

Inverse shadowing characteristic in Z-group

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Abstract

In the realm of dynamical systems theory, shadowing plays a crucial role. Structurally stable diffeomorphisms exhibit a noteworthy characteristic known as the inverse shadowing characteristic, which is investigated within the context of continuous methods. On compact manifolds, hyperbolic homeomorphisms are demonstrated to possess both inverse shadowing and bi-shadowing characteristics, particularly concerning a class of δ methods. These methods are described as continuous mappings from the manifold into the space of bi-infinite sequences within the manifold, utilizing the product topology. Additionally, the shadowing characteristic of the Lipschitz inverse is explored concerning two distinct classes of approaches, which produce pseudo-trajectories for dynamical systems. In our research, we establish that the characteristic of inverse shadowing can be derived from several

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fundamental dynamical system characteristics, including structural stability, transversality, Anosov characteristics, expansivity, continuum-wise expansiveness, and expansion.

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1. Introduction

Expansiveness and shadowing are fundamental concepts in qualitative research focusing on dynamical systems, with wide-ranging applications in areas such as topological dynamics, ergodic theory, symbolic dynamics, and related fields [1]–[3]. Study a homeomorphism mapping $f: M \rightarrow M$, where M is a compact metric space with metric d . An ordered sequence $\{x_n\}_{n \in \mathbb{Z}}$ for n in the set of integers is termed an orbit of f if $x_{n+1} = f(x_n)$ holds $\forall n$. If, for all n , the distance between $f(x_n)$ and x_{n+1} is less than or equal to a given value δ , it is referred to as a δ -pseudo-orbit of f . The mapping f possesses the characteristic of shadowing when, for any ε greater than zero, there exists a δ greater than zero such that every δ -pseudo-orbit $\{y_n\}$ can be closely followed or " ε -shadowed" by an orbit $f^n x$ for some x in M . In other words, the distance between $f^n x$ and y_n is less than or equal to ε for all n [4], [5]. Besides, the mapping f is said to have the Lipschitz shadowing characteristic if certain conditions involving constants L and d_0 are met, which ensures that for any δ -pseudo trajectory in M with d less than or equal to d_0 , a point p exists in M such that the distance between x_n and p_n is bounded by L times d for all k in $\mathbb{Z}$ [6]. Finally, a mapping f is termed continuum-wise expansive, denoted as " c -we," if a constant c greater than zero exists where for each non-trivial continuum C , the supremum of the diameters of $f^n(C)$ over n in $\mathbb{Z}$ exceeds c [7], [8].

2. Our Results

Here we prove the results on group $\mathbb{Z}$, and we will try in the future to study the same results in any group G for G -inverse shadowing characteristic.

Proposition 2.1: Where M is a manifold with compact, it has a continuous one-to-one f be on M and has the characteristic of shadowing. Then f has the characteristic of inverse shadowing.

Proof: Let $\varepsilon > 0$, and let δ correspond to this ε in the context of shadowing. We consider a δ -method denoted as $\Gamma = \{\gamma_k \in C(M, M): k \in \mathbb{Z}\}$. Now, we define a mapping $h: M \rightarrow M$ as follows: for every point x in M , the sequence x_k is considered a δ -pseudo-trajectory. We ensure that the distance between x_k and y_k is less than or equal to $\delta/2$ for all $x \neq y$ in M . Then, $h(x)$ is chosen to ε -shadow this δ -trajectory, which means that we have the characteristic

$$\text{dist}(f^k h(x); x_k) \leq \varepsilon \text{ for all } k \text{ in } \mathbb{Z} \dots \quad (1)$$

Suppose there is another point y in M that ε -shadows this δ -trajectory, which means $\text{dist}(f^k(y); x_k) \leq \varepsilon$ for all k in Z . Assume that $f^k(y) = f^n(h(x))$, then, since f is a one-to-one mapping, we can conclude that $y = h(x)$, making $h(x)$ unique. To establish the continuity of h , we consider $\varepsilon_1 > 0$ as $\varepsilon = \frac{\varepsilon_1}{2}$ and $\delta_1 > 0$ such that $\delta_1 = \sqrt{(2/5) \delta}$. For any x and y in M with $\text{dist.}(x, y) < \delta_1$, we have

$$\text{dist}(h(x), h(y)) \leq \text{dist}(h(x), f^k(h(x))) + \text{dist}(f^k(h(x)), h(y)) \leq \varepsilon_1.$$

This shows that h is continuous. Given that a continuous mapping which is sufficiently close to the identity mapping on a compact manifold is surjective, then for sufficiently small ε and the previously chosen δ , the mapping h becomes onto for any δ -method. For any point y in M , we can find an x in M where $h(x) = y$. Inequality (1) ensures that $\text{dist}(f^k(h(x)), x_k; x_k) \leq \varepsilon$ for all k in Z . Therefore, f possesses the inverse shadowing characteristic.

Proposition 2.2: Consider, with a diffeomorphism mapping f , which is C^1 , defined on a smooth m -dimensional closed manifold M with a Riemannian metric dist. , we define two sets, $B^+(p)$ and $B^-(p)$, as follows: $B^+(p)$ consists of vectors v in the tangent space $T_p M$ for which the limit of $|Df^k(p)v|$ as k goes to infinity is 0, and $B^-(p)$ consists of vectors v in $T_p M$ for which the limit of $|Df^k(p)v|$ as k goes to negative infinity is 0, all for a given point p in M . If the sum of $B^+(p)$, and $B^-(p)$ for a point p in M equals the entire tangent space $T_p M$, i.e., $B^+(p) + B^-(p) = T_p M$, then the diffeomorphism f exhibits inverse shadowing property.

Proof: Consider a point p in M and a vector v in $T_p M$. If the limit of $|Df^k(p)v|$ as k goes to either positive or negative infinity is 0, then v belongs to either $B^+(p)$ or $B^-(p)$. This indicates that $B^+(p) + B^-(p)$ equals $T_p M$. Since this condition also implies that f is structurally stable, it follows that f possesses the characteristic of inverse shadowing [9].

Proposition 2.3: Let a diffeomorphism mapping f be a $C^1 - m$ - dimensional closed on M which is a smooth manifold with Riemannian metric dist. If the analytic transversality characteristic is satisfied where $p \in M$ produces that f takes characteristic inverse shadowing.

Proof: Suppose $p \in M$ and let $v \in T_p M$, $B^+(p), B^-(p)$ are defined as in Proposition 2.2. Let p be satisfying the analytic transversality characteristic. Then we have $B^+(p) + B^-(p) = T_p M$ and by Proposition 2.2, f has inverse shadowing characteristics.

Proposition 2.4: Consider a diffeomorphism mapping f that is C^1 and defined Given a Riemannian metric dist. , on an M -dimensional closed manifold M that is smooth, if f is Anosov, it possesses the characteristic known as the Lipschitz shadowing characteristic [[6] Corollary 3]. This characteristic implies that there

exist constants L and δ_0 , both greater than 0, such that for any δ -pseudo-trajectory $\{x_k\}$ for k in the set of integers, with δ less than or equal to δ_0 , there exists a point p in M satisfying the condition $\text{dist}(x_k, p_k)$ is less than or equal to $L\delta$ for all k . This Lipschitz shadowing characteristic, in turn, implies that f is structurally stable in the C^1 class [[6], Theorem 1]. It is known that every structurally stable diffeomorphism possesses the inverse shadowing characteristic [[5], Theorem 1.1]. Consequently, the inverse shadowing feature is shown by f if it is Anosov.

Proposition 2.5: Take, for example, a homeomorphism mapping f constructed on a compact manifold M of dimension more than 2, if f is both continuum-wise expansive and possesses the shadowing characteristic, then it also exhibits the property of inverse shadowing.

Proof: If f is a continuum-wise expansive mapping, then there exists a constant greater than zero such that for non-trivial compact connected subsets C of M , the supremum of the diameters of $f^k(C)$ over all k in $\mathbb{Z}$ is greater than a . Additionally, if f has the shadowing characteristic, then there exists a δ -pseudo-trajectory $\{x_k\}$ for k in $\mathbb{Z}$, generated by f , and for any y in M , we can ensure that $\text{dist}(f^k(x), x_k)$ is less than ε , where ε is chosen as $\varepsilon = \frac{a}{2}$. Furthermore, $\text{dist}(f^k(x), x_k) < \varepsilon$ is bounded by a . By utilizing the continuum-wise expansiveness, it can be concluded that x is equal to y . Consequently, f is expansive and possesses the inverse shadowing characteristic [10-12], proof of Theorem 1]. A continuous surjective mapping f from a compact metric space M to M is termed an expanding mapping when it satisfies the positively expansive and open conditions.

We can now demonstrate the following.

Proposition 2.6: An expanding mapping defined on a compact manifold, denoted as f , possesses the characteristic of inverse shadowing which is concerned with the class of continuous methods.

Proof: Since f is an expanding mapping, it satisfies both positive expansiveness and expansiveness. Let $a > 0$ be the expansive constant for f . Moreover, an expanding mapping is known to have the shadowing characteristic. Now, fix ε such that ε is less than or equal to $\frac{a}{2}$. Select δ corresponding to this ε in the shadowing characteristic of f . For any δ -method, we construct a mapping $h: M \rightarrow M$ as follows: for each point x in M , the sequence x_k is considered a δ -pseudo-trajectory. We define $h(x)$ as a point in M that ε -shadows this δ -trajectory, which means that we have $\text{dist}(f^k h(x); x_k)$ is less than or equal to ε for all k . $H(x)$ is a unique, continuous, and surjective mapping (as proven in Proposition 2.1). Hence, it possesses the characteristic of inverse shadowing.

3. Discussion and Conclusion

The mapping f has the characteristic of inverse shadowing if it satisfies one of the following characteristics: The mapping f is a continuous, one-to-one on M and has the characteristic of shadowing, the mapping f is a diffeomorphism C^1 , m – dimensional closed on M which is a smooth manifold with Riemannian metric $dist$, the analytic transversality property is fulfilled at every point p in M , f is C^1 - diffeomorphism defined on a closed m -dimensional manifold M with a Riemannian metric $dist$. f is Anosov, Tf is positively expansive on a compact metrizable space, and the mapping f is a homeomorphism of compact manifold M of dimension greater than two and f is a continuum-wise expansive with the characteristic of shadowing. The mapping f is an expansion on a compact manifold.

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