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Double captive domination number (DCDN)

Zainab Yasir Alrikabi *
Department of Mathematics
Faculty of Education for Girls
University of Kufa
Najaf
Iraq

Ahmed A. Omran †
Department of Mathematics
College of Education for Pure Science
University of Babylon
Babylon
Iraq

Hassan Jiad Al-Hwaeer §
Department of Mathematics and Computer Application
College of Sciences
University of Al-Nahrain
Baghdad
Iraq

Abstract

Throughout this work, a new definition of domination is presented which is called The Double Captive Domination Number (DCDS). Also, some of the properties of this definition are been proved. Especially, to some certain graphs and some operation of two graphs.

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Keywords: *Double captive domination number, Double captive dominating set, Complement.*

* *E-mail:* **Zainabalrikabi76@gmail.com (Corresponding Author)**
† *E-mail:* **pure.ahmed.omran@uobabylon.edu.iq**
§ *E-mail:* **Hassan1167@yahoo.com**

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1. Introduction

Consider $G = (V, E)$ be an undirected simple finite graph [4]. In recent years, the notion of domination has become the most common concept used in all sciences and disciplines Therefore, many new definitions have emerged around this concept to find solutions to various problems [2-3] and [5-7]. A subset $D \subseteq \mathrm{V}(G)$ is a dominating set (DS) of G if $N(v) \cap D \neq \emptyset$; for all $v \in V - D$ [4].

A subset $D \subset \mathbb{V}(G)$ is a captive DS (CDS) if $\forall\, v \in D,\ N(v) \cap D \neq \emptyset N(v) \cap (\mathrm{V} - \mathrm{D})$. The cardinality of the smallest CDS of G, denoted by $\gamma_{ca}(G)$, is the captive domination number $(CDN)$of $\mathbb{G}$. The CDN is initiated by Ahmed and etc. [1], and they get many properties and bounded. After this Zainab and Ahmed determined the CDN, complement of CDN, and inverse of CDN of many kinds of graphs [8-9]. In this paper, the new properties are discussed, especially for some graphs and several bounds for $DCDS$ are stated.

2. Double CDN (DCDN)

A graph G and let $D \subseteq V(G)$, so D is called a double CDS $(D \subseteq V(G))$ in the graph G if the set D is a CDS and $|N(v) \cap D| \leq 2, \forall\, v \in (\mathrm{V} - \mathrm{D})$.

Definition 2.2: Consider D to be DCDS in graph G, so D is called a minimal DCDS (MDCDS) if there is no proper DCDS in the set D.

Definition 2.3: Consider a graph G and D ⊆ V(G), so |D| is called DCDN if $|D| = min\{|D_i||D_i \in MDCDS\}$, and is denoted by $\gamma_{Dca}(G)$.

Note 2.4: Every $DCDS$ is a CDS depending on Definition 1.1, but the converse is not true in general. As an example, take C_4, see Figure 1:

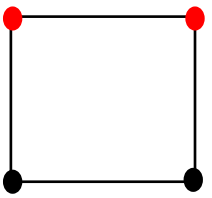


Figure 1
A graph G has γ_{ca}but has no γ_{Dca}

Observation 2.5: Consider G to be a graph of order n and has $DCDS$ and DN γ_{Dca}, then :

1. $|V(G)| \geq 3$.
2. $de\, g(v) \geq 2, \forall\, v \in \mathrm{V}(G)$.
3. $\Delta(G) \geq 2$ and $\delta(G) \geq 2$.
4. If $v \in \mathrm{V} - \mathrm{D}$, so the intersection of $N(v)$with D and $(\mathrm{V} - \mathrm{D})$ is non-empty.
5. $\gamma(G) \leq \gamma_t(G) \leq \gamma_{ca}(G) \leq \gamma_{Dca}(G)$.

Remark 2.6: G has no $DCDN$ (γ_{Dca}) if G has a pendant vertex.

Proof: The proof is straightforward from Definition 2.1.

Lemma 2.7: [3] *Consider G to be a graph such that the degree of every vertex in it is greater than or equal to two, then the graph G contains a cycle.*

Proposition 2.8: Consider G to be a graph and has $DCDN$ γ_{Dca}, thus G contains a cycle. The converse of this property is not true in general.

Proof: The graph G has $DCDN$ γ_{Dca} by hypothesis, so $de\,g(v) \geq 2, \forall\, v \in \mathrm{V}(G)$ according to Observation 1.5. (2). Therefore, the graph G contains a cycle according to Lemma 2.6.

The converse is not true in general, as an example, see Figure 1.

Proposition 2.9: Consider G ($K_{n,m}$) to be a bipartite graph with partite sets X and Y of cardinality n and m respectively, so $\gamma_{\mathrm{Dca}}(\mathrm{K}_{\mathrm{n,m}}) = 4$ for $\mathrm{n, m} \geq 3$.

Proof: Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of the set X and the set of vertices of the set Y are $\{u_1, u_2, \dots, u_m\}$. According to the definition of a bipartite set each vertex belonging to the set X is adjacent to each vertex in the set $\mathbb{Y}$ and vice versa. Now, to keep of the definition of $DCDS$ (D) two vertices of each of the partite sets X and Y must be chosen. Thus, $|\mathrm{D}| \geq 4$, so three cases are classified as below.

Case 1: If $|G| = 4$, then all vertices in G belong to D. Therefore the graph has no $DCDS$ since there is no vertex out of the set D.

Case 2: If $|G| = 5$, then the cardinal of partite sets X and Y are two and three. Without loss of generality assume that $|X| = 2$ and $|Y| = 3$. Thus, the two vertices in the set Y that belong to DS do not dominate any vertex in the set X. Therefore, the DS is not $DCDS$.

Case 3: If $|G| > 6$; $n, m \geq 3$, then If we choose two vertices from each partite sets X and Y and create from these four vertices a set D. One can easily show that set D satisfies all conditions of $DCDS$. Furthermore, the cardinal of the set D is the minimum cardinal which satisfies all conditions of $DCDS$, then $\gamma_{Dca}(G) = 4$.

Depending on the above classifications, the required is obtained.

Proposition 2.10: Consider G be a wheel graph W_n, so

$$\gamma_{Dca}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1.$$

Proof: Let $v_1, v_2, \dots, v_n$ are the vertices of the wheel graph, and let D be the subset of the vertex set such that $D = \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil\frac{n-1}{3}\right\rceil - 1\right\} \cup \{v_n\}$. It's known that $W_n \equiv K_1 + C_{n-1}$, so the wheel graph obtained from join operation of two graphs one of them is $K_1 = \{v_n\}$ and the other is $C_{n-1} = \{v_1, v_2, \dots, v_{n-1}\}$. Now, let $D_1 = \left\{v_{1+3i}, i = 0,1, \dots, \left\lceil\frac{n-1}{3}\right\rceil - 1\right\}$ it is easy to see that D_1 is DS of the wheel graph but it is not $DCDS$, since its not total. To keep the total property let $D = D_1 \cup S_1$, the set keeps to the total property and DS. Furthermore, each vertex in the set D dominates at least one vertex in $(\mathrm{V} - \mathrm{D})$. Thus, D is $DCDS$ and it's obvious that the cardinal of this set is minimum. Therefore, $\gamma_{Dca}(W_n) = \left\lceil\frac{n-1}{3}\right\rceil + 1$.

Proposition 2.11: Consider G to be a corona graph of two cycles $G \equiv C_n \odot C_m$ then $\gamma_{Dca}(G) = n(\left\lceil\frac{m}{3}\right\rceil + 1)$.

Proof: Suppose that $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_m\}$ are the vertices of C_n and C_m respectively. Furthermore, let $\{v_1, v_2, \dots, v_n, u_1^1, u_2^1, \dots, u_m^1, u_1^2, u_2^2, \dots, u_m^2, u_1^n, u_2^n, \dots, u_m^n\}$ are the vertices of the graph G (as an example, see Figure 2). In each copy of the cycle C_m, the minimum cardinality of DS of this copy is equal to $\left\lceil\frac{m}{3}\right\rceil$ and add the vertices of C_n to keep the property that each vertex in $(\mathrm{V} - \mathrm{D})$ dominates by two vertices. Thus, this set is $DCDS$ and the cardinal of this set is minimum, so $\gamma_{Dca}(G) = n(\left\lceil\frac{m}{3}\right\rceil + 1)$.

Example 2.12:

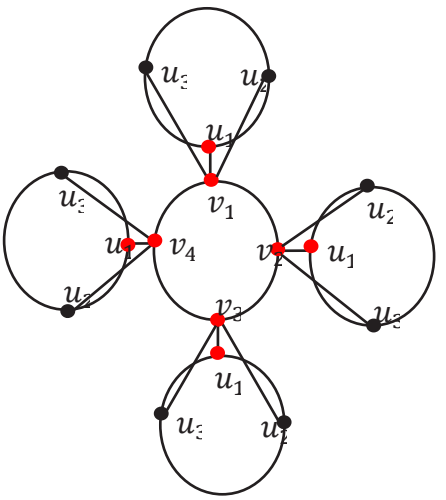


Figure 2
$\boldsymbol{G \equiv C_n \odot C_m}$

3. Conclusion

New important properties of a new definition have been discovered along the process of arriving at the results described in this article. Specially identified above are several graphs' inverse and complement.

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