

R^* -coessential submodule and R^* -coclosed submodule

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Abstract

To introduce new concepts called R^* -coessential submodule, R^* -coclosed submodule. which are generalizations of coessential, and coclosed submodule respectively. We investigate various form of such conditions and properties and their relations to R^* -supplemented module.

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1. Introduction

M considered as unitary left R -module and R is any ring with 1. A submodule E of M is referred to as small in M (denote by $E \ll M$) if whenever, $E + F = M$ for some $F \subseteq M$ implies that $F = M$, [1]. Many generalizations of small submodule see [2,3]. Özcan in [4] introduced the concept of consingular submodule of M as follow $Z^*(M) = \{m \in M; Rm \ll E(M)\}$, where $E(M)$ is the injective hull of M . if $Z^*(M) = M$, then M is called consingular. In [2] Mustafa and Khalid introduced new type of small submodule (adepends on cosingular submodules) if whenever $M = E + F$ and $\text{Rad}_R * \left(\frac{M}{F}\right)$

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$= \frac{M}{F}$ implies that $F = M$ where Rad_R^* is the sum of all e^* -small submodules in M , denoted by $E \ll_R^* M$. We use Rad_R^* -small submodule that appeared in [5] to introduce R^* -coessential submodule and R^* -coclosed submodule which are generalizations of coessential and coclosed respectively, and we illustrate the relation between them and other concepts. Main properties of these concepts are proved.

Definition 2.1: Let M be an R -module and let F and E be its submodules and $F \subseteq E \subseteq M$. When $\frac{E}{F} \ll_R^* \frac{M}{F}$, then F is referred to as the Rad_R^* -coessential submodule of E in M (for short, R^* -coessential submodule), shown by $F \subseteq_{R^*ce} E$ in M .

Remarks and Examples 2.2.

1. As the Z -module Z_6 . In Z_6 , obviously $\{0\}$ is R^* -coessential submodule of $\{0,3\}$.
2. As the Z -module Z_8 . $\{0,4\} \subseteq_{R^*ce} \{0,2,4,6\}$ in Z_8 since $\frac{\{0,2,4,6\}}{\{0,4\}} \ll_R^* \frac{Z_8}{\{0,4\}}$
3. As the Z -module Z . Within Z , $4Z$ is a R^* -coessential submodule of $2Z$.
4. As the Z -module M , $M = Z \oplus Z_{p^\infty}$. It is evident that in M , $\{0\}$ is not a R^* -coessential submodule of Z_{p^∞} . $\{0\} \subseteq Z_{p^\infty} \subseteq M$, but $\frac{Z_{p^\infty}}{\{0\}}$ is not R^* -small in $\frac{M}{\{0\}}$.
5. Assuming that M is an R -module and $E \subseteq M$. Then $E \ll_R^* M$ if and only if $\{0\} \subseteq_{R^*ce} E$ in M .
6. Observe that each coessential submodule is a R^* -coessential submodule; however, this need not always be the case, as Z_6 might also be a Z -module. In Z_6 , $\{0\}$ is a non-coessential submodule of $\{0,3\}$ that is R^* -coessential. The features of the R^* -coessential submodule that are required subsequently are given in the following assertion.

Proposition 2.3: Let $H \subseteq F \subseteq E \subseteq M$ and M be an R -module. Then $F \subseteq_{R^*ce} E$ in M iff $\frac{F}{H} \subseteq_{R^*ce} \frac{E}{H}$ in $\frac{M}{H}$.

Proof: Assume that $F \subseteq_{R^*ce} E$ in M , since $\frac{\frac{E}{H}}{\frac{F}{H}} \cong \frac{E}{F}$ and $\frac{\frac{M}{H}}{\frac{F}{H}} \cong \frac{M}{F}$ (by third Isomorphism theorem), due to $F \subseteq_{R^*ce} E$. Subsequence $\frac{E}{F} \ll_R^* \frac{M}{F}$, therefore $\frac{\frac{E}{H}}{\frac{F}{H}} \ll_R^* \frac{\frac{M}{H}}{\frac{F}{H}}$. Consequently $\frac{F}{H} \subseteq_{R^*ce} \frac{E}{H}$ in $\frac{M}{H}$.

In another hand, $\frac{F}{H} \subseteq_{R^*ce} \frac{E}{H}$ in $\frac{M}{H}$, due to $\frac{\frac{E}{H}}{\frac{F}{H}} \cong \frac{E}{F}$, $\frac{\frac{M}{H}}{\frac{F}{H}} \cong \frac{M}{F}$ (by third Isomorphism theorem). As a consequence of $\frac{\frac{E}{H}}{\frac{F}{H}} \ll_{R^*} \frac{\frac{M}{H}}{\frac{F}{H}}$, then $\frac{E}{F} \ll_{R^*} \frac{M}{F}$. For that reason, $F \subseteq_{R^*ce} E$.

Proposition 2.4: Suppose that $H \subseteq F \subseteq E \subseteq M$, M be an R -module. Then $H \subseteq_{R^*ce} E$ in M iff $H \subseteq_{R^*ce} F$ in M and $F \subseteq_{R^*ce} E$ in M .

Proof: " $\Rightarrow$ ": Suppose that $H \subseteq_{R^*ce} E$ in M . Simply, due to $\frac{F}{H} \subseteq \frac{E}{H} \subseteq \frac{M}{H}$ and $\frac{E}{H} \ll_{R^*} \frac{M}{H}$. Consequently $\frac{F}{H} \ll_{R^*} \frac{M}{H}$ (by proposition (2.6(1)) [5]). So $H \subseteq_{R^*ce} F$ in M . Now define $f: \frac{M}{H} \rightarrow \frac{M}{F}$ by $f(m+H) = m+F \forall m \in M$. f is well define, let $m_1+H = m_2+H$ iff $m_1-m_2 \in H$ so $f(m_1-m_2) \in f(H)$ iff $f(m_1+H) = f(m_2+H)$ iff $m_1+F = m_2+F$. Clearly f is an epimorphism. Because of $\frac{E}{H} \ll_{R^*} \frac{M}{H}$. Next $f\left(\frac{E}{H}\right) = \frac{E}{F} \ll_{R^*} \frac{M}{F}$ (by proposition (2.6(4)) [5]) Consequently $F \subseteq_{R^*ce} E$ in M .

" $\Leftarrow$ ": To show $H \subseteq_{R^*ce} E$ in M . Assume that $H \subseteq_{R^*ce} F$ in M and $F \subseteq_{R^*ce} E$ in M , to show $H \subseteq_{R^*ce} E$ in M . Let $\frac{E}{H} + \frac{X}{H} = \frac{M}{H}$ and $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$, then $M = E + X$, hence $\frac{M}{F} = \frac{E+X}{F} = \frac{E}{F} + \frac{X+F}{F}$, since $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$ (by corollary (2.6) [5]). Next $R^*\left(\frac{M}{X+F}\right) = \frac{M}{X+F}$. But $\frac{E}{F} \ll_{R^*} \frac{M}{F}$, therefore $\frac{X+F}{F} = \frac{M}{F}$, and. Hence $X+F = M$. Furthermore $\frac{X}{H} + \frac{F}{H} = \frac{M}{H}$. Because of $\frac{F}{H} \ll_{R^*} \frac{M}{H}$ and $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$. Therefore $\frac{X}{H} = \frac{M}{H}$.

Proposition 2.5: Let M be an R -module. If $H \subseteq M$ and $F \subseteq_{R^*ce} E$, then $F+H \subseteq_{R^*ce} E+H$ in M . The converse is true if $H \ll_{R^*} M$.

Proof: To show that $F+H \subseteq_{R^*ce} E+H$, where $H \subseteq M$. Let $\frac{E+H}{F+H} + \frac{U}{F+H} = \frac{M}{F+H}$, and $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$, then $M = E+H+U$, since $H \subseteq H+F$ contained in U , then $M = E+U$, so $\frac{E+U}{F} = \frac{M}{F} = \frac{E}{F} + \frac{U}{F}$, but $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$ and $\frac{E}{F} \ll_{R^*} \frac{M}{F}$, so $\frac{M}{F} = \frac{U}{F}$, then $\frac{U}{F+H} = \frac{M}{F+H}$, therefore $\frac{E+H}{F+H} \ll_{R^*} \frac{M}{F+H}$.

In other hand $F+H \subseteq_{R^*ce} E+H$ in M and $H \ll_{R^*} M$. To show that $F \subseteq_{R^*ce} E$ in M . Let $\frac{M}{F} = \frac{E}{F} + \frac{U}{F}$, and $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$. Subsequently $M = E+U$, so $\frac{M}{F+H} = \frac{E+H}{F+H} + \frac{U+H}{F+H}$. Consequently $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$, then by corollary ((2.5) [5]) state $R^*\left(\frac{M}{U+H}\right) = \frac{M}{U+H}$. But $\frac{E+H}{F+H} \ll_{R^*} \frac{M}{F+H}$, therefore $\frac{M}{F+H} = \frac{U+H}{F+H}$ and, so $M =$

$U + H$. Since $H \ll_R^* M$ and $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$. Consequently $M = U$, therefore $\frac{M}{F} = \frac{U}{F}$ and $F \subseteq_{R^*ce} E$ in M .

Proposition 2.6: Let be an R -module M and $H \ll_R^* M$. If $F \subseteq_{R^*ce} E$ in M , then $F \subseteq_{R^*ce} E + H$ in M .

Proof: Suppose that $F \subseteq_{R^*ce} E$ in M and $H \ll_R^* M$. To show that $F \subseteq_{R^*ce} E + H$ in M . Let $\frac{M}{F} = \frac{E+H}{F} + \frac{U}{F}$ and $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$, so $E + H + S = M$, (by corollary (2.5) [6]) stat that $R^*\left(\frac{M}{S+E}\right) = \frac{M}{S+E}$ and $H \ll_R^* M$. Subsequently $M = E + S$ and $\frac{M}{F} = \frac{E}{F} + \frac{U}{F}$. However, $\frac{E}{F} \ll_R^* \frac{M}{F}$ and $R^*\left(\frac{M}{U}\right) = \frac{M}{U}$. Therefore $\frac{M}{F} = \frac{U}{F}$. Thus $F \subseteq_{R^*ce} E + H$ in M .

Proposition 2.7: Consider both M and M' R -modules as well as $f: M \rightarrow M'$ be a homomorphism, then $f(F) \subseteq_{R^*ce} f(E)$ in $f(M)$ if $F \subseteq_{R^*ce} E$ in M .

Proof: Suppose that $F \subseteq_{R^*ce} E$ in M . Define $\phi: \frac{M}{F} \rightarrow \frac{f(M)}{f(F)}$ By $\phi(m + K) = f(m) + f(K)$, for each $m \in M$, since $\frac{E}{F} \ll_R^* \frac{M}{F}$, then (by proposition (2.6)(4) [5]) $\phi\left(\frac{E}{F}\right) = \frac{f(E)}{f(F)} \ll_{R^*} \phi\left(\frac{M}{F}\right) = \frac{f(M)}{f(F)}$. Thus we get the result.

Proposition 2.8: Assuming that M is R -module and $H \subseteq E \subseteq M$. If $E = H + S$ and $S \ll_R^* M$. Then $H \subseteq_{R^*ce} E$ in M .

Proof: Suppose that $E = H + U$ and $S \ll_R^* M$. Let $\frac{M}{H} = \frac{E}{H} + \frac{X}{H}$ for some $X \subseteq M$ and $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$, to prove $\frac{M}{H} = \frac{X}{H}$, then $M = E + X$, hence $M = H + S + X = X + S$, since $S \ll_R^* M$ and $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$, therefore $M = X$. Thus $H \subseteq_{R^*ce} E$ in M .

Proposition 2.9: Consider $H \subseteq E \subseteq M$ and M as R -module. If $M = H + E$, and $U \cap E \ll_R^* M$. Then $H \subseteq_{R^*ce} U$ in M .

Proof: Suppose that $H \subseteq U \subseteq M$, $M = H + E$ and $U \cap E \ll_R^* M$. Let $\frac{M}{H} = \frac{U}{H} + \frac{F}{H}$ and $R^*\left(\frac{M}{F}\right) = \frac{M}{F}$, to prove $\frac{M}{H} = \frac{F}{H}$, where $F \subseteq M$, then $M = U + F$. Because of this $U = U \cap M = U \cap (H + E) = H + (U \cap E)$ (by modular law). Subsequently $M = U + F = H + (U \cap E) + F$. So $M = (U \cap E) + F$. But $U \cap E \ll_R^* M$ and $R^*\left(\frac{M}{F}\right) = \frac{M}{F}$, therefore $\frac{M}{H} = \frac{F}{H}$. Thus $H \subseteq_{R^*ce} U$ in M .

The R^* -coessentail submodule is characterized by the following assertion.

Proposition 2.10: Let M be an R -module. If $H \subseteq_{R^*ce} F$ in M and $X \subseteq_{R^*ce} E$ in M . Then $H + X \subseteq_{R^*ce} F + E$ in M .

Proof: Suppose that $H \subseteq_{R^*ce} F$ in M and $X \subseteq_{R^*ce} E$ in M . To show that $H + X \subseteq_{R^*ce} F + E$ in M , let $f: \frac{M}{H} \rightarrow \frac{M}{H+X}$ be a map defined by $f(m + H) = m + (H + X)$ for each $m \in M$ and $g: \frac{M}{X} \rightarrow \frac{M}{H+X}$ be map defined by $g(m + X) = m + (H + X)$ for each $m \in M$. Clearly each f and g are epimorphism. Because of $\frac{F}{H} \ll_{R^*} \frac{M}{X}$, and $\frac{E}{X} \ll_{R^*} \frac{M}{X}$. Consequently $f\left(\frac{F}{H}\right) = \frac{(F+X)}{(H+X)} \ll_{R^*} \frac{M}{H+X}$. Also, $g\left(\frac{E}{X}\right) = \frac{(E+X)}{(H+X)} \ll_{R^*} \frac{M}{H+X}$ (by proposition (2.6(4)) [5]). Hence $\frac{F+X}{H+X} + \frac{E+X}{H+X} = \frac{F+E}{H+X} \ll_{R^*} \frac{M}{H+X}$ (by proposition (2.6, (2)) [5]). Thus $H + X \subseteq_{R^*ce} F + E$ in M .

Proposition 2.11: Let M be an R -module and let H, F, E , and X be submodules. Following are equivalent statements:

- 1) If $H \subseteq_{R^*ce} H + F$ in M , then $H \cap F \subseteq_{R^*ce} F$ in M .
- 2) If $H \subseteq_{R^*ce} F$ in M and $Y \subseteq M$, then $H \cap Y \subseteq_{R^*ce} F \cap Y$ in M .
- 3) If $H \subseteq_{R^*ce} F$ in M and $X \subseteq_{R^*ce} E$ in M , then $H \cap X \subseteq_{R^*ce} F \cap E$ in M .

Proof: “(1)” $\Rightarrow$ “(2)”:
Let $H \subseteq_{R^*ce} F$ in M and Y contained in M . Since $H + (F \cap Y) \subseteq F$, then $H \subseteq_{R^*ce} H + (F \cap Y)$ in M (by proposition 2.4), hence $H \cap (F \cap Y) \subseteq_{R^*ce} (F \cap Y)$ in M , by (1) $H \cap Y \subseteq_{R^*ce} F \cap Y$ in M .

“(2)” $\Rightarrow$ “(3)”:
Let $H \subseteq_{R^*ce} F$ in M and $X \subseteq_{R^*ce} E$ in M . By (2), $H \cap X \subseteq_{R^*ce} F \cap X$ in M . Also $X \subseteq_{R^*ce} E$ in M and $F \subseteq M$, then $F \cap X \subseteq_{R^*ce} F \cap E$ in M . Thus $H \cap X \subseteq_{R^*ce} F \cap E$ in M by proposition (2.4).

“(3)” $\Rightarrow$ “(1)”:
Let $H \subseteq_{R^*ce} H + F$ in M . Since $H \subseteq_{R^*ce} F$ by (3) $H \cap F \subseteq_{R^*ce} (H + F) \cap F$ in M . Thus $H \cap F \subseteq_{R^*ce} F$ in M .

We use the following lemma in next theorem.

Lemma 2.12: Assume M be an R -module such that $M = E + F$ and $M = (E \cap F) + H$ for each of its three submodules, A, B and L . That means $(E \cap H) + F = (F \cap H) + E$, [7,9].

Theorem 2.13: Assume that the module $M = F + X$ has $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$. Let X be contained in E and $X \subseteq_{R^*ce} E$ in M . Then $F \cap X \subseteq_{R^*ce} F \cap E$ in M .

Proof: To demonstrate that $\frac{M}{(F \cap X)} = \frac{H}{(F \cap X)}$. Let $\frac{M}{(F \cap X)} = \frac{(F \cap E)}{(F \cap X)} + \frac{H}{(F \cap X)}$ for $H \subseteq M$ and $R^*\left(\frac{M}{H}\right) = \frac{M}{H}$. For that reason, $M = (F \cap E) + H$, therefore $M = E + H$. By lemma (2.13) suggest that $M = (F \cap H) + E$. Consequently $\frac{M}{X} = \frac{(F \cap H) + X}{X} + \frac{E}{X}$ however, $R^*\left(\frac{M}{X}\right) = \frac{M}{X}$, then $R^*\left(\frac{M}{(F \cap H) + X}\right) = \frac{M}{(F \cap H) + X}$ According to corollary (2.5,[5]). But $\frac{E}{X} \ll_{R^*} \frac{M}{X}$. Therefore $M = (F \cap H) + X$

again by lemma(2.13), $M = (F \cap X) + H = H$ once more again. As a result,

$$\frac{M}{(F \cap X)} = \frac{H}{(F \cap X)}, \frac{F \cap E}{F \cap X} \ll_{R^*} \frac{M}{F \cap X}.$$

Not that if E and F are both submodules of M . E is considered to be a R^* -supplement of F in M if $M = E + F$ and $E \cap F \ll_{R^*} E$. Additionally, E is weak R^* -supplement of F in M if $M = E + F$ and $E \cap F \ll_{R^*} M$. When each submodule in M has a R^* -supplement, M is said to be R^* -supplemented; conversely, if each submodule in M has a weak R^* -supplement, M is considered weakly supplemented. [6].

Proposition 2.14: Consider F, E and U to be submodules of R -module M . If F is R^* -supplement, of E in M and if E is R^* -supplement of U in M with $F \subseteq U$, then $F \subseteq_{R^*ce} U$ in M .

Proof: Let us assume $\frac{M}{F} = \frac{U}{F} + \frac{V}{F}$, where F contained in U , and $R^*\left(\frac{M}{V}\right) = \frac{M}{V}$. To show that $F \subseteq_{R^*ce} U$ in M . In order to demonstrate $\frac{M}{F} = \frac{V}{F}$. Consequently $M = U + V$, due to F is R^* -supplement of E in M , following that $M = E + F$ and $E \cap F \ll_{R^*} E$. $V = V \cap M = V \cap (E + F) = F + (V \cap E)$, (by modular law). E is R^* -supplement of U in M , subsequently $M = E + U$ and $E \cap U \ll_{R^*} E$, $E = E \cap M = E \cap (U + V)$. $E = (E \cap U) + (E \cap V)$ follows. In order to demonstrate that $R^*\left(\frac{E}{E \cap V}\right) = \frac{E}{E \cap V}$, (by second Isomorphism) $\frac{E}{E \cap V} \cong \frac{E+V}{V} = \frac{M}{V}$ (because of this $F \subseteq V$ and $E + V = E + V + F = (E + F) + V$, however $R^*\left(\frac{M}{V}\right) = \frac{M}{V}$, hence $R^*\left(\frac{E}{E \cap V}\right) = \frac{E}{E \cap V}$, since $E \cap U \ll_{R^*} E$, then $E = E \cap V$, so $E \subseteq V$. Subsequently $V = F + E = M$. Thus $F \subseteq_{R^*ce} U$ in M .

3. R^* -coclosed submodule

Let E and F be a submodule of M a submodule E of M is called coclosed of M (notation $E \subseteq_{cc} M$) if whenever $\frac{E}{F} \ll \frac{M}{F}$ implies that $E = F$, [7].

Definition 3.1: Let M is R -module and E be a submodule of M , we say that E is R^* -coclosed submodule of M indicated by $(E \subseteq_{R^*CC} M)$ if whenever $X \subseteq_{R^*ce} E$ in M for some X of E , we have $X = E$.

Remarks and Examples 3.2.

1. $\{0\}$ is R^* -coclosed submodule in M .
2. $\{0, 2\}$ in Z_4 as Z -module is not R^* -coclosed submodule
3. Consider Z_{12} as Z -module, the R^* -small coessential submodule of $\langle \underline{2} \rangle = \{0, 2, 4, 6, 8\}$ is $\langle \bar{4} \rangle = \{0, 4, 8\}$, $(\langle \bar{4} \rangle \subseteq_{R^*ce} \langle \bar{2} \rangle)$ since $\frac{\langle \bar{2} \rangle}{\langle \bar{4} \rangle} \cong \langle \bar{4} \rangle \ll_{R^*} Z_{12}$, but $\langle \bar{2} \rangle \neq \langle \bar{4} \rangle$. So $\langle \bar{2} \rangle$ is not R^* -coclosed submodule of Z_{12} as Z -module.

4. Z_8 as Z -module, since $\langle \underline{2} \rangle = \{0, 2, 4, 6\} \ll_{R^*} Z_8$ and $\frac{\langle \underline{2} \rangle}{\langle \underline{0} \rangle} \cong \langle \underline{2} \rangle \ll_{R^*} Z_8$ so $\langle \underline{0} \rangle \ll_{R^*} \langle \underline{2} \rangle$, but $\langle \underline{2} \rangle \neq \langle \underline{0} \rangle$, so $\langle \underline{2} \rangle$ is not R^* -coclosed submodule of Z_8 as Z -module
5. For any module M every R^* -coclosed submodule is coclosed. To see this, let X be a submodule of E such that $\frac{E}{X} \ll \frac{M}{X}$, then $\frac{E}{X} \ll_{R^*} \frac{M}{X}$ by [5]. Since $E \subseteq_{R^*CC} M$, then $E = X$.

Proposition 3.3: Let M is R -module and E a submodule of M satisfies for every submodule U of E such that $R^*\left(\frac{E}{U}\right) = \frac{E}{U}$. Following are equivalent statements:

1. E is R^* -supplement submodule of M .
2. E is R^* -coclosed in M .
3. if $U \ll_{R^*} M$, then $U \ll_{R^*} E$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) If M is weakly R^* -supplemented, then (3) $\Rightarrow$ (1).

Proof: “(1)” $\Rightarrow$ “(2)”: To prove that E is R^* -coclosed. Let E be R^* -supplement of F in M . Consequently $M = E + F$ and $E \cap F \ll_{R^*} E$, Let $\frac{E}{U} \ll_{R^*} \frac{M}{U}$, for some submodule U of E . Since $M = E + F$, $\frac{M}{U} = \frac{E}{U} + \frac{F+U}{U}$, to show $R^*\left(\frac{M}{F+U}\right) = \frac{M}{F+U}$, since $\frac{M}{F+U} = \frac{E+F+U}{F+U} \cong \frac{E}{E \cap (F+U)} = \frac{E}{U+(E \cap F)}$ (by modular law). Consequently $R^*\left(\frac{M}{F+U}\right) = \frac{M}{F+U}$, but $\frac{E}{U} \ll_{R^*} \frac{M}{U}$, therefore $\frac{M}{U} = \frac{F+U}{U}$. Take note of that $E = E \cap M = E \cap (F + U) = U + (E \cap F)$ (again by modular law). Keep in mind that although $E \cap F \ll_{R^*} E$ and $R^*\left(\frac{E}{U}\right) = \frac{E}{U}$, therefore $E = U$.

“(2)” $\Rightarrow$ “(3)”: Suppose that E is R^* -coclosed in M and $U \ll_{R^*} M$, let $E = U + K$ and $R^*\left(\frac{E}{K}\right) = \frac{E}{K}$, since E is R^* -coclosed in M , it suffices to demonstrate that $\frac{E}{K} \ll_{R^*} \frac{M}{K}$, let $\frac{E}{K} + \frac{F}{K} = \frac{M}{K}$ and $R^*\left(\frac{M}{F}\right) = \frac{M}{F}$. Consequently $M = E + F = U + K + F = U + F$. However $U \ll_{R^*} M$, and $R^*\left(\frac{M}{F}\right) = \frac{M}{F}$, $M = F$.

“(3)” $\Rightarrow$ “(1)”: Thinking about that M is R^* -weakly supplemented means that, there exists a submodule F of M such that $M = E + F$, and $E \cap F \ll_{R^*} M$, by (3) consequently $E \cap F \ll_{R^*} E$. Thus E is R^* -supplemented submodule of M .

Corollary 3.4: Let M be an R -module, and let $E \subseteq F \subseteq M$ with $R^*\left(\frac{F}{E}\right) = \frac{F}{E}$. Then

1. If F is R^* -coclosed in M , then $\frac{F}{E}$ is R^* -coclosed in $\frac{M}{E}$.
2. If E is R^* -coclosed in M , then E is R^* -coclosed in F . The converse is true if F is R^* -coclosed in M .

3. Let C be R^* -coclosed in M , then for any $E \subseteq F \subseteq C \subseteq M$, $\frac{F}{E} \ll_{R^*} \frac{M}{E}$ if and only if $\frac{F}{E} \ll_{R^*} \frac{C}{E}$.
4. If $E \ll_{R^*} F$, and $\frac{F}{E}$ R^* -coclosed in $\frac{M}{E}$, then F is R^* -coclosed in M . Provided that every submodule is R^* -coclosed in M .

Proof:

1. Suppose F is R^* -coclosed in M . Let $\frac{X}{E} \subseteq_{R^*ce} \frac{F}{E}$, $\frac{\frac{F}{E}}{\frac{X}{E}} \cong \frac{F}{X}$ and $E \subseteq X \subseteq F$, then $X \subseteq_{R^*ce} F$ in M (by proposition 2.3) but F is R^* -coclosed in M , therefore $F = X$, so $\frac{F}{E} = \frac{X}{E}$. Thus $\frac{F}{E}$ is R^* -coclosed in $\frac{M}{E}$.
2. Assume that E is R^* -coclosed in M , and let S be a submodule of E such that $\frac{E}{S} \ll_{R^*} \frac{M}{S}$, then $\frac{E}{S} \ll_{R^*} \frac{M}{S}$ by proposition (2.6 (1) [5]) but E is R^* -coclosed in M , therefore $S = E$, thus E is R^* -coclosed in F .
3. Clearly by previous proposition.
4. Assume that $\frac{F}{K} \ll_{R^*} \frac{M}{K}$, $K \subseteq F$, since $\frac{F}{K+E} \cong \frac{\frac{F}{K}}{\frac{K+E}{K}} \ll_{R^*} \frac{\frac{M}{K}}{\frac{K+E}{K}} \cong \frac{M}{K+E}$ (by proposition 2.6), then $K+E \subseteq_{R^*ce} F$, but F is R^* -coclosed in M , hence $F = K+E$, but $E \ll_{R^*} F$ and $R^*\left(\frac{F}{E}\right) = \frac{F}{E}$, therefore $F = K$. Thus F is R^* -coclosed in M .

Recalling that if each proper submodule of M is R^* -small, then a non-zero module is said to be R^* -hollow, [5,8].

Proposition 3.5: The R^* -hollow submodule is R^* -hollow for all nonzero R^* -coclosed submodules.

Proof: Let M is R^* -hollow module and, H is nonzero R^* -coclosed submodule of M . Assume H be a proper submodule of E . Since M is R^* -hollow, then $H \ll_{R^*} M$. But H is R^* -coclosed in M , therefore $H \ll_{R^*} E$ by proposition (3.3). Thus E is be R^* -hollow.

Proposition 3.6: Let E is a nonzero R^* -hollow submodule of an R -module M , then either $E \ll_{R^*} M$ or E is R^* -coclosed submodule of M , but not both.

Proof: Suppose that E is a nonzero R^* -hollow submodule of M and E is not R^* -coclosed, we have to show that $E \ll_{R^*} M$. Since E is not R^* -coclosed in M , then there exist $H \subset E$ such that $\frac{E}{H} \ll_{R^*} \frac{M}{H}$. To prove that $E \ll_{R^*} M$, let $M = E + K$ and $R^*\left(\frac{M}{K}\right) = \frac{M}{K}$, then $\frac{M}{H} = \frac{E}{H} + \frac{H+K}{H}$. Subsequently $\frac{M}{H} = \frac{E+H+K}{H} \cong \frac{E}{E \cap (H+K)} = \frac{E}{H+(E \cap K)}$. However (by (2.5), [5]) $R^*\left(\frac{M}{H+K}\right) = \frac{M}{H+K}$. Consequently

$M = H + K$, but E is R^* -hollow and H is proper submodule of E , therefore $H \ll_{R^*} E$, so $H \ll_{R^*} M$, (by [5]), then $M = K$. As a result $E \ll_{R^*} M$. Consequently if $E \ll_{R^*} M$ and E is R^* -coclosed, then $\frac{E}{0} \ll_{R^*} \frac{M}{0}$ implies that $E = 0$ which is contradiction.

References

- [1] Kasch.F, Modules and rings, Londone: Press Inc (1982).
- [2] Wasan.A, On jacosban -small submodule, *Iraqi Journal of Science*, 60.7, pp. 1584-1591 (2019).
- [3] Wasan.H, e^* -essentail small submodules and e^* -hollow module, *European Journal of Pure and Applied Mathematcs* 15. 2, pp. 478-482 (2022).
- [4] Ozcan.A, Modules with small cyclic submodule in thier hall, Taylor, 30. 4, pp. 1575-1589 (2002).
- [5] Mustafa and Wassan.K, R^* -small submodule and R^* -hollow module, Accepted to *Iraqi Journal of Science* for puplication, 66.2 (2025).
- [6] Mustafa and Wassan.K, R^* -supplement module and related concept, Accepted for publication in AIP Conference Proceedings, pp. 1551-7616 (2023).
- [7] John.C, Supplement and projectiveity in module theory, Germany: Birkhauer verlage (2006).
- [8] Amira A. Abduljaleel and Sahira M. Yaseen, On large-hollow lifting modules, *Journal of Discrete Mathematical Sciences and Cryptography*, 25.8, 2739-2745 (2022).
- [9] Shaymaa E. Sarhan and Majid Mohammed Abed, On simple submodules and C-rings, *Journal of Discrete Mathematical Sciences and Cryptography*, 25.8, pp. 2747-2751 (2022).

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