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$\mathcal{L}$- open set and some related concepts

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Abstract

In this paper we argue that the set all L-open is a topology and broaden the idea of an open set by introducing the notion of an L-open set. In addition to exploring the concept of L-base, this paper aims to provide a thorough investigation of an L-open set and a plethora of new results. The connections between open sets, ordinary open sets, h-open sets, and L-open sets are also provided.

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1. Introduction

Mohammed and Ahmed [1] introduced some concepts plays a significant turn in algebra and cryptography. Some kinds of sets game a substantial turn in the study of several properties in topological space as well as some authors introduced different generalized of an open set [2].

In 1963, Levine [3] expanded on the concept of an open set by introducing the notion of a semi-open set. The concept of α-open, a generalization of open set, was initially put out by Njastad in 1965 [4]. An analysis of the regular open concept was conducted in 2014 by Ali et al. [5] which is a stronger form of an open set. The concept of h- open was studied in 2023 by [6]. There are many authors are studied some concepts related to open sets for example Abdulqader and Abdalbaqi [7] and Esmaeel and Nasir [8]. In this paper we generalized the concept of open set by introduce the concept of $\mathcal{L}-$open set and we prove that the set all $\mathcal{L}-$open is a topology.

2. The main results

Definition 2.1: Assume that $(\mathcal{X}, \mathfrak{F})$ is a topological space and $\mathcal{H} \subseteq \mathcal{X}$, then $\mathcal{H}$ is said to be $\mathcal{L}-$open set if $\mathcal{H} \subseteq T \cup Int(\mathcal{H})\ \forall T \in \mathfrak{F}\ \&\ T \neq \Phi$.

The set of all $\mathcal{L}-$ open sets is denoted by $\mathcal{L}O_{\mathcal{X}}$.

Definition 2.2: The complements of an $\mathcal{L}-$ open sets is said to be $\mathcal{L}-$ closed sets.

Example 2.3: Let $\mathcal{X} = \{v_1, v_2, v_3\}$ and $\mathfrak{F} = \{\Phi, \{v_1, v_2\}, \mathcal{X}\}$. Now, $\forall T \in \mathfrak{F}\ \&\ T \neq \Phi$, we have $\Phi \subseteq T \cup Int(\Phi) = T\ and\ \mathcal{X} \subseteq T \cup Int(\mathcal{X}) = \mathcal{X}$ .Hence, $\Phi, \mathcal{X}\ are\ \mathcal{L}-$open sets. Since, $Int(\{v_1\}) = \Phi\ \&\ \{v_1\} \subseteq \{v_1, v_2\} \cup Int(\{v_1\}) = \{v_1, v_2\}$ and $\{v_1\} \subseteq \mathcal{X} \cup Int(\{v_1\}) = \mathcal{X}$, then $\{v_1\}$ is an $\mathcal{L}-$open set. Since, $Int(\{v_2\}) = \Phi\ \&\ \{v_2\} \subseteq \{v_1, v_2\} \cup Int(\{v_2\}) = \{v_1, v_2\}$ and $\{v_2\} \subseteq \mathcal{X} \cup Int(\{v_2\}) = \mathcal{X}$, then $\{v_2\}$ is an $\mathcal{L}-$open set. Since, $Int(\{v_1, v_2\}) = \{v_1, v_2\}\ \&\ \{\eta_1, \eta_2\} \subseteq \{v_1, v_2\} \cup Int(\{v_1, v_2\}) = \{v_1, v_2\}$ and $\{\{v_1, v_2\} \cup Int(\{v_1\}) = \{v_1, v_2\}_1, v_2\} \subseteq \mathcal{X} \cup Int(\{v_1, v_2\}) = \mathcal{X}$, then $\{v_1, v_2\}$ is an $\mathcal{L}-$open set. Since, $Int(\{\eta_3\}) = \Phi\ \&\ \{\eta_3\} \nsubseteq \{v_1, v_2\} \cup Int(\{v_3\}) = \{v_1, v_2\}$, then $\{v_3\}$ is not $\mathcal{L}-$open set. Since, $Int(\{v_1, v_3\}) = \Phi\ \&\ \{v_1, v_3\} \nsubseteq \{v_1, v_2\} \cup Int(\{v_1, v_3\}) = \{\eta_1, \eta_2\}$, then $\{v_1, v_3\}$ is not $\mathcal{L}-$open set. Since, $Int(\{v_2, v_3\}) = \Phi\ \&\ \{v_2, v_3\} \nsubseteq \{v_1, v_2\} \cup Int(\{v_2, v_3\}) = \{v_1, v_2\}$, then $\{v_2, v_3\}$ is not $\mathcal{L}-$open set.

Therefore, $\mathcal{L}O_{\mathcal{X}} = \{\Phi, \{v_1\}, \{v_2\}, \{v_1, v_2\}, \mathcal{X}\}$.

Proposition 2.4:

1. $\Phi, \mathcal{X}\ are\ \mathcal{L}-$open sets.
2. If $\{\gamma\}_{\propto \in \mathrm{I}}$ is a family of $\mathcal{L}-$open sets. then $\bigcup_{\propto \in \mathrm{I}} \gamma_{\propto}$ is $\mathcal{L}-$open set.
3. Assume γ_1 and γ_2 are $\mathcal{L}-$open sets, then is a $\gamma_1 \cap \gamma_2$ $\mathcal{L}-$open set.

Proof: Assume $(\mathcal{X}, \mathfrak{F})$ is topological space, then

1. $\forall T \in \mathfrak{F}$ & $T \neq \Phi$, we have $\Phi \subseteq T \cup Int(\Phi) = T$ and $\mathcal{X} \subseteq T \cup Int(\mathcal{X}) = \mathcal{X}$. Hence, $\Phi, \mathcal{X}$ are $\mathcal{L}$ –open sets, thus $\Phi, \mathcal{X} \in \mathcal{LO}_{\mathcal{X}}$.
2. Let $\gamma_\propto \in \mathcal{LO}_{\mathcal{X}} \forall \propto \in \mathrm{I}$. To prove $\bigcup_{\propto \in \mathrm{I}} \gamma_\propto \in \mathcal{LO}_{\mathcal{X}}$, we must prove that $\bigcup_{\propto \in \mathrm{I}} \gamma_\propto \subseteq T \cup Int(\bigcup_{\propto \in \mathrm{I}} \gamma_\propto) \forall T \in \mathfrak{F}$ & $T \neq \Phi$. Since $\gamma_\propto \in \mathcal{LO}_{\mathcal{X}}$ $\forall \propto \in \mathrm{I}$, then $\gamma_\propto \subseteq T \cup Int(\gamma_\propto)$ and $\forall T \in \mathfrak{F}$ & $T \neq \Phi$. Hence $\bigcup_{\propto \in \mathrm{I}} \gamma_\propto \subseteq \bigcup_{\propto \in \mathrm{I}} \big(T \cup Int(\gamma_\propto)\big) = T \cup (\bigcup_{\propto \in \mathrm{I}} \big(Int(\gamma_\propto)\big)) \subseteq T \cup Int(\bigcup_{\propto \in \mathrm{I}} \gamma_\propto)$. thus $\bigcup_{\propto \in \mathrm{I}} \gamma_\propto \in \mathcal{LO}_{\mathcal{X}}$.
3. Let γ_1 and $\gamma_2 \in \mathcal{LO}_{\mathcal{X}}$, then $\forall T \in \mathfrak{F}$ & $T \neq \Phi$, we have $\gamma_1 \subseteq T \cup Int(\gamma_1)$ and $\gamma_2 \subseteq T \cup Int(\gamma_2)$, hence
$$\gamma_1 \cap \gamma_2 \subseteq \big(T \cup Int(\gamma_1)\big) \cap \big(T \cup Int(\gamma_2)\big) = T \cup [Int(\gamma_1) \cap Int(\gamma_2)] = T \cup Int(\gamma_1 \cap \gamma_2).$$
Therefore, $\gamma_1 \cap \gamma_2 \in \mathcal{LO}_{\mathcal{X}}$.

Theorem 2.5: *For any topological space* $(\mathcal{X}, \mathfrak{F})$*, then* $\mathcal{LO}_{\mathcal{X}}$ *is a* $topology$ on $\mathcal{X}$.

Definition 2.6: If $(\mathcal{X}, \mathfrak{F})$ is topological space, then an ordered pair $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is named $\mathcal{L}$ –space.

Theorem 2.7: *If* $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L} - space$. *Then:*

1. $\Phi, \mathcal{X}$ are $\mathcal{L} -$ closed.
2. If $\mathcal{H}_1, \mathcal{H}_2$ are $\mathcal{L} -$ closed, then $\mathcal{H}_1 \cup \mathcal{H}_2$ is a $\mathcal{L} -$ closed.
3. If $\{\mathcal{H}_\propto\}_{\propto \in \mathrm{I}}$ be a class of $\mathcal{L} -$ closed, then $\bigcap_{\propto \in \mathrm{I}} \mathcal{H}_\propto$ is a $\mathcal{L} -$ closed.

Definition 2.8: Suppose that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L} -$ space and $\eta \in \mathcal{X}$. A subset $\mathcal{H}$ of $\mathcal{X}$ is called $\mathcal{L} - neighbourhood$ of η (in short $\mathcal{H}$ is $\mathcal{L} - nhd$ of η) if there is $\mathcal{K} \in \mathcal{LO}_{\mathcal{X}}$ containing η such that $\mathcal{K} \subseteq \mathcal{H}$.

Example 2.9: Let $\mathcal{X} = \{g_1, g_2, g_3, g_4\}$ and $\mathfrak{F} = \{\Phi, \{g_1, g_2, g_3\}, \mathcal{X}\}$.

Therefore, $\mathcal{LO}_{\mathcal{X}} = \begin{Bmatrix} \Phi, \mathcal{X}, \{g_1\}, \{g_2\}, \{g_3\}, \{g_1, g_2\}, \\ \{g_1, g_3\}, \{g_2, g_3\}, \{g_1, g_2, g_3\} \end{Bmatrix}$. If we take $\eta = g_1$ and $\mathcal{H} = \{g_1, g_4\}$, then $g_1 \in \{g_1\} \in \mathcal{LO}_{\mathcal{X}}$ & $\{g_1\} \subseteq \mathcal{H}$. Hence $\mathcal{H}$ is $\mathcal{L} - nhd$ of g_1.

Remark 2.10: If $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L}$ –space and $\eta \in \mathcal{H} \subseteq \mathcal{X}$. If $\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}$, then $\mathcal{H}$ is $\mathcal{L} - nhd$ of η.

Theorem 2.11: Assume that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is $\mathcal{L}$ –space and $\mathcal{H} \subseteq \mathcal{X}$, then $\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}$ iff $\mathcal{H}$ includes $\mathcal{L} - nhd$ of each of its points.

Proposition 2.12: Suppose $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L} -$ space and $\mathcal{H}_1$ and $\mathcal{H}_2$ are $\mathcal{L} - nhd$ of η. Then $\mathcal{H}_1 \cap \mathcal{H}_2$ is a $\mathcal{L} - nhd$ of η.

Proposition 2.13: If $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is a $\mathcal{L}-$space and $\{\mathcal{H}_{\propto}\}_{\propto\in I}$ be a family of $\mathcal{L}-nhd$ of η. Then $\bigcup_{\propto\in I}\mathcal{H}_{\propto}$ is a $\mathcal{L}-nhd$ of η.

Proposition 2.14: Every open set is a $\mathcal{L}-$ open set.

Proof: Let $\mathcal{H}$ be an open set and $T \in \mathfrak{F}$ & $T \neq \Phi$. Then $\mathcal{H} = Int(\mathcal{H})$. So, we have $T \cup Int(\mathcal{H}) = T \cup \mathcal{H} \supseteq \mathcal{H}$, thus $\mathcal{H} is\ a\ \mathcal{L}-$ open set.

Example 2.16: Let $\mathcal{X} = \{\varkappa_1, \varkappa_2, \varkappa_3\}$ and $\mathfrak{F} = \{\Phi, \{\varkappa_1, \varkappa_3\}, \mathcal{X}\}$. Then, $\mathcal{L}O_{\mathcal{X}} = \{\Phi, \{\varkappa_1\}, \{\varkappa_3\}, \{\varkappa_1, \varkappa_3\}, \mathcal{X}\}$. $\{\varkappa_3\}$ $is\ \mathcal{L}-$ open set, but it is not open.

Proposition 2.17: An $\mathcal{L}-$ space $(\mathcal{X}, \mathcal{L}O_{\mathcal{X}})$ is finer than of a space $(\mathcal{X}, \mathfrak{F})$.

Theorem 2.18: *If $(\mathcal{X}, \mathfrak{F})$ is topological space, then every subset of a unique nonempty proper open set in $\mathcal{X}$ is a $\mathcal{L}-$ open set.*

Proof: Assume $\mathcal{H}$ is unique nonempty proper open set and $\mathcal{F} \subseteq \mathcal{H}$, then $Int(\mathcal{H}) = \mathcal{H}$ and $Int(\mathcal{F}) \subseteq Int(\mathcal{H})$, hence $Int(\mathcal{F}) \subseteq \mathcal{H}$. To prove $\mathcal{F}$ is $\mathcal{L}-$open set, we must prove $\mathcal{F} \subseteq T \cup Int(\mathcal{F})$ $\forall T \in \mathfrak{F}$ & $T \neq \Phi$. Since $\mathcal{H}$ is unique nonempty proper open set, then either $T = \mathcal{X}$ or $T = \mathcal{H}$. If $T = \mathcal{X}$, then $T \cup Int(\mathcal{F}) = \mathcal{X}$, hence $\mathcal{F} \subseteq T \cup Int(\mathcal{F})$. If $T = \mathcal{H}$, then $T \cup Int(\mathcal{F}) = \mathcal{H} \cup Int(\mathcal{F}) = \mathcal{H}$ because $Int(\mathcal{F}) \subseteq \mathcal{H}$. Hence $\mathcal{F} \subseteq T \cup Int(\mathcal{F})$ because $\mathcal{F} \subseteq \mathcal{H}$.Therefore, $\mathcal{F}$ is $\mathcal{L}-$open set.

Remark 2.19: The following example indicate that the converse of above Proposition 3is not true.

Example 2.20: Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and $\mathfrak{F} = \{\Phi, \{x_1, x_2, x_4\}, \mathcal{X}\}$.

Then, $\mathcal{L}O_{\mathcal{X}} = \begin{Bmatrix} \Phi, \mathcal{X}, \{x_1\}, \{x_2\}, \{x_4\}, \{x_1, x_2\}, \\ \{x_1, x_4\}, \{x_2, x_4\}, \{x_1, x_2, x_4\} \end{Bmatrix}$. Now, $\{x_2, x_4\}$ is an $\mathcal{L}-$open but not regular open because $Int(cl(\{x_2, x_4\})) = Int(\mathcal{X}) = \mathcal{X} \neq \{x_2, x_4\}$.

Proposition 2.21: Every $\mathcal{L}-$ open set is a h $-$ open set.

Proof: Let γ be an $\mathcal{L}-$open set and let $T \in \mathfrak{F}\ such\ that\ \Phi \neq T \neq \mathcal{X}$. Then $\mathcal{H} \subseteq T \cup Int(\gamma)$. Since $T \in \mathfrak{F}$, then $T = Int(T)$. So, we have $T \cup Int(\gamma) = Int(T) \cup Int(\gamma)$. Now, $Int(T)\ and\ Int(\gamma)$ are open sets contained in $T\ and\ \gamma$ respectively, then $Int(T) \cup Int(\gamma)$ is an open set which contained in $T \cup \gamma$. But $Int(T \cup \gamma)$ is a largest open set which contained in $T \cup \gamma$. So, we have $Int(T) \cup Int(\gamma) \subseteq Int(T \cup \gamma)$, hence $\gamma \subseteq Int(T \cup \gamma)$ thus $\gamma is\ a\ h-$ open set.

Remark 2.22: The following example investigate the reverse of above Proposition is not true.

Definition 2.23: Assume $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L}-$ space and $\Phi \neq \mathcal{Z} \subseteq \mathcal{X}$. Define a class $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$ as follows: $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} = \{\mathcal{S}: \mathcal{S} = \mathcal{H} \cap \mathcal{Z}, \text{for some } \mathcal{H} \in \mathcal{LO}_{\mathcal{X}}\}$. Then $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$ is called a restriction of $\mathcal{LO}_{\mathcal{X}}$ relative to $\mathcal{Z}$

Example 2.24: Let $\mathcal{X} = \{r, s, t\}$. Define a class $\mathfrak{F}$ as follows:

$\mathfrak{F} = \{\Phi, \{s, t\}, \mathcal{X}\}$. Then $(\mathcal{X}, \mathfrak{F})$ is topological space. Therefore, $\mathcal{LO}_{\mathcal{X}} = \{\Phi, \{s\}, \{t\}, \{s, t\}, \mathcal{X}\}$. Consider, $\mathcal{Z} = \{r, s\}$. Then $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} = \{\Phi, \mathcal{Z} = \{r, s\}, \{s\}\}$ is the restriction of $\mathcal{LO}_{\mathcal{X}}$ relative to $\mathcal{Z}$.

Proposition 2.25: Let $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ be a $\mathcal{L}-$space and $\Phi \neq \mathcal{Z} \subseteq \mathcal{X}$. Then $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$ is topology on $\mathcal{Z}$.

Note 2.26: Assume that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is $\mathcal{L}-$space and $\Phi \neq \mathcal{Z} \subseteq \mathcal{X}$. Then $(\mathcal{Z}, (\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}})$ is called $\mathcal{L}-$ subspace of $(\mathcal{X}, \mathcal{SLO}_{\mathcal{X}})$.

Proposition 2.27: If $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L}-$ space and $\mathcal{H} \subseteq \mathcal{Z} \subseteq \mathcal{X}$ such that $\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}$, then $\mathcal{H} \in (\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$.

Remark 2.28: Assume $(\mathcal{X}, \mathcal{SLO}_{\mathcal{X}})$ is a $\mathcal{L}-$space and $\Phi \neq \mathcal{Z} \subseteq \mathcal{X}$, then not necessarily that $\mathcal{LO}_{\mathcal{X}} \subseteq (\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$.

Remark 2.29: If $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L}-$ space and $\Phi \neq \mathcal{Z} \subseteq \mathcal{X}$, then not necessarily that $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} \subseteq \mathcal{LO}_{\mathcal{X}}$.

Example 2.30: Let $\mathcal{X} = \{\iota, \varsigma, \mathrm{e}, d\}$ and $\mathfrak{F} = \{\Phi, \{\iota, \varsigma, \mathrm{e}\}, \mathcal{X}\}$. Then, $\mathcal{LO}_{\mathcal{X}} = \begin{Bmatrix} \Phi, \{\iota\}, \{\varsigma\}, \{\mathrm{e}\}, \{\iota, \varsigma\}, \\ \{\iota, \mathrm{e}\}, \{\varsigma, \mathrm{e}\}, \{\iota, \varsigma, \mathrm{e}\}, \mathcal{X} \end{Bmatrix}$.Put, $\mathcal{Z} = \{n, \mathrm{e}, d\}$. Then the restriction of $\mathcal{LO}_{\mathcal{X}}$ relative to $\mathcal{Z}$ is $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} = \{\Phi, \mathcal{Z}, \{\iota, \mathrm{e}\}, \{\mathrm{e}\}, \{\iota\}\}$. $\{\iota, \mathrm{e}, d\} \in (\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$, but $\{\iota, \mathrm{e}, d\} \notin \mathcal{LO}_{\mathcal{X}}$and $\{\iota, \varsigma\} \in \mathcal{LO}_{\mathcal{X}}$ but $\{\iota, \varsigma\} \notin (\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}}$.

Remark 2.31: The following Theorem we put the necessarily condition to obtain $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} \subseteq \mathcal{LO}_{\mathcal{X}}$.

Theorem 2.32: Assume that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L}-$ space and $\Phi \neq \mathcal{Z} \subseteq \mathcal{X}$. If $\mathcal{Z} \in \mathcal{LO}_{\mathcal{X}}$. Then $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} = \{\mathcal{S} \subseteq \mathcal{Z}: \mathcal{S} \in \mathcal{LO}_{\mathcal{X}}\}$ and $(\mathcal{LO}_{\mathcal{X}})|_{\mathcal{Z}} \subseteq \mathcal{LO}_{\mathcal{X}}$.

Definition 2.33: Assume that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is a $\mathcal{L}-$ space. A family $\mathcal{A}_{\mathcal{X}}$ of subsets of $\mathcal{X}$ is said to be $\mathcal{L}-base$ for $\mathcal{LO}_{\mathcal{X}}$ if the following holds:

1. $\mathcal{A}_{\mathcal{X}} \subseteq \mathcal{LO}_{\mathcal{X}}$
2. $\forall \varkappa \in \mathcal{X}$, there is $\mathcal{U} \in \mathcal{A}_{\mathcal{X}}$ such that $\varkappa \in \mathcal{U}$.
3. $\forall\, \mathcal{U}_1, \mathcal{U}_2 \in \mathcal{A}_{\mathcal{X}}$ and $\forall\, \varkappa \in \mathcal{U}_1 \cap \mathcal{U}_2$ there is $\mathcal{U} \in \mathcal{A}_{\mathcal{X}}$ such that $\varkappa \in \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$.

Example 2.34: Let$\mathcal{X} = \{\iota, \varsigma, \mathrm{e}, d\}$and $\mathfrak{F} = \{\Phi, \{\iota, \varsigma\}, \{n\iota, \varsigma, \mathrm{e}\}, \{\iota, \varsigma, d\}, \mathcal{X}\}$.

Then, $\mathcal{LO}_{\mathcal{X}} = \{\Phi, \{\iota\}, \{\varsigma\}, \{\iota, \varsigma\}, \{\iota, \varsigma, \mathrm{e}\}, \{\iota, \varsigma, d\}, \mathcal{X}\}$.

If $\mathcal{A}_{\mathcal{X}} = \{\{\iota, \varsigma\}, \iota, \varsigma, \{n\iota, \varsigma, d\}\}$. then $\mathcal{A}_{\mathcal{X}}$ is a $\mathcal{L} - base$ for $\mathcal{LO}_{\mathcal{X}}$.

Theorem 2.37: Let $\mathcal{A}_{\mathcal{X}}$ be a $\mathcal{L} - base$ for a $\mathcal{L} - space$ $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$. Then a class $\mathcal{G}_{\mathcal{A}_{\mathcal{X}}} = \{\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}: \forall h \in \mathcal{H}$, there is $\mathcal{U} \in \mathcal{A}_{\mathcal{X}}$ such that $h \in \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{H}\}$ is topology on $\mathcal{X}$ that contains $\mathcal{A}_{\mathcal{X}}$.

Note 2.38: Any class is defined of the form $\mathcal{G}_{\mathcal{A}_{\mathcal{X}}}$ in Theorem (2.37) is called topology generated by $\mathcal{L} - base$ $\mathcal{A}_{\mathcal{X}}$.

Proposition 2.39: Let $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ be a $\mathcal{L} - space$ and $\mathcal{A}_{\mathcal{X}} \subseteq \mathcal{LO}_{\mathcal{X}}$. If for every $\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}, \forall h \in \mathcal{H}$ there is $\mathcal{U} \in \mathcal{A}_{\mathcal{X}}$ such that $h \in \mathcal{U} \subseteq \mathcal{H}$. Then $\mathcal{A}_{\mathcal{X}}$ is a $\mathcal{L} - base$ for $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$.

Theorem 2.40: *If $\mathcal{A}_{\mathcal{X}}$, $\mathcal{B}_{\mathcal{X}}$ $\mathcal{L} - base$ for $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ and*
$\mathcal{G}_{\mathcal{A}_{\mathcal{X}}} = \{\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}: \forall h \in \mathcal{H}$, *there is* $\mathcal{U} \in \mathcal{A}_{\mathcal{X}}$ *such that* $h \in \mathcal{U}$ *and* $\mathcal{U} \subseteq \mathcal{H}\}$,
$\mathcal{F}_{\mathcal{A}_{\mathcal{X}}} = \{\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}: \forall h \in \mathcal{H}$, *there is* $\mathcal{R} \in \mathcal{B}_{\mathcal{X}}$ *such that* $h \in \mathcal{R}$ *and* $\mathcal{R} \subseteq \mathcal{H}\}$. *Then the following are equivalent:*

1. $\mathcal{G}_{\mathcal{A}_{\mathcal{X}}}$ *is finer than* $\mathcal{F}_{\mathcal{B}_{\mathcal{X}}}$.
2. $\forall \varkappa \in \mathcal{X}$ *and* $\forall \mathcal{R} \in \mathcal{B}_{\mathcal{X}}$ *with* $\varkappa \in \mathcal{R}$, *there is* $\mathcal{U} \in \mathcal{A}_{\mathcal{X}}$ *such that* $\varkappa \in \mathcal{U} \subseteq \mathcal{R}$.

Conclusion

The main results are obtained are:

1. $\mathcal{LO}_{\mathcal{X}}$ is a topology on $\mathcal{X}$.
2. Assume that $(\mathcal{X}, \mathcal{LO}_{\mathcal{X}})$ is $\mathcal{L}$ −space and $\mathcal{H} \subseteq \mathcal{X}$, then $\mathcal{H} \in \mathcal{LO}_{\mathcal{X}}$ iff $\mathcal{H}$ includes $\mathcal{L} - nhd$ of each of its points.
3. Every open set is a $\mathcal{L} -$ open set.
4. If $(\mathcal{X}, \mathfrak{F})$ is topological space, then every subset of a unique nonempty proper open set in $\mathcal{X}$ is a $\mathcal{L} -$ open set.
5. Every regular open set is a $\mathcal{L} -$ open set.
6. Every $\mathcal{L} -$ open set is a $\mathrm{h} -$ open set.

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