Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 27 (2024), No. 5, pp. 1695–1704
DOI : 10.47974/JDMSC-2011

On using of weird number sequence in counting Beurling prime systems

Hussein A. Hussein *
Department of Mathematics
College of Education for Pure Science
University of Babylon
Babylon
Iraq

Faez A. Al-Maamori †
Department of Security
College of Information Technology
University of Babylon
Babylon
Iraq

Abstract

Beurling introduces the generalisation of the primes in the third decade of last century. He said in his work that any positive , increasiy real sequences $\{r_i\}_{i=1}^{\infty}$ with r_1 precisely qreatar than one called Beurling primes the fundamental Furthermore, the series of Beurling integers (or B-integers) could be generated by using the F-T-A. This work adapted the generalization Balanzarios work "Exampls of the generalised prime system" where using the sequence of weird numbers to show that size of $|N(x) - ax|$, $a > 0$ is always less that the asymptotic behavior of $\pi_q(x)$.

Subject Classification: *11D45, 54G20.*

Keywords: *F-T-A and B-prime systems.*

1. Introduction

Introducing a nonempty set M of all arithmetic functions , m : R→ ℂ .
Now , $M^+ \subseteq M$. where $M^+ = \{m : (1, \infty) \to \mathbb{C}\}$, m is increasy
And $M_0^+ = \{m \in M \; s.t \; m \; is \; increasy$ and m(1)= 0 $\}$ where as

* *E-mail:* **hussein.jaslm.pure438@student.uobabylon.edu.iq**
(Corresponding Author)

† *E-mail:* **faez@itnetuobabylon.edu.iq**

© Taru Publications

m_1^+ ={m∈M s.t m is increasy & m(1) =1}

Here $M_0^+ = M_0 \cap M^+$.

In a simple case of a generalised prime system , the set of Beurling prims can be defined as the sequence of real prims that occur twice in each step, or r={2,2,3,3,5,5,...}. So, in this case when applying the fundamental theorem of arithmetic to get the set of the generalized integers implies the set of naturel elements where each element (n) occurs as d(n) times. Therefor

$$\lim_{x\to\infty} \left|\frac{\pi(x)\log x}{x}\right| = 2 \tag{1}$$

where as

$$N_q \sim \sum_{n\in N_q} d(n) \tag{2}$$

In Beurling's work (precisely during 1937), he showed that as x goes to infinity

$\lim_{x\to\infty} \left|\frac{N_q(x)-bx}{x((\log x)^a)^{-1}}\right|$ =k …… (1) for som b > 0 and k > 0 with a > $\frac{3}{2}$

And $\lim_{x\to\infty} \left|\frac{\pi_q(x)\log x}{x}\right|$= 1 …… (2)

the Beurling system mentioned above only exists for values exactly greater than 3/2 and does not exist for values equal to 3/2 (refer to diamond [1, 2, 3]). Furthermore, the aforementioned is known as the Prime Number Theorem Generalisation.

Later Hall (see [6]) showed that for each (s_1, s_2, d) with $s_1 \in [0,1)$, $s_2 \in (1, +\infty]$ and d ∈ [0,1], than in Beurling prims system such that

i) $\lim_{x\to\infty} \left|\frac{N_q(x)-bx}{x[(\log x)^d]^{-1}}\right|$ = k ii) $\lim_{x\to\infty} inf\ [\frac{\pi_q(x)\log x}{x}] = s_1$

iii) $\lim_{x\to\infty} sup\ [\frac{\pi_q(x)\log x}{x}] = s_2$

From the foregoing, it is clear that the Beurling zeta function is involved in the estimation Brosess, which begins with $N_q(x)$ (or $\pi_q(x)$) and ends with the approximate value of the other. In fact $\zeta_q(x) = \sum_{n\geq 1} n^{-s}$ for s = $\sigma + it$. follow:

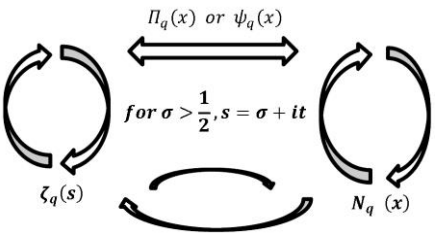

Figure 1

Explain the link between these functions.

In the seventh decade of the last century, Diamond indicated in his work (see [5]) where he introduce the modification of ζ (s) for s = σ +it, for σ> 1 as follows:

$$\xi_q(s) = \sum_{n\geq 1} n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (2)$$

Here, p is a prime for which the product is running over all primes. It is worth to notify that equation (2) converges absolutely for $\sigma > 1$ (see[5]).

Define the space of arithmetical functions M, $M: R \to ¢$ where, $\boldsymbol{M}$(x) continuous from the right hand side. Here, the function should be of local bounded variation. Let $M^+ \subseteq M$ and for any m in M , m is increasing function. So, if $M_0 \subseteq M$ (where $M_0=\{m \in M: m(1)=0\}$), then $M_0^+ = M \cap M^+$ .It is worth to show some challenged Beurling's prime systems:

(i) Define K as a set of actual primes apart from 2. In fact, K here satisfying the conditions of Beurling. That is; K represents Beurling's primes. So, following "the fundamental theorem of arithmetic" one can get N_q to be the Beurling's integers where N_q taking only the odd numbers. Furthermore, this addresses that $\lim_{x\to\infty} \left|\frac{(\pi_q(x)-1)\log x}{x}\right| = 1$. While, $\lim_{x\to\infty} \left|N(x) - \frac{1}{2}x\right| \leq k$, for some k>0. This is indicating that there is a discrete B-prime system (π_q, N_q).So, from what have been done by Balanzario in his work (see [@]) introducing a continuous B-counting function $\prod_q$ as shown here behind: (1) $\pi(a) \sum_{P<x} 1$; where P is prime (2) $N(a) = \sum_{n<a} 1$It is worth to notify that seeing the sequence of B-primes doesnot leeds to see the sequence of B-integers by applying the F-T-A in general. In fact some time is impossible to pick up the B-integer sequence from what have got it of B-primes. The related link below reflecting the relation between these functions involving B-zeta as in Figure 1 above.

$$\zeta_{g(s)} = \int_{1-}^{\infty} x^{-s}\ dN_g(x)\ \ and \qquad -\frac{\zeta_g'(s)}{\zeta_g(s)} = \int_{1-}^{\infty} x^{-s}\ \ d\psi_g(x)$$

Where $\pi_g(x) \sim \psi_g(x)\ for\ \ \sigma > \frac{1}{2}$ (see [5]).So with the above its obvious that: $\pi_q M_0^+$ {m:m is increasing function with m(1)=0} and $N_q M_1^+$ {m:m is increasing function with m(1)=1} $N_q(x) = e^{\pi_q}$. Then the order pair (π_q, N_q) is called Beurling's prime system (see [5]).

Example: For instance, one can take the sequance of B-primes such that each prime occure twicely. this inform us that the counting function of B-primes assymptoticly to $\frac{2x}{logx}$. whereas, when applying the fundamental theorem of arithmetic, one could get a sequence of B-integers numbers for which each element n occure as d(n), where d(n) is the divisor function. Here, the counting function of B-integers be have as a ssymptoticly of $\sum_{n\in N} d(n)$. So, in the above example means there is a B- primes system such that $\pi_B(x) \sim \frac{2x}{logx}$ & $N_B(x) = \sum_{n\in N_B} d(n)$.

2. Weird numbers

Studying weird numbers is always surprising and full of unexpected difficulties. Until now, we don't know if there are odd weird numbers at least less than10^{28}. Benkoski defined the weird numbers in 1972 and he proved with

Erdos that weird numbers have positive density. Factually, the weird numbers are infinite because weird number multiplied by a prime larger than its sum of divisors is again weird and this was proved . Now some definitions related to the main subject "weird numbers" will be stated as:

Definition 1 [7]: If x is a positive in N and $\sigma(x) = 2x$ implies x is named perfect. The first perfect numbers are 6, 28, 496, …

Definition 2 [7]: If x is a positive integer and $\sigma(x) \geq 2x$ we say x is called an abundant. The well known abundant started with 12, 18, 20, …

$\Delta(x) = \sigma(x) - 2x$ is called the abundance of x..

Definition 3 [7]: If x is a positive in N and $\sigma(x) < 2x$ where x is called "deficient". The first deficient numbers are 1, 2, 3, 4, 5, 7,…

Definition 4 [7]: If x is a positive integer and equal to sum of all or some of its proper divisors then x is the pseudo perfect number. The first few pseudo perfect numbers are 6, 12, 18, …

Definition 5 [7]: A positive integer x is weird if it is abundant but not pseudo perfect. The first few weird numbers are 70, 836, 4030, 5830, 7192, 7912, 9272 …

Kravitz in 1976 showed that for a natural number n and prime number p. if

$$q = \frac{2^n p-(p+1)}{(p+1)-2^n}$$

is a prime then $w=2^{n-1}$ is weird. Kravitz used Mersenne prime (the prime number of the form $2^n - 1$) $2^{16}-1$ as p and 57 as n and get a large weird number $w = 2.55 \times 10^{57}$.

To here are some exemplification about weird number:

Cardinal number 945:

The established divisors about 945 are 1, 3, 5, 7, 9, 15, 21, 27, 35, 45, 63, 105, 189, 315..

The sum of these divisors is 1 + 3 + 5 + 7 + 9 + 15 + 21 + 27 + 35 + 45 + 63 + 105 + 189 + 315 = 793, which is sizeable than 945..

Be that as it may, there is no subset about these divisors that quantity up to 945.

On that account , 945 is a weird number.

3. Primitive weird numbers

Definition 6 [8]: A weird number is a primitive if none of its proper divisors is weird. Actually there are only 24 primitive weird numbers less than million.

Until now, the infiniteness of primitive weird numbers is unverified but, in 2014, Giuseppe Melfi gave a way to build a weird to be primitive as follows

Theorem 9 [9]: *Let* $z \in Z^{+}$ *and with* m , $n \in$ *{odd integers} where* $p=2^{z+2}-m$ *and* $q=2^{z+2}+n$ *primes. If* $n+3 < m < 2^{\frac{z-2}{2}}$ *then* $w=2^{z}$ *pq is primitive weird number.*

This theorem did not proved the "unconditional infiniteness".

More generally, in 2018 Melfi again with three others presented an algorithm to generate numbers as a "primitive weird " $mp_1p_2\dots p_n$ as m should be deficient and $p_1, p_2,\dots, p_n$ are primes as the following

Theorem 10: *Suppose that k>1 is d-number and n>1, assuming* $p_1, p_2, \dots, p_n$ *be primes with* $\sigma(k)+1 < p_1 < \dots < p_n$. *Let*

$$V^{*} = [p_1 - \sigma(k)(p_n - p_1)^{-1}]$$

Let w = K $p_1\, p_2 \dots p_n$ and

$$U_{K,P_1,P_n} = \bigcup_{j=0}^{v^{*}} \{a \in \mathbb{N}: jp_n + \sigma(k) < a < (j+1)p_1\}$$

For more details the reader could see page (3,4,5) in [10] then for all a ∈K, a can't be expressible as a sumation of distinct divisors of x.

Theorem 11 [11]: *Generating a primitive weird number*

Let $c = \prod_{j=1}^{r} p_i$ such that $\Delta(c) \leq -1$ (c is deficient) and there exist a prime number $q > p_r$ such that cq is abundant and for n > 1 let $\{qi\}_{i=1}^{n}$ be an increasing sequence of primes such that $q_1 > \coprod_{j=1}^{r}(p_i + 1)$. Let $x = c\coprod_{i=1}^{n} q_i$, if x is abundant and $\Delta(x) \in K$ then x is primitive weird.

4. Some known results from literatures

(1) In 1903, Landu **[12]** proved that:

$$\lim_{x\to\infty} \left|\frac{N_q(x)-ax}{x^{\theta}}\right| \leq k \text{ for } \theta < 1 \rightarrow \quad \lim_{x\to\infty} \left|\frac{\pi_q(x)\log x}{x}\right| = 1$$

Moreover

$$\lim_{x\to\infty} \left|\frac{\pi_q(x)-\mathrm{li}(x)}{x\exp\{-c(\log x)^{\frac{1}{2}}\}}\right| \leq k \quad \text{For some c, k >0}$$

(2) In 1961 , Malliavin **[13]** proved that :

$\lim_{x\to\infty} \left|\frac{N_q(x)-ax}{x\exp\{-A(\log x)^{\alpha}\}}\right| \leq k$ for some A > 0 and $0 < \alpha < 1$. →

$\lim_{x\to\infty} \left|\frac{\pi_q(x)-\mathrm{li}(x)}{x\exp\{-b(\log x)^{\beta}\}}\right| \leq k$For some $\beta > 0$ *and* $b > 0$ & $\beta = \frac{\alpha}{10}$

(3) In 1970 , Diamond **[5]** improved Malliavin′s result and conversely he proved that if :

$\lim_{x\to\infty} \left|\frac{\pi_q(x)-\mathrm{li}(x)}{x\exp\{-c(\log x)^{\alpha}\}}\right| \leq k$ holds for $0 < \alpha < 1$, $c > 0$

→ $\lim_{x\to\infty} \left|\frac{N_q(x)-px}{x\exp\{-b(\log x \log\log\log x)^{\beta}\}}\right| \leq k$, For some b , k >0 & $\beta = \frac{\alpha}{1+\alpha}$.

(4) In 1998 , Balanzario **[4]** proved that :

$\lim_{x\to\infty}\left|\frac{\pi_q(x)-\mathrm{li}(x)}{x\exp\{-(logx)^{\alpha}\}}\right| \le k$ for $k>0$ and $\lim_{x\to\infty}\left|\frac{N_q(x)-px}{x\exp\{-c(logx)^{\beta}\}}\right| \ge w$,
For some $w>0$ & $\alpha=\beta=\frac{1}{2}$.

(5) In 2006, Diamond, Montgomery and Vorhauer **[14]** proved Landau′s work is the optimal. I.e., they addressed that ∃ a discrete B – prime system as:

$$\lim_{x\to\infty}\left|\frac{\pi_q(x)-\mathrm{li}(x)}{x\exp\{-c(logx)^{\frac{1}{2}}\}}\right| \ge w \text{ for } c,w>0$$

(6) In 2006, Hilberdink extended Diamond′s [15] result (to α = 1 case) as follows : suppose

$\lim_{x\to\infty}\left|\frac{\Psi_q(x)-x}{x^{\alpha}}\right| \le k$ for some

$0<\alpha<1 \rightarrow \lim_{x\to\infty}\left|\frac{N_q(x)-px}{x\exp\{-c(logxloglogx)^{\frac{1}{2}}\}}\right| \le k$, for $p,c,k>0$.

(7) in 2013 , F . AL- Maamori [16] improved proved that:

$$\lim_{x\to\infty}\left|\frac{\pi_q(x)-li(x)}{x\exp\{-c(logx)^{\alpha}\}}\right| \le k \quad \ldots\ldots(\#) \text{ and}$$

$$\lim_{x\to\infty}\left|\frac{N_q(x)-px}{x\exp\{-c(logx)^{\alpha}\}}\right| \ge w \quad \ldots..(\#\#)$$

holds for some w, p and c >0 also $0<\beta=\alpha<1$.

Adapting same strategyof the auther Balanzario in 1998 where later medicated by AL- Maamori in 2015. Note that the error terms of the arithmetical functions usually of the form :

(a) $O(\frac{X}{(logx)^{\gamma}})$ (b) $O(x\,e^{c(logx)^{\alpha}})$ (c) $O(x^{\alpha})$

In our work dealing with the part (a) above.

$$\gamma(t)=1-\sum_{n>n_0}\alpha_n\frac{cos\,(b_n\,logt)}{t^{a_n}} \ldots\ldots(\#)$$

$$\zeta_{y(s)}=\int_{1-}^{\infty}x^{-s}\,dN_y(x)=\int_{1-}^{\infty}x^{-s}\,e^{d\pi(x)}=\exp\{\int_{1-}^{\infty}x^{-s}\,d\pi\,(x)\} \quad \ldots\ldots(\#\#)$$

in this work, the Balinzarios methods generalization is modified by assuming that x takes a sequence of w- numbers, as mentioned in (7) above. Since the weird numbers (as x approaches infinity) behaviours have multiple general forms, we can use the sense of the w- numbers to explain the behaviors of the Beurling counting functions from a different perspective [15-18].

Suppose that we have $\pi_q(x)=\int_1^x\frac{1-t^{-k}}{logt}\gamma(t)dt$,

where $\lim_{x\to\infty}\left|\frac{\pi_{q(x)}-\mathrm{li}(x)}{xe^{\left[-c(logx)\right]^{\alpha}}}\right| \le k$

$c>0$ with $0<\alpha<1$, k = 4 , n_0= 3 , x = e^{10}

$b_n=\exp\{(logx_n)^{\frac{1}{2}}\}$, $a_n=\frac{1}{log\,b_n}=\frac{1}{(logx_n)^{\frac{1}{2}}}=(logx_n)^{-\frac{1}{2}}$, $x_{n+1}=\exp\{(logx_n)^2\}$

$$x_1=e^{10},\ T_n=\exp\{(logx_n)^{3/4}\},\quad \alpha_n=\frac{2}{n^2},\quad \alpha=\sum_{n>n_0}\alpha_n.$$

Keeping the same style of Balanzario′s method.

Proposition 1: This is the diminishing of idea

If $\pi_q(x)$ is as in (#), then $\pi_q(x) = li(x) + O(xe^{-c(\sqrt{\log x}})$, Here $\alpha = \frac{1}{2}$, c = 4.

Proof: We have:

$$\pi_q(x)=\int_1^x \frac{1-t^{-k}}{logt}\gamma(t)dt= \int_1^x \frac{1-t^{-k}}{logt}\ (1 - \sum_{n>n_0}\alpha_n\ \frac{cos\ (b_n\ logt)}{t^{a_n}})\ \ dt$$

$= (m\prod_{i=1}^{k} p_i)e^{-2(logm+\sum_{i=1}^{k} logp_i)^{\frac{1}{2}}} \leq (mkp_i)\ e^{-2(\sum_{i=1}^{k} logp_i)^{\frac{1}{2}}} \leq (mkp_k)$ $e^{-2^{\alpha+1}(logp_k)^{\alpha}}$ Therefore , $\pi\left(m\prod_{i=1}^{k} p_i\right)$ =li(p_k) +O($p_k e^{-2^{\alpha+1}(logp_k)^{\alpha}}$) , $\alpha < 1$.

The Estimation of N_q(x) enable us to see the effecting of π_q(x) N_q(x) Starting calculating $M_q(x) = \int_1^X N_q(x)$ dt.

Since dealing with M_q(x) is doable than dealing with) N_q(x).

So, $$M_q(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \zeta_{\ q}(x)\frac{s^{s+1}}{s(s+1)}\,ds, \quad b > 1. \tag{1}$$

Calculating M_q(x) by following the same strategy of the mention paper to calculate $\zeta_{\ q}(x)$.

Using the arguments as it was follows in the mention paper, getting : $\left|\zeta_{\ q}(x)\right| \leq 45$.

The value in equation (1), there are in 0 and 1, singularities points. So calculating the integral for [b - i∞, b + i∞], splitting the path into the following steps for avoiding that singularities as follws:

Γ_1 : is the integral on [b-i∞, b-iT],

Γ_2 : is the integral on [b-iT, - $\frac{3}{2}$ - iT], Γ_3 : is the integral on [-$\frac{3}{2}$ -iT , -$\frac{3}{2}$ +iT],

Γ_4 : is the integral on [-$\frac{3}{2}$ +iT, b + iT], Γ_5 : is the integral on[b+iT, b+i∞],

Thus, we can write (1) as follows :

$$M_q(X) = I_1 + \cdots + I_5 + J_{-n} + \cdots + J_n\text{+R-}\{0,1\},$$

Since, $I_m=\frac{1}{2\pi i}\int_{\Gamma_m} \zeta_{\ q}(x)\frac{s^{s+1}}{s(s+1)}\,ds$, m= 1,…,5

$$J_m =\frac{1}{2\pi i}\int_{C_m} \zeta_{\ q}(x)\frac{s^{s+1}}{s(s+1)}\,ds\ , \quad n_0<|m| \leq n\ .$$

Here Γ_m would be the integral of (1) and Cm is the m-Loops that behorizontal $\ddot{a}$(s)= b_m.

Its obvious that , estimating I_1 is similar to estimating I_5 which is the same case for I_2 and I_4.

Where the stratygy of the mention paper tells us that the upper bound of should be O ($\frac{X^2}{T_n}$),

Similarly for I_2 and I_4.O$(\frac{x}{T_n})^2$ While the estimating of I_3 should be O$(\frac{1}{\sqrt{x}})$. From the above one could get

$|I_m| \ll \frac{X^2}{T_n}$, m =1, …,5 and then the equation (1) could reform it to be:

$$M_q(x) = O\left(\frac{X^2}{T_n}\right) \sum_{m=-n}^{n} J_m + \text{residues}\{0,1\} \quad (2)$$

Calculating the R- { 0 , 1 } in (2) to be =k$\emptyset(1)\frac{x^2}{2}$ + (1-k) $\emptyset(0)x$

Enable us to rewrite equation (2) as:

$$M_q(x) = k\emptyset(1)\frac{x^2}{2} + (1-k)\,\emptyset(0)x + \sum_{m=-n}^{n} J_m + O\left(\frac{X^2}{T_n}\right) \quad (3)$$

The fraction $\frac{X^2}{T_n}$ in equation (3) = $X^2 \exp\{-(logx_n)^{\frac{3}{4}}\}$, so the equation (3) would be :

$$M_q(x) = k\emptyset(1)\frac{x^2}{2} + \sum_{m=-n}^{n} J_m + O(X^2 \exp\{-(logx_n)^{\frac{3}{4}}\}) \quad (4)$$

The following sum $\sum_{m=-n}^{n} J_m$ as in equation (4) would be :

$$\left|\sum_{n_0<|m|\le n-1}^{n} J_m\right| \le 120x^2 e^{-(logx)^{\frac{3}{4}}}.$$

Proof: Estimating J_m:

$|J_m| = \left|\frac{1}{2\pi i}\int_{C_m} \zeta_P(x)\frac{s^{s+1}}{s(s+1)}\,ds\right|$, and we see that by [Balanzario, 1998] we get : $|J_m| \le \frac{15}{b_m^2} x^2 e^{-a_m logx}$

Secondly , we estimate $\left|\sum_{n_0<|m|\le n-1}^{n} J_m\right|$ as follows :

If $|m| \le n-1$, then $e^{-a_m logx} \le e^{-a_{n-1} logx}$

$= \exp\{-\frac{logx}{(log_{n-1})^{\frac{1}{4}}}\} \le 2\, e^{-(logx)^{\frac{3}{4}}}$. Hence $\left|\sum_{n_0<|m|\le n-1}^{n} J_m\right| \le 30\, x^2\, e^{-(logx)^{\frac{3}{4}}} \sum_{|m|>n_0}^{n} \frac{1}{b^2{}_m}$

The rest of the proof would be: $\sum_{|m|>n_0}^{n} \frac{1}{b_m^2} \le \sum_{|m|>n_0}^{n} e^{-\sqrt{logx_m}} \le 4$.

Therefore, $\left|\sum_{n_0<|m|\le n-1}^{n} J_m\right| \le 120x^2 e^{-(logx)^{\frac{3}{4}}}$, So the (4) could be written as:

$$M_q(x) = k\varphi(1)\frac{1}{2}x^2 + (J_{-n} + J_n) + O(x^2 e^{-(logx)^{\frac{3}{4}}}) \quad (5)$$

The rest of the proof should estimating : $J_{-n} + J_n$. Following the same strategy one could have

$J_n = J_n' + J_n''$, we get → $J_n = \frac{sin\frac{\pi\alpha_n}{2}}{\pi} x^{2-a_n+ib_n}\left(\frac{1}{logx}\right)^{\frac{\alpha_n}{2}+1} s_n + O\left(x^2 e^{-(logx)^{\frac{3}{4}}}\right)$ (7)

Hence equation (7) as $|J_n| \ge \frac{x^2}{\pi 1600} \cdot \frac{e^{-3(logx_n)^{\frac{1}{2}}}}{4(logx)^4} \ge \frac{10^{-5}}{(logx)^4} e^{-3(logx)^{\frac{1}{2}}}$

We already know that $|J_n|$ is large , but still it can be that $N(J_n)=0$.

Equation (7),with $x=x_n(1+\frac{\theta_1}{logx_n}),|\theta_1|<1$. Then , we get by[Balanzario , 1998] : $(J_n) \geq \frac{10^{-5}}{2(logx)^4} e^{-c(logx)^{\frac{1}{2}}}$ When $x= m\prod_{i=1}^{k} p_i < mkp_k$. Therefore,

$$J_n \geq \frac{10^{-5}}{2(logp_k)^4} e^{-c(logp_k)^{\frac{1}{2}+\varepsilon}}, \quad \forall \varepsilon, c>0, \quad \alpha>\beta, \theta_1=\theta(+)$$

$$J_n \leq \frac{-10^{-5}}{2(logp_k)^4} e^{-c(logp_k)^{\frac{1}{2}+\varepsilon}}, \quad \forall \varepsilon, c>0, \quad \alpha>\beta, \theta_1=\theta(-)$$

These inequalities and equation :

$$M_q(m\prod_{i=1}^{k} p_i)=2\varphi(1)(m\prod_{i=1}^{k} p_i)^2+2(J_n)+O((m\prod_{i=1}^{k} p_i)^2 e^{-(logp_k)^{\frac{3}{4}}})$$

We have $M_q(m\prod_{i=1}^{k} p_i)=2\varphi(1)(m\prod_{i=1}^{k} p_i)^2+\Omega_{\pm}((logp_k)^2 e^{-c(logp_k)^{\frac{1}{2}+\varepsilon}})$, $\forall \varepsilon, c>0$. Therefore, the nice enhancing in this manner is concentrated in showing that for equation (#) in result (7) holds for $0<\alpha<1$ and (##) exists as well but for β very closed to a half.

References

[1] E. P. Balanzario, "On Chebyshev's inequalities for Beurling's generalized primes", *Mathematica Slovaca* 50 (4), 415-436.

[2] A. Beurling, "Analyse de la loi asymptotique de la distribution des nombres premiers généralisés. I," Acta Math., vol. 68, no. 1, pp. 255–291 (1937).

[3] G. Amato et al., "Primitive weird numbers having more than three distinct prime factors To cite this version : HAL Id : hal-01684543," 20 18.

[4] E. P. Balanzario, "An example in Beurling's theory of primes," Acta Arith., vol. 87, no. 2, pp. 121–139 (1998).

[5] H. G. Diamond, "A set of generalized numbers showing Beurling's theorem to be sharp," *Illinois J. Math.*, vol. 14, no. 1, pp. 29–34 (1970).

[6] R. S. Hall, "Beurling Generalized Prime Number Systems in Which the Chebyshev Inequalities Fail," Proc. Am. Math. Soc., vol. 40, no. 1, p. 79 (2006).

[7] S. J. Benkoski, P. Erdos, On weird and pseudo perfect numbers, *Mathematics of computation*, 126, 617-623 (1974).

[8] S. Pajunen, On primitive weird numbers, In: A Collection of Manuscripts Related to the Fibonacci Sequence (18th Anniversary Volume), V. Hoggat, ed., Fibonacci Association, 162-166 (1980).

[9] G. Melfi, On the conditional infiniteness of primitive weird numbers, *Journal of Number Theory.* 147, 508-514 (2015).

[10] G. Amato, M. Hasler, G. Melfi, M. Parton, Primitive weird numbers having more than three distinct prime factors, Riv. Mat. Univ. Parma, 7(1), 153-163 (2016).

[11] H. Sara, Beurling prime systems and the Riemann hypothesis, PhD thesis, University of Baghdad (2019).

[12] A. Landau, Newer primers and primers, Math. Ann, 56, 645-670 (1903).

[13] P. Malliavin, Sur le rest de la lio asymptotique de repartition des premiers nombres generalises de Beurling, Acta Math, 106, 281-298 (1961).

[14] H. Diamond, H. Mongomery and U. Vorhauer, Beurling primes with large oscillation, Math. Ann., 334,1-36 (2006).

[15] T. W. Hilberdink and Michel L. Lapidus, Beurling Zeta Functions Generalised primes,and Fractal Membranes, *Acta Appl. Math.*, 94, 21-48 (2006).

[16] Faez Ali AL-Maamori, An example in Beurling's theory of generalized primes, ACTA ARITHMETICA, 168.4, 383-395 (2015).

[17] Abbood, Mohaimen M., Ebrahim, Hassan H., Al-Fayadh, Ali & Obaid, Ahmed J. Measure defined on Γ–algebra and some of their generalizations, *Journal of Interdisciplinary Mathematics*, 26:5, 821–827 (2023), DOI: 10.47974/JIM-1502.

[18] Al-Ameedee, Sarah A. & Obaid, Ahmed J. () New results and application of differential quasi subordinations for higher-order derivatives of meromorphic multivalent functions, *Journal of Interdisciplinary Mathematics*, :, 1-8, DOI: 10.47974/JIM-1483.

Received January, 2024