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Gamma semigroup whose gamma acts are quasi–injective

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Abstract

In this paper, we interpose the notion tools, Noetherian gamma acts, and γ – idempotent elements, to describe gamma semigroups in which all gamma acts are quasi – injective. We get the following equivalent (i) a gamma semigroup S is Noetherian and each finite direct sum of quasi – injective S_Γ-act is quasi – injective (ii) direct sum of quasi – injective S_Γ-act is quasi – injective. Also we prove that an γ – monoid is ΓRQI if and only if all right Γ- ideal generated by γ – idempotent element.

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1. Introduction

Let V be a Γ – semigroup, 1_γ in V is said to be γ – identity which γ in Γ, if (i) $t\gamma 1_\gamma = t = 1_\gamma \gamma t$ (ii) $t\beta 1_\gamma = 1_\gamma \beta t$ for all $t \in S$ and $\beta \in \Gamma \setminus \{\gamma\}$, and V is said to γ be γ -monoid if having –identity 1_γ. A subset W of V is said to be a right ideal of V if $W\Gamma V \subseteq W$. For a generalization of Γ – semi group. A nonempty set A is termed a right V_Γ-act if there exists a mapping from $A \times \Gamma \times V$ in to A written as $(a,\alpha,t) \mapsto a\alpha t$ satisfies $a\alpha(t\beta r) = (a\alpha t)\beta r$ for all $t, r \in V$, $\alpha, \beta \in \Gamma$, and $a \in A$ [1]. A right V_Γ-act A is said to be γ – unitary (for an fixed γ in Γ), if V is an γ -monoid such that $c = c\gamma 1_\gamma$ for all c in A. A non empty subset B of A is said to be V_Γ-subact ($B \leq A$) if $B\Gamma V \subseteq B$. An element Θ in is said to be fixed if $\Theta = \Theta\alpha t$ for all $\alpha \in \Gamma$ and $t \in S$, in this case Θ is said zero in A. An equivalence relation ε on A is said to be V_Γ-congruence, if $(c\alpha t, v\alpha t) \in \varepsilon$ for all $t \in S$, $\alpha \in \Gamma$, and $(c, v) \in \varepsilon$. A set $M/\pi = \{c\varepsilon \mid c \in M\}$ is said to be factor of A by ε, where $c\pi$ is equivalent class of c under ε, and A/ε is a V_Γ-act by the same structure of A: $(c\varepsilon)\alpha t \mapsto (c\alpha t)\varepsilon$ for all $c \in A$, $\alpha \in \Gamma$, and $t \in S$. Let Z be V_Γ-subact of A a relation $\sigma_Z = \{(m_1, m_2) \in A \times A \mid m_1$, and m_2 in Z or $m_1 = m_2\}$ is V_Γ-congruence on A and A/σ_Z is said to be **Rees factor** of M with respect to Z, (simply the symbol M /Z mean M/σ_Z). A mapping $g : A \to D$ (A and D are right V_Γ-acts) is called V_Γ-homomorphism, if $f(c\alpha h) = f(c)\alpha h$ for all $c \in A$, $h \in V$ and $\alpha \in \Gamma$, and $\ker(g) = \{(w_1, w_2) \in A \times A \mid g(w_1) = g(w_2)\}$ is said the kernel of g, and ker(g) is V_Γ-congruence for any V_Γ-homomorphism. $\mathrm{End}_{V_\Gamma}(A) = \{ f \mid f : A \to A$ is V_Γ-homomorphisms$\}$ is the endomorphism of A. The authors in ([3], [2]) show that $\mathrm{End}_{V_\Gamma}(A)$ need not α – monoid in general. By setting a condition said Γ-commute: $c\alpha(r\beta t) = c\beta(r\alpha t)$ for all $t, r \in V$, $\alpha, \beta \in \Gamma$ and $c \in A$ hence $\mathrm{End}_{V_\Gamma}(A)$ is being a structure of α – monoid for an α – unitary V_Γ-act A by $(g, \gamma, f) \mapsto g\gamma f$ which $g\gamma f(c) = g(f(c)\gamma 1_\alpha)$ for any $c \in A$. The concept of injectivity is introduces: An V_Γ-act A is said to be injective if for any V_Γ-monomorphism $f : W \to E$ where W and E are any two V_Γ-acts, any V_Γ-homomorphism $p : W \to A$, there is an V_Γ-homomorphism $\bar{p} : E \to A$ which satisfies $\bar{p}f = p$. For V_Γ-act W, there is a unique injective extension of W, denoted by E(W). For a generalization. In [2], the authors introduced the notion of quasi-injective gamma act : a V_Γ-act A is said to be quasi – injective, if any V_Γ-subact M of A and any $g \in \mathrm{Hom}_{V_\Gamma}(M, A)$, then an extension $\bar{g} \in \mathrm{End}_{S_\Gamma}(A)$ exist to g. Now for a collection $\{A_i \mid i \in I\}$ of right V_Γ-acts, the direct sum $\bigoplus_{i\in I} A_i$ of $\{A_i \mid i \in I\}$ is the set of elements $(w_i)_{i\in I} \in \prod_{i\in I} A_i$ such that $\{i \mid w_i \neq$

$\Theta_{A_i}\}$ is finite set [4, 5, 6]. In this paper, all V_Γ-act is Γ- commute α – unitary right having a unique zero.

2. Quasi – Injective and Noetherian Gamma Act Property

A V_Γ-subact B of A is called finite generator if and only if there is a finite subset $\{a_1, a_2, \ldots a_n\}$ of A and B = $a_1\Gamma V \cup a_2\Gamma V \ldots a_n\Gamma V$ [1]. If any ascending chain of V_Γ-subacts of A is stationary, then A is said Noetherian. It easy to prove a V_Γ-act A is Noetherian if and only if any V_Γ-subact of A is finite generating. Let A be a α – unitary V_Γ-act with T = $\mathrm{End}_{V_\Gamma}((A)$ and H = $\mathrm{End}_{V_\Gamma}(E(A))$ (proposition 3.9, [2]) proved $H \times \Gamma \times A = \{h\beta a \in E(A) \mid h \in H, \beta \in \Gamma \text{ and } a \in A\} = \{h(a\beta 1_\alpha) \in E(A) \mid h \in H, \beta \in \Gamma\ a \in A\}$. is quasi – injective V_Γ-subact of E(A) contain A [7].

Proposition 2.1: If direct sum of any set of quasi - injective V_Γ -acts remains quasi - injective, it implies that V is a right Noetherian Γ-semigroup.

Proof: Consider the ascending chain $\Theta = I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k \subseteq \ldots$ of a Γ – ideals and V / I_i be the Rees factorm let $Q_i = \mathrm{End}_{S_\Gamma}(I_i) \times \Gamma \times E(I_i)$ by (proposition 3.9, [2]) Q_i is quasi – injective for all i > 0. By the hypothesis $Q = \oplus_{i=1,2..} Q_i$ is quasi – injective. Let $I = \cup_{i=1,2,..} I_i$ and $\rho_k : I \to V / I_k$ be the canonical V_Γ-homomorphism from I into Q_k (since S / I_k embedding into Q_k). Now if w $\in$ I then a $\in I_t$ for some t, hence $\rho_k(w) = \Theta$ for any $k \geq t$. Let $f : I \to Q$ be a function defined by $f(w) = (\rho_k(w), \rho_k w), \ldots, \rho_k(w), \Theta, \ldots)$. Clearly that f is V_Γ-homomorphism, and since $I \subseteq S \subseteq Q_t \subseteq Q$, then there exists V_Γ-homomorphism $g : Q \to Q$ which is an extension to f. Suppose $g(1_\alpha) = h$. Then for all s $\in$ V, $g(s) = g(1_\alpha \alpha s) = h\alpha s$ and we can write $h = (h_1, h_2, \ldots, h_k, \Theta, \Theta, \ldots) \in Q$, there exists $k \in \mathbb{N}$ which $h_t = \Theta$ for all $t \geq k$. Then any a $\in$ I, f(a) = g(a) = hαa since $(h\alpha a)_t = h_t\alpha a = \Theta$ for all $t \geq k$ it follow that $I \subseteq I_k$ then above chain is stationary

Recalled. V_Γ-subact W of A is said to be Γ-retract in A, if there are V_Γ-homomorphisms $h : A \to W$ and $f : W \to A$ with $hf = 1_W$ [2]

Lemma. 2.2: *Let D and H be V_Γ-acts and $f : D \to H$ be V_Γ-monomorphism. If $D \oplus H$ is quasi – injective then D is Γ- retract of H.*

Proof: by quasi – injectivity of $D \oplus H$ there exist an V_Γ-homomorphism $g : D \oplus H \to D \oplus H$ such that $g(i_H f) = i_D$ and hence $1_D = p_D i_D = p_D(g(i_H f)) = (p_D g i_H) f = i_D f$ .Thus D is Γ - retract of H.

A V_Γ-subact E of A is called $\Gamma - \cap$- large in A, if any V_Γ-subact $R \neq \Theta$, then $E \cap R \neq \Theta$. Now for a Γ-ideal T of V. we defined the set $\Omega(M) = \{h : T \to V \mid$ there is $w \in A$ and $\gamma \in \Gamma$ such that $R_s^\gamma(w) \mid_T \subseteq \ker(h)\}$, $\Psi_M^\alpha = \{(s, t) \in A \times A \mid$ there is Γ-∩-large V_Γ-subact W such that $s\alpha w = t\alpha w$ for any $w \in W\}$ and $R_s^\gamma(w) \mid_I = R_s^\gamma(w) \cap (I \times I)$, and $R_s^\gamma(w) = \{(w, e) \in V \times V \mid h\gamma w = h\gamma e\}$. [2]

Theorem 2.3: *For a Γ-semigroup V. The following statements are equivalent*

(i) *direct sum of any finite set of quasi - injective V_Γ -acts remains quasi - injective, and V is a right Noetherian Γ-semigroup* **(ii)** *direct sum of any set of quasi - injective V_Γ -acts remains quasi - injective*

Proof: **(i) → (ii).** Let $\{A_i \mid i \in I\}$ be a collection of quasi – injective V_Γ-acts, and I be finitely generated right Γ- ideal of V with $f \in \Omega(\bigoplus_{i\in I} A_i)$, then there is a finite set of elements in V $\{x_1, \ldots, x_n\}$ with $I = x_1\Gamma V \cup \ldots \cup x_n\Gamma V$. Now $f(x_i) \in \bigoplus_{i\in I} A_i$ for all i = 1, …, n then there exist $c \in \mathbb{N}$ such that $f(x_i) \in \bigoplus_{i=1,..,c} M_i$ for all i = 1, 2, …, n. Then by (theorem 3.11, [2]) there exists an V_Γ-homomorphism $\bar{f}: S \to \bigoplus_{i=1,..,c} M_i \subseteq \bigoplus_{i\in I} M_i$ which is an extension of f. **(ii) → (i)** flows from **proposition 2.1**

3. ΓRQI - Semigroup

In this part we will consider Γ-semigroups in which all S_Γ-act is quasi – injective.

Definition 3.1: A Γ - semigroup V is called Γ – right quasi – injective (simply ΓRQI - Semigroup) if any right S_Γ-act is quasi – injective.

Examples 3.2:

(i) The Γ – semigroup S in (example 4.13, [2]) is not ΓRQI – Semigroup, since the right S_Γ-act M in this example is not quasi – injective.

(ii) Let $\Gamma = \{\alpha, \beta\}$ and S = {1, x, y, 0} such that 1 is α- identity and $w\alpha e = w$ for all w, e ∈ {x, y} and $r\beta t = 0$ for all r, t ∈ S. Then S is ΓRQI – Semigroup.

Proposition 3.3: A Γ - semigroup V is ΓRQI - Semigroup if and only if any right V_Γ-act is injective.

Proof: $A \oplus E(A)$ is quasi – injective for a V_Γ-act A. and lemma 2.3 A provided that A is Γ- retract of E(A), and it is obvious that any Γ – retract of E(A) is injective. The converse is obvious.

From proposition **2.2,** ΓRQI - Semigroup is a right Noethereian. But reverse may not be hold, for example the $\mathbb{N}$ – semigroup $\mathbb{Z}$ with multiplication mapping is right Noethereian but not ℕRQI– semigroup. An element r in Γ – semigroup V is called β – idempotent (where $\beta \in \Gamma$) if $r\beta r = r$. The following proposition gives other property of ΓRQI - Semigroup

Proposition 3.4: Let V be $\alpha_\circ$ - monoid. If V is ΓRQI - Semigroup, then any right Γ – ideal of V generated by $\alpha_\circ$ - idempotent element.

Proof: Let W a right Γ –ideal of V. Then by theorem 3.3 W is injective V_Γ-subact of V, by injectivity of W there exist an V_Γ-epimorphism $g : S \to W$ such that $g\ W = 1_W$. Thus $W = g(V) = g(1\alpha_\circ V) = g(1)\alpha_\circ V$ and $g(1)\alpha_\circ g(1) = g(1\alpha_\circ g(1)) = g(g(1)) = g(1)$ that $g(1)$ is $\alpha_\circ$ - idempotent

Proposition 3.5: Let V be $\alpha_\circ$ - monoid. If any right Γ – ideal of V is generated by $\alpha_\circ$ - idempotent element, implies V is ΓRQI - Semigroup.

Proof: Let A be an V_Γ-act, and I be a right Γ – ideal of V and $h : I \to A$ an V_Γ-homomorphism with $h \in \Omega(M)$. By hypotheses $I = x\Gamma V$ for some $\alpha_\circ$ - idempotent element x in V. Define $g : S \to A$ by $g(s) = h(x)\alpha_\circ s$ for all $s \in V$, it is clear that g is V_Γ-homomorphism and let $i = x\gamma t \in I$ where $\gamma \in \Gamma$ and $t \in V$. Then $g(i) = h(x)\alpha_\circ i = h(x)\alpha_\circ(x\gamma t) = (h(x)\alpha_\circ x)\gamma t = h(x\alpha_\circ x)\gamma t = h(x)\gamma t = h(x\gamma t) = h(i)$. this implies V is ΓRQI – Semigroup

The next theorem provides a characterization of ΓRQI-Semigroups.

Theorem 3.6: *Let V be $\alpha_\circ$ - monoid. Then V is ΓRQI – Semigroup if and only if any right Γ- ideal of V is generated by $\alpha_\circ$ - idempotent element*

Proof: Is directly from propositions 3.4 and 3.5

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