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f-coessential submodules

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Abstract

In this work, we introduce the concept of *f*-coessential submodule and definition (Let W be an R-module and N≤M≤W, then N is called a *f*-coessential submodule of M in W (symbolize $N\leq_{fcoe}M$ in W) if $M/N <<_f W/N$) was presented and its relationship to the concept of the coessential submodule, δ-coessential submodule and *f*-small were presented and some of the characteristics were given.

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Keywords: *f-coessential submodules, f-small submodules, f-supplement submodules, δ-coessential submodule.*

1. Introduction

Let T be a unit right R module where P is an identity ring. We will review some concepts and mention their source. Zhou in [1] introduces a small essential submodule: A is a non-zero small essential submodule of T (Brief $A\leq_{se}T$), if $A \cap N \neq 0 \ \forall \ (0 \neq)N \ll T$. In 2016 the small-singular submodules concept was introduced by [3]. The following provides a definition: let T be a P-module, then $Z^s(T)=\{a\in T|\text{Ann}(a) \leq_{se} P\}$. If $Z^s(T)$ =T, then T is called the s-singular module. For S≤M≤T, then S is called a coessential submodule of M in

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T (symbolize S$\leq_{co}$M in T) if $M/_S \ll T/_S$ [4]. And let P $\ll_f$T be a f-small submodule of T if, P+S=T and T/S s-singular[5]. Following Zhou [1], a submodule X of a module W is called a δ-small submodule (notation $X \ll_\delta T$) if T = X +S with T/S singular, then T = S. And let T be an R-module and L≤N≤T, then L is called a δ-coessential submodule of N in T (symbolize L$\leq_{\delta ce}$N in T) if $N/_L \ll_\delta T/_L$ [7-10].

Before starting with the details of the research topic, I need the following property and the prove is clear by [3] and [5]:

Proposition 1.2: Consider a P-module T with two submodules, S and K.

If $\frac{T}{S}$ is s-singular module, then $\frac{T}{S+K}$ is s-singular.

Proposition 1.3: Let H and K be an P-modules and $\alpha: H \to K$ be a homomorphism. If $\frac{K}{A}$ is a s-singular module, then $\frac{H}{\alpha^{-1}(A)}$ is a s-singular module.

Corollary 1.4 [3]: *Every submodules of s-singular is s-singular module.*

Corollary 1.5 [5]: *Every submodules of f-small submodules is f-small.*

Lemma 1.6: *Consider T as an R-module with two submodules, S and U. Then $S + U \ll_f T$ if and only if $S \ll_f T$ and $U \ll_f T$.*

2. f-coessential Submodules

Definition 2.1: Let T be an P-module & N≤H≤T, then N is called a f-coessential submodule of H in T if $H/_N \ll_f T/_N$, symbolize N$\leq_{fcoe}$H in T.

Examples and Remarks 2.2:

1. The relation between submodules f-coessential and f-small clear by (N $\ll_f$ T $\Leftrightarrow$ 0$\leq_{fcoe}$N in T)
2. Every f -coessential submodules of an R-module W is δ-coessential. But the converse is not true for example, consider $\mathbb{Z}_6$ as $\mathbb{Z}_6$-modules where $0 \leq_{\delta coe} 3\mathbb{Z}_6$ since $3\mathbb{Z}_6 \ll_\delta \mathbb{Z}_6$ but $0 \not\leq_{fcoe} 3\mathbb{Z}_6$ since $3\mathbb{Z}_6 + 2\mathbb{Z}_6 = \mathbb{Z}_6$ and $\mathbb{Z}_6/_{3\mathbb{Z}_6}$ is s-singular but $2\mathbb{Z}_6 \neq \mathbb{Z}_6$.
3. Clearly by Cor.(1.5) every submodule of f-coessential is f-coessential submodule.

In the following property, we prove the transitive property of f-coessential submodules by using the class of coessential submodules.

Proposition 2.3: For any chain of submodules K≤ N≤ O≤ T of an R-module T, then K$\leq_{fcoe}$N in T and N $\leq_{fcoe}$O in U if and only if K$\leq_{fcoe}$O in T.

Proof: $\Leftarrow$) Since $K \leq_{fcoe} O$ in T, then $O/_K \ll_f T/_K$. But $N/_K \subseteq O/_K \subseteq T/_K$ thus by Coll.(1.5)$N/_K \ll_f T/_K$ That is $K \leq_{fcoe} N$ in T. Now, define α: $T/_K \to T/_N$ by α (w+K) = w + N $\forall$ w $\in$ T. It is clear that α is an epimorphism, therefore by [5] α $(O/_K)$ = $O/_N \ll_f T/_N$. Thus N $\leq_{fcoe}$O in T.($\Longrightarrow$) Claim that K $\leq_{fcoe}$O in T, thus $T/_K = O/_K + X/_K$, where $\frac{T/_K}{X/_K}$ is an s-singular by 3^{th}isomorphism theorem , $T/_X$ s-singular. Then T = O + X and $T/_N = O/_N + X + N/_N$ and $\frac{T/_N}{X+N/_N} \cong T/_{X + N}$ is an s-singular by 3^{th}isomorphism theorem. So $T/_N$ = $X + N/_N$ and hence T = X + N. Then $T/_K = X/_K + N/_K$. and since $N/_K \ll_f T/_K$ & $\frac{T/_K}{X/_K}$ is an s-singular, $T/_K = X/_K$ thus T = X.

Proposition 2.4: Let T be an P-module and S $\subseteq$ M $\subseteq$ H $\subseteq$ T. Then $M \leq_{fcoe} H$ in T if and only if $\frac{M}{S} \leq_{fcoe} \frac{H}{S}$ in $\frac{T}{S}$.

Proof: Assume that $M \leq_{fcoe} H$ inT. Since $\frac{H}{M} \cong \frac{H/S}{M/S}$ and $\frac{T}{M} \cong \frac{T/S}{M/S}$ (by 3^{th}isomorphism theorem). We get $\frac{H/S}{M/S} \ll_f \frac{T/S}{M/S}$ so $\frac{M}{S} \leq_{fcoe} \frac{H}{S}$ in $\frac{T}{S}$.($\Leftarrow$) Suppose that $\frac{M}{S} \leq_{fcoe} \frac{H}{S}$ in $\frac{T}{S}$, since $\frac{H/S}{M/S} \cong \frac{H}{M}$ and $\frac{T/S}{M/S} \cong \frac{T}{M}$ (by 3^{th}isomorphism theorem), then $\frac{H}{M} \ll_f \frac{T}{M}$. So $M \leq_{fcoe} H$ in T.

Proposition 2.5: Let T be an P-module. If $M \leq_{fcoe} N$ and $H \leq_{fcoe} K$ in T, then M + $H \leq_{fcoe} N$+K in T.

Proof: Suppose that $M \leq_{fcoe} N$ in T and $H \leq_{fcoe} K$ in T. To show that $M+H \leq_{fcoe} N+K$ in T. Let $\alpha: \frac{T}{M} \to \frac{T}{M+H} \ni \alpha(t + M) = t + (M + H)\ \forall t \in T. And\ \beta: \frac{T}{H} \to \frac{T}{M+H} \ni \alpha(t + H) = t + (M + H)\ \forall t \in T$. Clearly each α and β an epimorphisms. Since $\frac{N}{M} \ll_f \frac{T}{M}$ and $\frac{K}{K} \ll_f \frac{T}{K}$ then $\alpha(\frac{N}{M})$= $\frac{N+H}{M+H} \ll_f \frac{T}{M+H}$and $\beta(\frac{K}{H})$= $\frac{K+H}{M+H} \ll_f \frac{T}{M+H}$, by[5] and hence $\frac{N+H}{M+H} + \frac{M+K}{M+H}$= $\frac{N+K}{M+H} \ll_f \frac{T}{M+H}$ by lemma(1.6). Therefore $M+H \leq_{fcoe} N+K$ in T.

Corollary 2.6: *Let T be an P-module and $H \ll_f T$. Then $M \leq_{fcoe} N$ in T, if and only if $M+H \leq_{fcoe} N+H$ in T.*

Proof: It is clear by using above proposition and converse part by using 3^{th}isomorphism theorem

Proposition 2.7: Let H and K be an P-modules and $\alpha: H \rightarrow K$ be a epimorphism. If $X \leq_{fcoe} Y$ in H, then $\alpha(X) \leq_{fcoe} \alpha(Y)$ in K.

Proof: Suppose that $X \leq_{fcoe} Y$ in H . To prove that $\alpha(X) \leq_{fcoe} \alpha(Y)$ in K. Let $\frac{K}{\alpha(X)}=\frac{\alpha(Y)}{\alpha(X)}+\frac{A}{\alpha(X)}$ where $\frac{K}{\alpha(X)}\Big/\frac{A}{\alpha(X)}$ is a s-singular module. Thus K= $\alpha(Y)+A$ it is clear H=Y+$\alpha^{-1}(A)$, so $\frac{H}{X}=\frac{Y}{X}+\frac{\alpha^{-1}(A)+X}{X}$, since $\frac{K}{\alpha(X)}\Big/\frac{A}{\alpha(X)}$ is a s-singular module, then $\frac{K}{A}$ is a s-singular module and hence by Pro.(1.3) we get $\frac{H}{\alpha^{-1}(A)}$ is a s-singular module. And by 3^{th}isomorphism theorem $\frac{H}{X}\Big/\frac{\alpha^{-1}(A)+X}{X} \cong \frac{H}{\alpha^{-1}(A)+X}$ is a s-singular module by Pro.(1.2). Therefore $\frac{H}{X}\Big/\frac{\alpha^{-1}(A)+X}{X}$ s-singular module. Since $\frac{Y}{X} \ll_f \frac{H}{X}$, then H=$\alpha^{-1}(A)+X$ so K=A+ $\alpha(X)$, but $\alpha(X) \subseteq A$, thus K=A. Hance $\alpha(X) \leq_{fcoe} \alpha(Y)$ in K

Proposition 2.8: Let T be an P-module with P is hollow and S $\subseteq$ N $\subseteq$ T, then S$\leq_{fcoe}$N in T if and only if S$\leq_{co}$N in T.

Proof: ($\leftarrow$) Assume that S $\leq_{co}$ N in T. To prove that then S$\leq_{fcoe}$N. Let $\frac{T}{S}=\frac{N}{S}+\frac{X}{S}$ where $\frac{T/S}{X/S}$ is s-singular module and since P is hollow ring then $\frac{T/S}{X/S}$ is singular module by [3]. But S $\leq_{co}$ N in T i.e $\frac{N}{S} \ll \frac{T}{S}$, therefore $\frac{T}{S}=\frac{X}{S}$. Thus S$\leq_{fcoe}$N. ($\rightarrow$) Assume that S $\leq_{fcoe}$ N in T. To prove that S$\leq_{co}$N in T ($\frac{N}{S} \ll \frac{T}{S}$). Let $\frac{T}{S}=\frac{N}{S}+\frac{X}{S}$ where $\frac{T/S}{X/S}$ is singular module and by relation between singular module and small-singular [3]we get $\frac{T/S}{X/S}$ is s-singular module but since $\frac{N}{S} \ll_f \frac{T}{S}$ so $\frac{T}{S}=\frac{X}{S}$.

Proposition 2.9: Let K $\subseteq$ H be submodules of an R-module T. If H = K + X and X$\ll_f T$, then K$\leq_{fcoe} H$in T.

Proof: Assume that H=K+X and X$\ll_f T$. Let $\frac{T}{K}=\frac{H}{K}+\frac{P}{K}$ where $\frac{T/K}{P/K}$ is s-singular module and by 3^{th}isomorphism theorem $\frac{T/K}{P/K} \cong \frac{T}{P}$. Then T=H+P=K+X+P and since $\frac{T}{P}$ is s-singular, and since X$\ll_f T$ and $\frac{T}{K+P}$ is small-singular by Pro.(1.2) therefore T=K+P and K$\subseteq P$. Thus T=X and hence K$\leq_{fcoe} H$in T.

Proposition 2.10: Let M, S, O and P be submodules of an R-module W. The following statements are equal:

i. If M$\leq_{fcoe} S \text{ in } W, \text{and } K \subseteq W, \text{then } M \cap K \leq_{fcoe} S \cap K \text{ in } W.$

ii. If M$\leq_{fcoe} S \text{ in } W \text{ and }$ O$\leq_{fcoe}$P in W, then M$\cap O \leq_{fcoe} S \cap P$.

iii. If M $\leq_{fcoe}$M+S in W, then M ∩ S $\leq_{fcoe}$ S in W.

Proof: ($i\rightarrow ii$) Let M$\leq_{fcoe}$ S in W, and O$\leq_{fcoe}$P in W. Then by (i), $M \cap O \leq_{fcoe} S \cap O$ in W. And O$\leq_{fcoe}$P in W and $S \subseteq W$, then S∩ $O \leq_{fcoe} S \cap P$ in W. So M∩ $O \leq_{fcoe} S \cap P$ in W by Pro.(2.8). ($ii\rightarrow iii$) Let M $\leq_{fcoe}$M+S in W. Since S$\leq_{fcoe}$S in W, then by (2) M ∩ S $\leq_{fcoe}$ $(M+S) \cap S$ in W. ($iii\rightarrow i$) Let M$\leq_{fcoe}$ S in W, and $K \subseteq W$. Since $M + (S \cap K) \subseteq S$, then M $\leq_{fcoe}$ $M + (S \cap K)$ in W by Pro.(2.8). Hence M∩ $(S \cap K) \leq_{fcoe}$ $S \cap K$ in W, by (iii). We get $M \cap K \leq_{fcoe} S \cap K$ in W.

Recall that S is named f −supplement of M in T if M + S=T & M ∩ S $\ll_f$ T[5].

Proposition 2.11: Let U, N and O be submodules of an R-module T. If U is a f −supplement of N in T and N is a f −supplement of O in T with U⊆O, then U$\leq_{fcoe}$ O in T.

Proof: Let U is a f −supplement of N in T and N is a f −supplement of O in T with U⊆O. To prove that U$\leq_{fcoe}$ O in T. Let $\frac{T}{U} = \frac{O}{U} + \frac{X}{U}$, where $\frac{T/U}{X/U}$ is s-singular submodule (T.P. $\frac{T}{U} = \frac{X}{U}$). So T=O+X by modular law X=$X \cap T$=X∩ $(N+U) = (X \cap N) + U$. Hence T=O+X=O+(X∩ N)+U= O+(X∩ N) by the second isomorphism theorem $\frac{N}{X \cap N} \cong \frac{N+X}{X} = \frac{T}{X}$, but $\frac{T}{X}$ is s-singular submodule, so $\frac{N}{X \cap N}$ is s-singular submodule. And since N is a f −supplement of O in T and X∩ $N \subseteq N$ then X∩ $N = N$ and hence N⊆ X. Thus T=X , therefore $\frac{T}{U} = \frac{X}{U}$. We get U$\leq_{fcoe}$ O in T.

Recall that a submodule H is called f −coclosed in T if $\frac{H}{A}$ is small-singular and $\frac{H}{A} \ll_f \frac{T}{A}$ for some A⊆ H, implies H = A (Shortly H$\leq_{fcoc}$ T), see [6].

Proposition 2.12: Let S $\leq_{fcoc}$ T. If K is a proper submodule of S s.t. K$\leq_{fcoe}$ S in T, then S = K ⊕ X for some submodule X of T.

Proof: Assume that K is a proper submodule of S s.t. K$\leq_{fcoe}$ S in T. By [2, Cor.(5.2.2)], ∃ X$\leq T$ s.t. K⊕$X \leq_e$A & by [1] we get K⊕$X \leq_{se}$A then by [3]$\frac{A}{K \oplus X}$ is s-singular . Let a map g:$\frac{T}{K} \rightarrow \frac{T}{K \oplus X}$ ∋ g(t)=t+ (K⊕X) ∀ $t \in T$, clear that g is an epimorphism. And $\frac{S}{K} \ll_f \frac{T}{K}$,so g($\frac{S}{K}$)= $\frac{S}{K \oplus X} \ll_f \frac{T}{K \oplus X}$ see[5]. But S$\leq_{fcoc}$ T, thus S=K⊕X.

3. Discussion and Conclusion

In the present paper, we introduced a new definition called f- coessential submodule, and showed its relationship with the following concepts f-small submodules, coessential submodules, and δ-coessential submodules with some of characteristics.

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