{24} + \eta_{24} = PA_{21}\eta_{14} + PA_{22}\xi_{24} + PA_{23}\eta_{34} + PA_{24}\eta_{44}$$
$$\eta_{31} + \eta_{33}\eta_{31} = PA_{31}\eta_{11} + PA_{32}\eta_{21} + PA_{33}\xi_{31} + PA_{34}\eta_{41}$$
$$\eta_{32} + \eta_{33}\eta_{32} = PA_{31}\eta_{12} + PA_{32}\eta_{22} + PA_{33}\xi_{32} + PA_{34}\eta_{42}$$
$$\eta_{33}^2 = PA_{31}\eta_{13} + PA_{32}\eta_{23} + PA_{33}\xi_{33} + PA_{34}\eta_{43}$$
$$\eta_{34} + \eta_{33}\eta_{34} = PA_{31}\eta_{14} + PA_{32}\eta_{24} + PA_{33}\xi_{34} + PA_{34}\eta_{44}$$
$$\eta_{41} + \eta_{44}\eta_{41} = PA_{41}\eta_{11} + PA_{42}\eta_{21} + PA_{43}\eta_{31} + PA_{44}\xi_{41}$$
$$\eta_{42} + \eta_{44}\eta_{42} = PA_{41}\eta_{12} + PA_{42}\eta_{22} + PA_{43}\eta_{32} + PA_{44}\xi_{42}$$
$$\eta_{43} + \eta_{44}\eta_{43} = PA_{41}\eta_{13} + PA_{42}\eta_{23} + PA_{43}\eta_{33} + PA_{44}\xi_{43}$$
$$\eta_{44}^2 = PA_{41}\eta_{14} + PA_{42}\eta_{24} + PA_{43}\eta_{34} + PA_{44}\xi_{44}$$

conversely

F is P-self-semi-homogenous of order greater than one if there exists a matrix H such that $F(\mathrm{H}x(n)) = P\mathrm{H}F(x(n))$ we claim that this matrix is defined as:

$$\boldsymbol{B} = \begin{pmatrix} \frac{\eta_{11}^2 - PA_{12}\eta_{21} - PA_{13}\eta_{31} - PA_{14}\eta_{41}}{PA_{11}} & \frac{\eta_{11}\eta_{12} + \eta_{12} - PA_{12}\eta_{22} - PA_{13}\eta_{32} - PA_{14}\eta_{42}}{PA_{11}} & \frac{\eta_{11}\eta_{13} + \eta_{13} - PA_{12}\eta_{23} - PA_{13}\eta_{33} - PA_{14}\eta_{43}}{PA_{11}} & \frac{\eta_{11}\eta_{14} + \eta_{14} - PA_{12}\eta_{24} - PA_{13}\eta_{34} - PA_{14}\eta_{44}}{PA_{11}} \\ \frac{\eta_{22}\eta_{21} + \eta_{21} - PA_{21}\eta_{11} - PA_{23}\eta_{31} - PA_{24}\eta_{41}}{PA_{22}} & \frac{\eta_{22}^2 - PA_{21}\eta_{12} - PA_{23}\eta_{32} - PA_{24}\eta_{42}}{PA_{22}} & \frac{\eta_{22}\eta_{23} + \eta_{23} - PA_{21}\eta_{13} - PA_{23}\eta_{33} - PA_{24}\eta_{43}}{PA_{22}} & \frac{\eta_{22}\eta_{24} + \eta_{24} - PA_{21}\eta_{14} - PA_{23}\eta_{34} - PA_{24}\eta_{44}}{PA_{22}} \\ \frac{\eta_{33}\eta_{31} + \eta_{31} - PA_{31}\eta_{11} - PA_{32}\eta_{21} - PA_{34}\eta_{41}}{PA_{33}} & \frac{\eta_{33}\eta_{32} + \eta_{32} - PA_{31}\eta_{12} - PA_{32}\eta_{22} - PA_{34}\eta_{42}}{PA_{33}} & \frac{\eta_{33}^2 - PA_{31}\eta_{13} - PA_{32}\eta_{23} - PA_{34}\eta_{43}}{PA_{33}} & \frac{\eta_{33}\eta_{34} + \eta_{34} - PA_{31}\eta_{14} - PA_{32}\eta_{24} - PA_{34}\eta_{44}}{PA_{33}} \\ \frac{\eta_{44}\eta_{41} + \eta_{41} - PA_{41}\eta_{11} - PA_{42}\eta_{21} - PA_{43}\eta_{31}}{PA_{44}} & \frac{\eta_{44}\eta_{42} + \eta_{42} - PA_{41}\eta_{12} - PA_{42}\eta_{22} - PA_{43}\eta_{32}}{PA_{44}} & \frac{\eta_{44}\eta_{43} + \eta_{43} - PA_{41}\eta_{13} - PA_{42}\eta_{23} - PA_{43}\eta_{33}}{PA_{44}} & \frac{\eta_{44}^2 - PA_{41}\eta_{14} - PA_{42}\eta_{24} - PA_{43}\eta_{34}}{PA_{44}} \end{pmatrix}$$

$$F(\mathrm{H}x(n)) = \mathrm{PH}^n F(x(n))$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\left(\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\ \eta_{41} & \eta_{42} & \eta_{43} & \eta_{44} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = P\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(z) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} f_1(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) \\ f_2(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) \\ f_3(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) \\ f_3(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix} = P \begin{pmatrix} A_{11}f_1(x) + A_{12}f_2(y) + A_{13}f_3(y) + A_{14}f_4(w) \\ A_{21}f_1(x) + A_{22}f_2(y) + A_{23}f_3(y) + A_{24}f_4(w) \\ A_{31}f_1(x) + A_{32}f_2(y) + A_{33}f_3(y) + A_{34}f_4(w) \\ A_{41}f_1(x) + A_{42}f_2(y) + A_{43}f_3(y) + A_{44}f_4(w) \end{pmatrix}$$

By using equation (6)

$$\begin{pmatrix} \xi_{11}(\eta_{11}x + \eta_{12}y + \eta_{13}z + \eta_{14}w) + \xi_{12}y + \xi_{13}z + \xi_{14}w \\ \xi_{21}x + \xi_{22}(\eta_{21}x + \eta_{22}y + \eta_{23}z + \eta_{24}w) + \xi_{23}z + \xi_{24}w \\ \xi_{31}x + \xi_{32}y + \xi_{33}(\eta_{31}x + \eta_{32}y + \eta_{33}z + \eta_{34}w) + \xi_{34}w \\ \xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}(\eta_{41}x + \eta_{42}y + \eta_{43}z + \eta_{44}w) \end{pmatrix}$$

$$= P\begin{pmatrix} A_{11}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{12}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{13}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{14}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ A_{21}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{22}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{23}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{24}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ A_{31}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{32}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{33}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{34}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \\ A_{41}(\xi_{11}x + \xi_{12}y + \xi_{13}z + \xi_{14}w) + A_{42}(\xi_{21}x + \xi_{22}y + \xi_{23}z + \xi_{24}w) + A_{43}(\xi_{31}x + \xi_{32}y + \xi_{33}z + \xi_{34}w) + A_{44}(\xi_{41}x + \xi_{42}y + \xi_{43}z + \xi_{44}w) \end{pmatrix}$$

Therefore:

$$\eta_{11}^2 = PA_{11}\xi_{11} + PA_{12}\eta_{21} + PA_{13}\eta_{31} + PA_{14}\eta_{41}$$

$$\eta_{11}^2 = PA_{11}\frac{\eta_{11}^2 - PA_{12}\eta_{21} - PA_{13}\eta_{31} - PA_{14}\eta_{41}}{PA_{11}} + PA_{12}\eta_{21} + PA_{13}\eta_{31} + PA_{14}\eta_{41}$$

$$\Longrightarrow \therefore \eta_{11}^2 = \eta_{11}^2$$

Also,

$$\eta_{11}\eta_{12} + \eta_{12} = PA_{11}\xi_{12} + PA_{12}\eta_{22} + PA_{13}\eta_{32} + PA_{14}\eta_{42}$$

$$\eta_{11}\eta_{12} + \eta_{12} = PA_{11}\frac{\eta_{11}\eta_{12} + \eta_{12} - PA_{12}\eta_{22} - PA_{13}\eta_{32} - PA_{14}\eta_{42}}{PA_{11}} + PA_{12}\eta_{22} + PA_{13}\eta_{32} + PA_{14}\eta_{42}$$

$$\therefore \eta_{11}\eta_{12} + \eta_{12} = \eta_{11}\eta_{12} + \eta_{12}$$

Also,

$$\eta_{11}\eta_{13} + \eta_{13} = PA_{11}\xi_{13} + PA_{12}\eta_{23} + PA_{13}\eta_{33} + PA_{14}\eta_{43}$$

$$\eta_{11}\eta_{13} + \eta_{13} = PA_{11}\frac{\eta_{11}\eta_{13} + \eta_{13} - PA_{12}\eta_{23} - PA_{13}\eta_{33} - PA_{14}\eta_{43}}{PA_{11}} + PA_{12}\eta_{23} + PA_{13}\eta_{33} + PA_{14}\eta_{43}$$

$$\therefore \eta_{11}\eta_{13} + \eta_{13} = \eta_{11}\eta_{13} + \eta_{13}$$

Also,

$$\eta_{11}\eta_{14} + \eta_{14} = PA_{11}\xi_{14} + PA_{12}\eta_{24} + PA_{13}\eta_{34} + PA_{14}\eta_{44}$$

$$\eta_{11}\eta_{14} + \eta_{14} = PA_{11}\frac{\eta_{11}\eta_{14} + \eta_{14} - PA_{12}\eta_{24} - PA_{13}\eta_{34} - PA_{14}\eta_{44}}{PA_{11}} + PA_{12}\eta_{24} + PA_{13}\eta_{34} + PA_{14}\eta_{44}$$

$$\therefore \eta_{11}\eta_{14} + \eta_{14} = \eta_{11}\eta_{14} + \eta_{14}$$

Also,

$$\eta_{21} + \eta_{22}\eta_{21} = PA_{21}\eta_{11} + PA_{22}\xi_{21} + PA_{23}\eta_{31} + PA_{24}\eta_{41}$$

$$\eta_{21}+\eta_{22}\eta_{21}=PA_{21}\eta_{11}+PA_{22}\frac{\eta_{22}\eta_{21}+\eta_{21}-PA_{21}\eta_{11}-PA_{23}\eta_{31}-PA_{24}\eta_{41}}{PA_{22}}+PA_{23}\eta_{31}+PA_{24}\eta_{41}$$

$\therefore \eta_{21}+\eta_{22}\eta_{21}=\eta_{21}+\eta_{22}\eta_{21}$

Also,

$$\eta_{22}^2=PA_{21}\eta_{12}+PA_{22}\xi_{22}+PA_{23}\eta_{32}+PA_{24}\eta_{42}$$

$$\eta_{22}^2=PA_{21}\eta_{12}+PA_{22}\frac{\eta_{22}^2-PA_{21}\eta_{12}-PA_{23}\eta_{32}-PA_{24}\eta_{42}}{PA_{22}}+PA_{23}\eta_{32}+PA_{24}\eta_{42}$$

$$\Longrightarrow\therefore \eta_{22}^2=\eta_{22}^2$$

Also,

$$\eta_{22}\eta_{23}+\eta_{23}=PA_{21}\eta_{13}+PA_{22}\xi_{23}+PA_{23}\eta_{33}+PA_{24}\eta_{43}$$

$$\eta_{22}\eta_{23}+\eta_{23}=PA_{21}\eta_{13}+PA_{22}\frac{\eta_{22}\eta_{23}+\eta_{23}-PA_{21}\eta_{13}-PA_{23}\eta_{33}-PA_{24}\eta_{43}}{PA_{22}}+PA_{23}\eta_{33}+PA_{24}\eta_{43}$$

$\therefore \eta_{22}\eta_{23}+\eta_{23}=\eta_{22}\eta_{23}+\eta_{23}$

Also,

$$\eta_{22}\eta_{24}+\eta_{24}=PA_{21}\eta_{14}+PA_{22}\xi_{24}+PA_{23}\eta_{34}+PA_{24}\eta_{44}$$

$$\eta_{22}\eta_{24}+\eta_{24}=PA_{21}\eta_{14}+PA_{22}\frac{\eta_{22}\eta_{24}+\eta_{24}-PA_{21}\eta_{14}-PA_{23}\eta_{34}-PA_{24}\eta_{44}}{PA_{22}}+PA_{23}\eta_{34}+PA_{24}\eta_{44}$$

$\therefore \eta_{22}\eta_{24}+\eta_{24}=\eta_{22}\eta_{24}+\eta_{24}$

Also,

$$\eta_{31}+\eta_{33}\eta_{31}=PA_{31}\eta_{11}+PA_{32}\eta_{21}+PA_{33}\xi_{31}+PA_{34}\eta_{41}$$

$$\eta_{31}+\eta_{33}\eta_{31}=PA_{31}\eta_{11}+PA_{32}\eta_{21}+PA_{33}\frac{\eta_{33}\eta_{31}+\eta_{31}-PA_{31}\eta_{11}-PA_{32}\eta_{21}-PA_{34}\eta_{41}}{PA_{33}}+PA_{34}\eta_{41}$$

$\therefore \eta_{31}+\eta_{33}\eta_{31}=\eta_{31}+\eta_{33}\eta_{31}$

Also,

$$\eta_{32}+\eta_{33}\eta_{32}=PA_{31}\eta_{12}+PA_{32}\eta_{22}+PA_{33}\xi_{32}+PA_{34}\eta_{42}$$

$$\eta_{32}+\eta_{33}\eta_{32}=PA_{31}\eta_{12}+PA_{32}\eta_{22}+PA_{33}\frac{\eta_{33}\eta_{32}+\eta_{32}-PA_{31}\eta_{12}-PA_{32}\eta_{22}-PA_{34}\eta_{42}}{PA_{33}}+PA_{34}\eta_{42}$$

$\therefore \eta_{32}+\eta_{33}\eta_{32}=\eta_{32}+\eta_{33}\eta_{32}$

Also,

$$\eta_{33}^2=PA_{31}\eta_{13}+PA_{32}\eta_{23}+PA_{33}\xi_{33}+PA_{34}\eta_{43}$$

$$\eta_{33}^2=PA_{31}\eta_{13}+PA_{32}\eta_{23}+PA_{33}\frac{\eta_{33}^2-PA_{31}\eta_{13}-PA_{32}\eta_{23}-PA_{34}\eta_{43}}{PA_{33}}+PA_{34}\eta_{43}$$

$$\Longrightarrow\therefore \eta_{33}^2=\eta_{33}^2$$

Also,

$$\eta_{34}+\eta_{33}\eta_{34}=PA_{31}\eta_{14}+PA_{32}\eta_{24}+PA_{33}\xi_{34}+PA_{34}\eta_{44}$$

$$\eta_{34}+\eta_{33}\eta_{34}=PA_{31}\eta_{14}+PA_{32}\eta_{24}+PA_{33}\frac{\eta_{33}\eta_{34}+\eta_{34}-PA_{31}\eta_{14}-PA_{32}\eta_{24}-PA_{34}\eta_{44}}{PA_{33}}+PA_{34}\eta_{44}$$

$\therefore \eta_{34}+\eta_{33}\eta_{34}=\eta_{34}+\eta_{33}\eta_{34}$

Also,

$$\eta_{41}+\eta_{44}\eta_{41}=PA_{41}\eta_{11}+PA_{42}\eta_{21}+PA_{43}\eta_{31}+PA_{44}\xi_{41}$$

$$\eta_{41}+\eta_{44}\eta_{41}=PA_{41}\eta_{11}+PA_{42}\eta_{21}+PA_{43}\eta_{31}+PA_{44}\frac{\eta_{44}\eta_{41}+\eta_{41}-PA_{41}\eta_{11}-PA_{42}\eta_{21}-PA_{43}\eta_{31}}{PA_{44}}$$

$\therefore \eta_{41}+\eta_{44}\eta_{41}=\eta_{41}+\eta_{44}\eta_{41}$

Also,

$$\eta_{42}+\eta_{44}\eta_{42}=PA_{41}\eta_{12}+PA_{42}\eta_{22}+PA_{43}\eta_{32}+PA_{44}\xi_{42}$$

$$\eta_{42}+\eta_{44}\eta_{42}=PA_{41}\eta_{12}+PA_{42}\eta_{22}+PA_{43}\eta_{32}+PA_{44}\frac{\eta_{44}\eta_{42}+\eta_{42}-PA_{41}\eta_{12}-PA_{42}\eta_{22}-PA_{43}\eta_{32}}{PA_{44}}$$

$\therefore \eta_{42}+\eta_{44}\eta_{42}=\eta_{42}+\eta_{44}\eta_{42}$

Also,

$$\eta_{43}+\eta_{44}\eta_{43}=PA_{41}\eta_{13}+PA_{42}\eta_{23}+PA_{43}\eta_{33}+PA_{44}\xi_{43}$$

$$\eta_{43}+\eta_{44}\eta_{43}=PA_{41}\eta_{13}+PA_{42}\eta_{23}+PA_{43}\eta_{33}+PA_{44}\frac{\eta_{44}\eta_{43}+\eta_{43}-PA_{41}\eta_{13}-PA_{42}\eta_{23}-PA_{43}\eta_{33}}{PA_{44}}$$

$\therefore \eta_{43}+\eta_{44}\eta_{43}=\eta_{43}+\eta_{44}\eta_{43}$

Also,

$$\eta_{44}^2=PA_{41}\eta_{14}+PA_{42}\eta_{24}+PA_{43}\eta_{34}+PA_{44}\xi_{44}$$

$$\eta_{44}^2=PA_{41}\eta_{14}+PA_{42}\eta_{24}+PA_{43}\eta_{34}+PA_{44}\frac{\eta_{44}^2-PA_{41}\eta_{14}-PA_{42}\eta_{24}-PA_{43}\eta_{34}}{PA_{44}}$$

$$\Longrightarrow\therefore \eta_{44}^2=\eta_{44}^2$$

Consequently, considering the order greater than one, system (2) is p-self-semi-homogeneous due to the equality of the left and right hands.

Corollary 4.5: *A homogenous system of differences Equation (2) is P-self-semi homogenous of order greater than one if and only if the following conditions hold:*

$$\left.\begin{aligned}
&\eta_{11}^2 = PA_{11}\xi_{11} + PA_{12}\eta_{21} + PA_{13}\eta_{31} + PA_{14}\eta_{41}\\
&\eta_{11}\eta_{12} + \eta_{12} = PA_{11}\xi_{12} + PA_{12}\eta_{22} + PA_{13}\eta_{32} + PA_{14}\eta_{42}\\
&\eta_{11}\eta_{13} + \eta_{13} = PA_{11}\xi_{13} + PA_{12}\eta_{23} + PA_{13}\eta_{33} + PA_{14}\eta_{43}\\
&\eta_{11}\eta_{14} + \eta_{14} = PA_{11}\xi_{14} + PA_{12}\eta_{24} + PA_{13}\eta_{34} + PA_{14}\eta_{44}\\
&\eta_{21} + \eta_{22}\eta_{21} = PA_{21}\eta_{11} + PA_{22}\xi_{21} + PA_{23}\eta_{31} + PA_{24}\eta_{41}\\
&\eta_{22}^2 = PA_{21}\eta_{12} + PA_{22}\xi_{22} + PA_{23}\eta_{32} + PA_{24}\eta_{42}\\
&\eta_{22}\eta_{23} + \eta_{23} = PA_{21}\eta_{13} + PA_{22}\xi_{23} + PA_{23}\eta_{33} + PA_{24}\eta_{43}\\
&\eta_{22}\eta_{24} + \eta_{24} = PA_{21}\eta_{14} + PA_{22}\xi_{24} + PA_{23}\eta_{34} + PA_{24}\eta_{44}\\
&\eta_{31} + \eta_{33}\eta_{31} = PA_{31}\eta_{11} + PA_{32}\eta_{21} + PA_{33}\xi_{31} + PA_{34}\eta_{41}\\
&\eta_{32} + \eta_{33}\eta_{32} = PA_{31}\eta_{12} + PA_{32}\eta_{22} + PA_{33}\xi_{32} + PA_{34}\eta_{42}\\
&\eta_{33}^2 = PA_{31}\eta_{13} + PA_{32}\eta_{23} + PA_{33}\xi_{33} + PA_{34}\eta_{43}\\
&\eta_{34} + \eta_{33}\eta_{34} = PA_{31}\eta_{14} + PA_{32}\eta_{24} + PA_{33}\xi_{34} + PA_{34}\eta_{44}\\
&\eta_{41} + \eta_{44}\eta_{41} = PA_{41}\eta_{11} + PA_{42}\eta_{21} + PA_{43}\eta_{31} + PA_{44}\xi_{41}\\
&\eta_{42} + \eta_{44}\eta_{42} = PA_{41}\eta_{12} + PA_{42}\eta_{22} + PA_{43}\eta_{32} + PA_{44}\xi_{42}\\
&\eta_{43} + \eta_{44}\eta_{43} = PA_{41}\eta_{13} + PA_{42}\eta_{23} + PA_{43}\eta_{33} + PA_{44}\xi_{43}\\
&\eta_{44}^2 = PA_{41}\eta_{14} + PA_{42}\eta_{24} + PA_{43}\eta_{34} + PA_{44}\xi_{44}
\end{aligned}\right\} \tag{15}$$

Example 4.6: Let $F(X(n)) = \Xi X(n)$ be a system of difference equation then its P-self-semi-homogenous of order two such that

$$\Xi = \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}, \text{ there exists } H = \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{pmatrix} \text{ With P=9.}$$

$$F(X(n)) = \Xi X(n) \Rightarrow \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix} \Rightarrow \begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(Z) \\ f_4(w) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9}x \\ \frac{1}{9}y \\ \frac{1}{9}z \\ \frac{1}{9}w \end{pmatrix}$$

$$F\big(\mathrm{H}x(n)\big) = \mathrm{PH}F\big(x(n)\big) \Rightarrow \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{pmatrix} \begin{pmatrix} x \\ y \\ Z \\ w \end{pmatrix}$$

$$= 9 \begin{pmatrix} \frac{1}{81} & 0 & 0 & 0 \\ 0 & \frac{1}{81} & 0 & 0 \\ 0 & 0 & \frac{1}{81} & 0 \\ 0 & 0 & 0 & \frac{1}{81} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(y) \\ f_3(Z) \\ f_4(w) \end{pmatrix}$$

$$\begin{pmatrix} f_1\left(\frac{1}{9}x\right) \\ f_2(\frac{1}{9}y) \\ f_3(\frac{1}{9}z) \\ f_4(\frac{1}{9}w) \end{pmatrix} = 9 \begin{pmatrix} \frac{1}{81}f_1(x) \\ \frac{1}{81}f_2(y) \\ \frac{1}{81}f_3(z) \\ \frac{1}{81}f_4(w) \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{81}x \\ \frac{1}{81}y \\ \frac{1}{81}z \\ \frac{1}{81}w \end{pmatrix} = 9 \begin{pmatrix} \frac{1}{81}\frac{1}{9}x \\ \frac{1}{81}\frac{1}{9}y \\ \frac{1}{81}\frac{1}{9}z \\ \frac{1}{81}\frac{1}{9}w \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{81}x \\ \frac{1}{81}y \\ \frac{1}{81}z \\ \frac{1}{81}w \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{81}x \\ \frac{1}{81}y \\ \frac{1}{81}z \\ \frac{1}{81}w \end{pmatrix}$$

$$\therefore F\big(\mathrm{H}x(n)\big) = P\mathrm{H}^2F\big(x(n)\big)$$

Alternatively, solve via corollary (4-5).

5. Conclusion

In this paper, we introduced a novel definition of a homogeneous system, which we refer to as a P-self-semi-homogenous system of order m with (4*4) dimensions. In many instances where p and m were positive integers, we looked at them. The values of P and m were obtained twice in our work: once when they were equal to one and once more when they were more than one.

6. Future work

Extending the aforementioned investigation to encompass a system of time-varying (or non- autonomous) homogeneous difference equations, this study also seeks to examine the generalization of the (n*n)-dimensional case.

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