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Generated new classes of BCK-algebras using permutation quasi-ordered sets

Ali Abbas Asmael *
Shuker Khalil †
Department of Mathematics
College of Science
University of Basrah
Basrah
Iraq

Abstract

This work aims to develop new techniques for introducing new *BCK*-algebra classes via permutation in symmetric groups. permutation generalized *BCK*-algebras are introduced as a generalization of positive implicative permutation *BCK*-algebras. It shows how to create a permutation generalized *BCK*-algebra from a permutation quasi-ordered set. The concept of permutation generalized ideals of permutation generalized *BCK*-algebras is introduced, followed by examining the relationships between such ideals and congruence. Permutation generalized *BCK*-ideals are described in detail. It is established that a set generates a permutation generalized *BCK*-ideal.

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Keywords: *Symmetric groups, Permutation sets, BCK-algebra, Quasi-ordering, Congruence, Compatible.*

1. Introduction

Logical systems are fundamental components of artificial intelligence that are utilized to make human decisions. Many−valued logic and fuzzy logic are also used. A BCK positive logic and a BCI positive logic prompted the study of BCK-algebras [1]and BCI −algebras [2], respectively. The study of abstract algebras and their generalizations justified by known logics is of significant interest to academics working in the fields of algebraic structures, artificial

* *E-mail:* **ali_abbas.math@utq.edu.iq (Corresponding Author)**
† *E-mail:* **shuker.khalil@uobasrah.edu.iq**

intelligence, and computing. Thus, several scholars have examined generalizations of BCK/BCI −algebras [3,4]. For any permutation β in the symmetric group S_n, the product of disjoint cycles $\beta = (b_1^1, b_2^1, \dots, b_{\alpha_1}^1)$ $(b_1^2, b_2^2, \dots, b_{\alpha_2}^2) \dots (b_1^{c(\beta)}, b_2^{c(\beta)}, \dots, b_{\alpha_{c(\beta)}}^{c(\beta)})$ can be decomposed essentially uniquely [5]. As a result, we can write β as $\lambda_1\lambda_2 \dots \lambda_{c(\beta)}$. The division is known as the cycle type of β [6]. The permutation set is non−classical set like soft set, neutrosoph set and so on, they are studied in topology and algebra and other fields [7-11]. In this work, the permutation sets are used to introduce new classes of BCK −algebra. As a generalization of positive implicative BCK −algebras, permutation generalized BCK −algebras are introduced. A permutation generalized BCK −algebra is constructed from a permutation quasi ordered set. The concept of permutation generalized BCK −ideals of permutation generalized BCK −algebras is proposed, and the linkages between such ideals and congruence are investigated. Permutation generalized BCK −ideals are thoroughly described. It has been demonstrated that a set produces a permutation generalized BCK −ideal.

2. Preliminaries

Definition 2.1[9]: Suppose β is permutation in symmetric group S_n on the set $\Omega = 1,2, \dots, n\}$ and the cycle type of β is $\alpha(\beta) = (\alpha_1, \alpha_2, \dots, \alpha_{c(\beta)})$ then β composite of pairwise disjoint cycles $\{\lambda_i\}_{i=1}^{c(\beta)}$ where $\lambda_i = (b_1^i, b_2^i, \dots, b_{\alpha_i}^i)$ $1 \leq i \leq c(\beta)$. For any k −cycle $\lambda = (b_1, b_2, \dots, b_k)$ in S_n we define β −set as $\lambda^\beta = \{b_1, b_2, \dots, b_k\}$ and it is called $\beta -$ set of cycle λ . So the $\beta -$ sets of $\{\lambda_i\}_{i=1}^{c(\beta)}$ are defined by $\{\lambda_i^\beta = \{b_1^i, b_2^i, \dots, b_{\alpha_i}^i\} | 1 \leq i \leq c(\beta)\}$.

Definition 2.2 [1]: A BCK −algebra is an algebra $(X,*,0)$ of type (2,0) satisfying the following axioms (1)$((x * y) * (x * z)) * (z * y) = 0$, (2) $(x * (x * y)) * y = 0$, (3) $x * x = 0$, (4) $0 * x = 0$, (5) $x * y = 0$ and $y * x = 0$ imply $x = y$, $\forall x, y, z \in X$.

3. Permutation Generalized BCK −Algebras

In this section, we study some novel notions of $PBCK/\ PGBCK$ −algebras and their ideals, and we provide a method for generating $PGBCK$ −algebras using permutation quasi-ordering set.

Definition 3.1: Let $X = \left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)} = \{t_1^i, t_2^i, \dots, t_{\alpha_i}^i\} | 1 \leq i \leq c(\beta)\}$ be a collection of β-sets, where β is a permutation in the symmetric group $G = S_n$. Let T= λ_k^β , for some $\lambda_k^\beta \in X$, where λ_k^β such that $\sum_{s=1}^{\alpha_k} t_s^k \leq \sum_{s=1}^{\alpha_i} t_s^i$ & $|\lambda_k| \leq |\lambda_i|, \forall (1 \leq i \leq c(\beta))$ .... (*). If there are at least two disjoint β-sets say λ_h^β, λ_g^β satisfy (*). Let T= λ_h^β if $\exists t_r^h \in \lambda_h^\beta$ with $t_r^h < t_s^g, \forall\ (1 \leq s \leq \alpha_g)$ or T= λ_g^β if $\exists t_r^g \in \lambda_g^\beta$ with $t_r^g < t_s^h, \forall\ (1 \leq s \leq \alpha_h)$. Also, let $\#: X \times X \longrightarrow X$ be a binary operation. We say $(X, \#, T)$ is a permutation BCK-algebra $(PBCK-A)$, if $\#$ such that the conditions: (1) $((\lambda_i^\beta \# \lambda_j^\beta) \#((\lambda_i^\beta \# \lambda_m^\beta))\#(\lambda_m^\beta \# \lambda_j^\beta) = T$, (2) $(\lambda_i^\beta \# (\lambda_i^\beta \# \lambda_j^\beta)) \# \lambda_j^\beta = T$, (3) $\lambda_i^\beta \# \lambda_i^\beta = T$, (4)$T\# \lambda_i^\beta = T$, (5) If $\lambda_i^\beta \# \lambda_j^\beta = T$ and $\lambda_j^\beta \# \lambda_i^\beta = T$, then $\lambda_i^\beta = \lambda_j^\beta, \forall \lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in X$.

Definition 3.2: Let $(X, \#, T)$ be a $(PBCK-A)$ and $\emptyset \neq M \subseteq X$. We say M is a permutation BCK-subalgebra $(PBCK-SA)$ if $\lambda_i^\beta \# \lambda_j^\beta \in M, \forall \lambda_i^\beta, \lambda_j^\beta \in M$.

Definition 3.3: Let $(X, \#, T)$ be a $(PBCK-A)$ and $\emptyset \neq M \subseteq X$. We say M is a permutation BCK-ideal $(PBCK-I)$ if it such that; (1) $T \in M$, and (2) If $\lambda_i^\beta \# \lambda_j^\beta \in M, \lambda_j^\beta \in M$, then $\lambda_i^\beta \in M, \forall \lambda_i^\beta, \lambda_j^\beta \in X$.

Example 3.4: Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 1 & 2 & 5 & 7 & 10 & 12 & 9 & 8 & 6 & 11 \end{pmatrix}$ be a permutation in S_{12}. So, $\beta = (1\ 3)(5)(2\ 4)(9)(6\ 7\ 10\ 8\ 12\ 11)$ and $X = \left\{\lambda_i^\beta\right\}_{i=1}^{5} = \{\{1,3\}, \{5\}, \{2,4\}, \{9\}, \{6,7,8,10,11,12\}\}$ and $T = \{1,3\}$. Define $\# : X \times X \longrightarrow X$ by Table (1).

Table 1
$(X, \#, T)$ **is a** $(PBCK-A)$.

#	{1,3}	{5}	{2,4}	{9}	{6, 7, 8, 10, 11, 12}
{1,3}	{1,3}	{1,3}	{1,3}	{1,3}	{1,3}
{5}	{5}	{1,3}	{5}	{1,3}	{1,3}
{2,4}	{2,4}	{2,4}	{1,3}	{1,3}	{1,3}
{9}	{9}	{9}	{9}	{1,3}	{1,3}
{6, 7, 8, 10, 11, 12}	{6, 7, 8, 10, 11, 12}	{9}	{6, 7, 8, 10, 11, 12}	{5}	{1,3}

We consider that $(X, \#, T)$ is a $(PBCK-A)$. Also, $H = \{\{1,3\}, \{2,4\}\}$ is a $(PBCK-I)$ & $(PBCK-SA)$. However, $D = \{\{1,3\}, \{5\}, \{9\}\}$ is a $(PBCK-SA)$, but is not $(PBCK-I)$. Since $\{6,7,8,10,11,12\}\#\{5\} = \{9\} \in D$ & $\{5\} \in D$, but $\{6,7,8,10,11,12\} \notin D$.

Definition 3.5: Let $X = \left\{\lambda_i^\beta\right\}_{i=1}^{c(\beta)}$ be a collection of β-sets in S_n. We say $(X, \#, T)$ is a permutation generalized BCK-algebra $(PGBCK-A)$, if $\#: X \times X \longrightarrow X$ such that: (1) $\lambda_i^\beta \# T = \lambda_i^\beta$, (2) $\lambda_i^\beta \# \lambda_i^\beta = T$, (3) $(\lambda_i^\beta \# \lambda_j^\beta)\# \lambda_m^\beta = (\lambda_i^\beta \# \lambda_m^\beta)\# \lambda_j^\beta$, (4) $(\lambda_i^\beta \# \lambda_j^\beta)\# \lambda_m^\beta = (\lambda_i^\beta \# \lambda_m^\beta)\#(\lambda_j^\beta \# \lambda_m^\beta)$, $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in X$.

Example 3.6: Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 8 & 4 & 1 & 3 & 7 & 6 & 9 & 2 & 11 & 10 \end{pmatrix} \in S_{11}$. So, $\beta = (1\ 5\ 3\ 4)\ \ (2\ 8\ 9)\ (6\ 7)\ (10\ 11)$ and $X = \left\{\lambda_i^\beta\right\}_{i=1}^4 = \{\{1,5,3,4\}, \{2,8,9\}, \{10,11\}, \{6.7\}\}$, so $T = \{6,7\}$. Let $\#_1, \#_2 : X \times X \longrightarrow X$ be defined by Tables 2 and 3.

Table 2

$(X, \#_1, T)$ is a $(PGBCK - A)$.

$\#_2$	{6,7}	{1,5,3,4}	{2,8,9}	{10,11}
{6,7}	{6,7}	{6,7}	{6,7}	{6,7}
{1,5,3,4}	{1,5,3,4}	{6,7}	{1,5,3,4}	{1,5,3,4}
{2,8,9}	{2,8,9}	{2,8,9}	{6,7}	{6,7}
{10,11}	{10,11}	{10,11}	{6,7}	{6,7}

Table 3

$(X, \#_2, T)$ is a $(PGBCK - A)$.

$\#_1$	{6,7}	{1,5,3,4}	{2,8,9}	{10,11}
{6,7}	{6,7}	{6,7}	{6,7}	{6,7}
{1,5,3,4}	{1,5,3,4}	{6,7}	{6,7}	{6,7}
{2,8,9}	{2,8,9}	{6,7}	{6,7}	{6,7}
{10,11}	{10,11}	{6,7}	{6,7}	{6,7}

We consider that $(X, \#_1, T)$ and $(X, \#_2, T)$ are permutation generalized BCK-algebras $(PGBCK - As)$. But they are not permutation BCK-algebras $(PBCK - As)$. Because $\{2,8,9\}\#_1\{10,11\} = \{10,11\}\#_1\{2,8,9\} = T$ and $\{2,8,9\}\#_2\{10,11\} = \{10,11\}\#_2\{2,8,9\} = T$, but $\{10,11\} \neq \{2,8,9\}$.

Definition 3.7: For any $(PBCK - A)$ X, let $\leq$ be defined by $\lambda_i^\beta \leq \lambda_j^\beta \leftrightarrow \lambda_i^\beta \# \lambda_j^\beta = T$, it is a partial order on X. Also, for any $(PBCK - A)$ X, if $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_m^\beta = (\lambda_i^\beta \# \lambda_m^\beta) \# (\lambda_j^\beta \# \lambda_m^\beta)$, $\forall \lambda_i^\beta, \lambda_j^\beta, \lambda_m^\beta \in X$. We say X is positive implicative permutation BCK-algebra $(PIPBCK - A)$.

Remark 3.8: Every $(PIPBCK - A)$ is $(PGBCK - A)$ [From Definition 3.5]. However, the converse maybe not true. See $(X, \#_1, T)$ and $(X, \#_2, T)$ in Example (3.6), they are $(PGBCK - As)$, but they are not $(PIPBCK - As)$.

Proposition 3.9: Let $(X, \#, T)$ be a $(PGBCK - A)$. Then; **(i)** $T \# \lambda_i^\beta = T$, **(ii)** $(\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_i^\beta = T$, **(iii)** $\lambda_i^\beta \# \lambda_j^\beta = T$ implies $(\lambda_i^\beta \# \lambda_r^\beta) * (\lambda_j^\beta \# \lambda_r^\beta) = T$.

Proof: **(i)** It is hold from [(2) & (4) in Definition (3.5)], **(ii)** It is hold from (i) and [Definition (3.5)-(4),(1)]. **(iii)** Suppose that $\lambda_i^\beta \# \lambda_j^\beta = T \rightarrow (\lambda_i^\beta \# \lambda_r^\beta) \# (\lambda_j^\beta \# \lambda_r^\beta) = (\lambda_i^\beta \# \lambda_j^\beta) \# \lambda_r^\beta = T \# \lambda_r^\beta = T$. This brings the proof to the end.

Definition 3.10: Let $X = \left\{\lambda_i^{\beta}\right\}_{i=1}^{c(\beta)}$ be a collection of β-sets in S_n and R be a relation on X. We say (X, R) is a permutation quasi-ordering set $(PQ - OS)$ and R is permutation quasi-ordering $(PQ - O)$ of X, if R is a transitive and reflexive on X.

Proposition 3.11: Let $(X, \#, T\ )$ be a $(PGBCK - A)$ and R_X be a relation on X, then R_X is $(PQ - O)$ of X if R_X such that $\left(\lambda_i^{\beta}, \lambda_j^{\beta}\right) \in R_X \leftrightarrow \lambda_j^{\beta} \# \lambda_i^{\beta} = T, \forall \lambda_i^{\beta}, \lambda_j^{\beta} \in X$.

Proof: Since $\lambda_i^{\beta} \# \lambda_i^{\beta} = T, \forall \lambda_i^{\beta} \in X$, thus $(\lambda_i^{\beta}, \lambda_i^{\beta}) \in R_X$, then R_X is reflexive. Also, by [Definition (3.5)-(1),(4)]&[Proposition (3.9)], so R_X is transitive. R_X is a $(IPQ - O)$ of X.

Proposition 3.12: Let $(X, \#, T\ )$ be a $(PGBCK - A)$ and R_X be a relation on X, if R_X such that $\left(\lambda_i^{\beta}, \lambda_j^{\beta}\right) \in R_X \leftrightarrow \lambda_j^{\beta} \# \lambda_i^{\beta} = T, \forall \lambda_i^{\beta}, \lambda_j^{\beta} \in X$. Then; (i) (λ_i^{β}, T) $\in R_X, \forall \lambda_i^{\beta} \in X$, (ii) If $\lambda_i^{\beta} \in X$ with $(T, \lambda_i^{\beta}) \in R_X$, then $\lambda_i^{\beta} = T$.

Proof: (i) It is hold from Proposition (3.9)-(i)], (ii) It is hold from (1) in Definition (3.5)].

Note: If $\left(\lambda_i^{\beta}, \lambda_j^{\beta}\right) \in R_X \leftrightarrow \lambda_j^{\beta} \# \lambda_i^{\beta} = T, \forall \lambda_i^{\beta}, \lambda_j^{\beta} \in X$, and $(X, \#, T\ )$ is a $(PGBCK - A)$, then R_X is called induced permutation quasi-ordering $(IPQ - O)$ of $(PGBCK - A)\ X$.

Lemma 3.13: *Assume that R_X is $(IPQ - O)$ of $(PGBCK - A)\ X$. Let $\lambda_i^{\beta}, \lambda_j^{\beta} \in X$ with $(\lambda_i^{\beta}, \lambda_j^{\beta}) \in R_X$. Then $(\lambda_i^{\beta} \# \lambda_r^{\beta}, \lambda_j^{\beta} \# \lambda_r^{\beta}) \in R_X$ and $(\lambda_r^{\beta} \# \lambda_j^{\beta}, \lambda_r^{\beta} \# \lambda_i^{\beta}) \in R_X$, $\forall \lambda_r^{\beta} \in X$.*

Proof: Let $\lambda_i^{\beta}, \lambda_j^{\beta}, \lambda_r^{\beta} \in X$ be such that $(\lambda_i^{\beta}, \lambda_j^{\beta}) \in R_X$. Then $\lambda_j^{\beta} * \lambda_i^{\beta} = T$, and so $(\lambda_j^{\beta} \# \lambda_r^{\beta}) \# \left(\lambda_i^{\beta} \# \lambda_r^{\beta}\right) = (\lambda_j^{\beta} \# \lambda_i^{\beta}) \# \lambda_r^{\beta} = T \# \lambda_r^{\beta} = T$, and $(\lambda_r^{\beta} \# \lambda_i^{\beta})$ $\# (\lambda_r^{\beta} \# \lambda_j^{\beta}) = \lambda_r^{\beta} \# (\lambda_r^{\beta} \# \lambda_j^{\beta}) \# \lambda_i^{\beta} = (\lambda_r^{\beta} \# \lambda_i^{\beta}) \# (\lambda_r^{\beta} \# \lambda_j^{\beta}) \# \lambda_i^{\beta} = (\lambda_r^{\beta} \# \lambda_i^{\beta})$ $\# (\lambda_r^{\beta} \# \lambda_i^{\beta}) \# (\lambda_j^{\beta} \# \lambda_i^{\beta}) = (\lambda_r^{\beta} \# \lambda_i^{\beta}) \# (\lambda_r^{\beta} \# \lambda_i^{\beta}) \# T = (\lambda_r^{\beta} \# \lambda_i^{\beta}) \# (\lambda_r^{\beta} \# \lambda_i^{\beta})$ $= T$. Hence $(\lambda_i^{\beta} \# \lambda_r^{\beta}, \lambda_j^{\beta} \# \lambda_r^{\beta}) \in R_X$ and $\left(\lambda_r^{\beta} \# \lambda_j^{\beta}, \lambda_r^{\beta} \# \lambda_i^{\beta}\right) \in R_X, \forall \lambda_r^{\beta} \in X$.

Proposition 3.14: Let R_X is $(IPQ - O)$ of $(PGBCK - A)\ X$. Then; (i) $(\lambda_j^{\beta}, \lambda_i^{\beta} \# \left(\lambda_i^{\beta} \# \lambda_j^{\beta}\right)) \in R_X, \forall \lambda_i^{\beta}, \lambda_j^{\beta} \in X$,(ii)$(\lambda_i^{\beta} \# \lambda_j^{\beta}, \left(\lambda_i^{\beta} \# \lambda_r^{\beta}\right) \# \left(\lambda_j^{\beta} \# \lambda_r^{\beta}\right) \in R_X, \forall \lambda_i^{\beta}, \lambda_j^{\beta}, \lambda_r^{\beta} \in X$.

Proof: (i) It is hold from [(2) and (3) in Definition (3.5)], (ii) It is hold from [Proposition (3.9)-(ii)] and [Lemma (3.13)].

For any permutation quasi-ordering R of X, refer as Ω_R the relation on X defined by $(\lambda_i^\beta, \lambda_j^\beta) \in \Omega_R$ if and only if $(\lambda_i^\beta, \lambda_j^\beta) \in R$ and $(\lambda_j^\beta, \lambda_i^\beta) \in R$. Ω_R is clearly an equivalence relation on X, which is also recognized as an equivalence relation generated by R. Let $[\lambda_i^\beta]_{\Omega_R}$ be equivalence class containing λ_i^β, where $[\lambda_i^\beta]_{\Omega_R} = \{\lambda_j^\beta \in X \mid (\lambda_j^\beta, \lambda_i^\beta) \in \Omega_R\}$, and $X/\Omega_R = \{[\lambda_i^\beta]_{\Omega_R} \mid \lambda_i^\beta \in X\}$. Define a relation $\leq_R$ on X/Ω_R by $[\lambda_i^\beta]_{\Omega_R} \leq_R [\lambda_j^\beta]_{\Omega_R} \leftrightarrow (\lambda_i^\beta, \lambda_j^\beta) \in R$. Hence $\leq_R$ is a partial order on X/Ω_R, thus $(X/\Omega_R, \leq_R)$ is a poset, it is said to be a poset assigned to the quasi-ordered set (X, R). A relation $R \, on \, X$ is said to be permutation compatible if $(\lambda_i^\beta \# \lambda_k^\beta, \lambda_j^\beta \# \lambda_d^\beta) \in R$ whenever $(\lambda_i^\beta, \lambda_j^\beta) \in R$ and $\left(\lambda_k^\beta, \lambda_d^\beta\right) \in R, \forall\, \lambda_i^\beta, \lambda_j^\beta, \lambda_k^\beta, \lambda_d^\beta \in X$. Also, the equivalence relation induced by R on X is said to be a permutation congruence relation on X if it is permutation compatible relation on X. The set $[T]_R = \{\lambda_i^\beta \in X \mid (\lambda_i^\beta, T) \in R\}$ is called the kernel of R.

Theorem 3.15: *Suppose that R_X is $(IPQ-O)$ of $(PGBCK-A)$ X and $\pi = \Omega_{R_X}$ is the equivalence relation induced by R_X. Then*; *(i) π is a congruence relation on X with kernel $[T]_\pi$ = {T}, (ii) the quotient algebra (X/π, $*$, $[T]_\pi$) is a $(PGBCK-A)$, where $*$ is operation on X/π is given as $[\lambda_i^\beta]_\pi * [\lambda_j^\beta]_\pi = [\lambda_i^\beta \# \lambda_j^\beta]_\pi, \forall [\lambda_i^\beta]_\pi, [\lambda_j^\beta]_\pi \in X/\pi$*

Proof: (i) It is obvious from [Lemma (3.13)] and [proposition (3.12)-(ii)], (ii) Since $(X, \#, T)$ is a $(PGBCK-A)$.Then, for any $[\lambda_i^\beta]_\pi, [\lambda_j^\beta]_\pi, [\lambda_m^\beta]_\pi \in X/\pi$ we have the four conditions are hold and hence (X/π, $*$, $[T]_\pi$) is a $(PGBCK-$A).

Remark 3.16: Let $(X, \#, T)$ be a $(PGBCK-A)$ and $\emptyset \neq D \subseteq X$. We refer by $\mathcal{R}_D$ the relation on X given by $(\lambda_i^\beta, \lambda_j^\beta) \in \mathcal{R}_D$ if and only $if\ \lambda_i^\beta \# \lambda_j^\beta \in D$ and $\lambda_j^\beta \# \lambda_i^\beta \in D$.

4. Permutations Generalized BCK-Ideals

In the following, let X signify a $(PGBCK-A)$ unless otherwise specified.

Definition 4.1: Let X be a $(PGBCK-A)$ and $\emptyset \neq F \subseteq X$. We say M is a permutation generalized BCK-ideal $(PGBCK-I)$ if it such that; (F_1) If $\lambda_i^\beta \in F$ and $\lambda_j^\beta \in X$, then $\lambda_i^\beta \# \lambda_j^\beta \in F$,($F_2$)If $\lambda_i^\beta, \lambda_j^\beta \in F$ and $\lambda_r^\beta \in X$, then $\lambda_r^\beta \#((\lambda_r^\beta \# \lambda_i^\beta) \# \lambda_j^\beta) \in F, \forall\, \lambda_i^\beta, \lambda_j^\beta, \lambda_r^\beta \in X$. Combining [(2) in Definition (3.5)] and (F_1), so $T \in F$.

Example 4.2: Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 1 & 2 & 5 & 7 & 10 & 12 & 9 & 8 & 6 & 11 \end{pmatrix} \in S_{12}$. So, $= (1\ 3)(5)(2\ 4)(9)(6\ 7\ 10\ 8\ 12\ 11)$, $X = \left\{\lambda_i^\beta\right\}_{i=1}^{5} = \{\{1,3\},\{5\},\{2,4\},\{9\},\{6,7,8,10,11,12\}\}$ and $\mathrm{T} = \{1,3\}$. Let $\# : X \times X \longrightarrow X$ be defined by Table 4.

Table 4

$(X, \#, T)$ is a $(PGBCK - A)$.

#	{1,3}	{5}	{2,4}	{9}	{6,7,8,10,11,12}
{1,3}	{1,3}	{1,3}	{1,3}	{1,3}	{1,3}
{5}	{5}	{1,3}	{5}	{1,3}	{5}
{2,4}	{2,4}	{2,4}	{1,3}	{1,3}	{2,4}
{9}	{9}	{2,4}	{5}	{1,3}	{9}
{6,7,8,10,11,12}	{6,7,8,10,11,12}	{6,7,8,10,11,12}	{6,7,8,10,11,12}	{6,7,8,10,11,12}	{1,3}

We consider that $(X, \#, T)$ is $(PGBCK - A)$. Also, $F = \{\{1,3\},\{5\},\{6,7,8,10,11,12\}\}$ and $C = \{\{1,3\},\{2,4\},\{6,7,8,10,11,12\}\}$ are $(PGBCK - Is)$.

Proposition 4.3: If F is a$(PGBCK - I)$ of X & $\lambda_j^\beta \in F, \lambda_i^\beta \in X, \lambda_i^\beta * \lambda_j^\beta = T \rightarrow \lambda_i^\beta \in F$.

Proof: Put $\lambda_j^\beta = T$ in (F_2) and from [(2) in Definition (3.5)], we have $\lambda_i^\beta \in F$.

Corollary 4.4: *Let F be a $(PGBCK - I)$ of X and let R_X be $(IPQ - O)$ of X. If $\lambda_j^\beta \in F$ and $(\lambda_j^\beta, \lambda_i^\beta) \in R_X$, for $\lambda_i^\beta \in X$, then $\lambda_i^\beta \in F$.*

Remark 4.5: Let D be a $(PGBCK - I)$ of X. Then the relation $\mathcal{R}_D$ is a congruence relation on X with the kernel $[T]_{\mathcal{R}_D} = D$.

Theorem 4.6: *If R is a reflexive and compatible relation on X. Then the kernel $[T]_R$ is a $(PGBCK - I)$ of X.*

Proof: By proposition 3.9-(i) and Definition 3.5-(1), we get $[T]_R$ is a $(PGBCK - I)$ of X.

Lemma 4.7: Let $(X,*,T)$ be $(PGBCK - A)$, $\emptyset \neq F \subseteq X$ and such that: (F_3) $T \in F$, (F_4) If $\lambda_i^\beta * \lambda_j^\beta \in F$ and $\lambda_j^\beta \in F$ imply $\lambda_i^\beta \in F$. $\forall \lambda_i^\beta \in F \rightarrow \lambda_j^\beta * \left(\lambda_j^\beta * \lambda_i^\beta\right) \in F, \forall \lambda_j^\beta \in X$.

Proof: Let $\lambda_i^\beta \in F$ and $\lambda_j^\beta \in X$, then $x * (\lambda_j^\beta * \lambda_i^\beta) * \lambda_i^\beta = (\lambda_j^\beta * \lambda_i^\beta) * (\lambda_j^\beta * \lambda_i^\beta) = T \in F$. It follows from (F_4) then $\lambda_j^\beta * (\lambda_j^\beta * \lambda_i^\beta) \in F$.

Conclusion

Some new extension of BCK-algebra using permutation are given in this work. Also, in future work, we will study the extension of our results using other non−classical sets like, fuzzy, soft and neutrosophic.

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