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A new type of fuzzy soft topological space via binary relation fuzzy soft set

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Abstract

We propose in this work a novel idea to construct a kind of fuzzy soft topological space via utilizing the definition of Binary Relation-FS-set, which was presented recently, and it is called Binary Relation-FS-topological spaces. We center around the foundational definitions and main properties of the Binary Relation-FS-topology, such as, the Binary Relation -FS-interior, and Binary Relation -FS-closure, with some basic theorems. This new kind of fuzzy-soft topology is more functional and advantageous, and it also contributes to some scientific applications. Furthermore, an advantageous and important applied example is presented.

Subject Classification: *Primary 03E72, Secondary 06D72.*

Keywords: *Binary relation-FS-topology, Binary relation-FS-open (closed) set, Binary relation-FS-closure (interior).*

1. Introduction

Theories such as fuzzy set and soft set have their own frameworks. Fuzzy theory was considered an advantageous and suitable tool before the topic of soft set theory, but after the presentation of set theory, the soft set is reflected as more appropriate between them. Maji and others. [1] first proposed the definition of fuzzy soft sets (briefly FS-set) via embedding the concepts of F-sets (fuzzy set). Yang and others [2] presented some main operations on FS- set

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theory. Later scientific researchers studied the concepts of FS- sets, see [3,4]. Moreon, Kandemir and Tanay [5] first proposed the definition of FS-topological space utilizing the FS-set.

Recently, in the reference [6], the authors introduced a new idea of FS-set by linking the notions of Binary, Ternary and N-ary relation S-sets definitions together with the F-sets definitions with respect to power set and then arising the novel kinds which they named as (Binary relation-FS-set, Ternary relation-FS-set, N-ary relation-FS-set). The reader can find the application of above new concept in digital image similarity and related ideas in references [7,8,9].

The main purpose of this work is to propose a novel kind of FS-topological space (namely Binary Relation-FS- topological space). Firstly, we will recall some foundational definitions in Binary Relation-FS-sets. Later we will define and investigate the basic and important properties of Binary Relation-FS-interior and Binary Relation-FS-closure of Binary Relation-FS- open(closed)-sets.

The paper is organized as follows. In section 2, main foundational definitions and concepts that are central in this our work are viewed. Section 3 This is the main section in this work, it presents the characterization of Binary Relation-FS-topology, Binary Relation-FS-open-set, Binary Relation-FS-closure and Binary Relation-FS-interior, with useful applied example and some main theorems. Finally, in section 4 we draw conclusions and consider future work.

2. Preliminaries

In this section, we view the important notions of Binary Relation-FS-sets and some operations of them, that are important and necessary for subsequent visions and which are central to our paper.

Definition 2.1 [6]: $\mathfrak{T}$ is an initial universe set, $\mathfrak{E}_1, \mathfrak{E}_2$ are sets of parameters. And $\mathcal{P}(\mathfrak{T})$ represents the power set of $\mathfrak{T}$. Binary Relation-FS-set $\mathfrak{B}$ on $\mathfrak{T}$ is representing by a composition of member-ship mapping of fuzzy-set with r. to power set together with the Binary Relation-Soft-set. Then $f_{\mathfrak{B}}: \mathfrak{E}_1 \times \mathfrak{E}_2 \to \mathbb{I} = [0,1]$, is a member-ship mapping of binary Relation-FS-set $\mathfrak{B}$ where; $f_{\mathfrak{B}}(\mathfrak{h}_i, \mathfrak{h}_j) = (\mathfrak{F}o\mathfrak{S})(\mathfrak{c}_i, \mathfrak{c}_j)$ i.e. $\mathfrak{B} = \{((\mathfrak{c}_i, \mathfrak{c}_j), (f_{\mathfrak{B}}(\mathfrak{c}_i, \mathfrak{c}_j))), \mathfrak{c}_i \in \mathfrak{E}_1 \text{ and } \mathfrak{c}_j \in \mathfrak{E}_2, i = 1, \dots, \mathfrak{s}; j = 1, \dots, \mathfrak{r}\}$

Definition 2.2 [6]: A Binary Relation-FS-set $\mathfrak{B}$ is called a null Binary Relation-FS-set iff $f_{\mathfrak{B}}(\mathfrak{c}_I, \mathfrak{c}_j) = 0$,i.e. $0(\mathfrak{c}_i, \mathfrak{c}_j) = \left\{\left((\mathfrak{c}_i, \mathfrak{c}_j), 0\right), \forall (\mathfrak{c}_i, \mathfrak{c}_j) \in \mathfrak{E}_1 \times \mathfrak{E}_2 \text{ ; } i = 1, ., \mathfrak{s} \text{ ; } j = 1, \dots, \mathfrak{r}\right\}$

Definition 2.3 [6]: A Binary Relation-FS-set $\mathfrak{B}$ is called a universal Binary Relation-FS-set iff $f_{\mathfrak{B}}(\mathfrak{c}_i, \mathfrak{c}_j) = 1$,i.e. $1(\mathfrak{c}_i, \mathfrak{c}_j) = \left\{\left((\mathfrak{c}_i, \mathfrak{c}_j), 1\right), \forall (\mathfrak{c}_i, \mathfrak{c}_j) \in \mathfrak{E}_1 \times \mathfrak{E}_2 \text{ ; } i = 1, .., \mathfrak{s} \text{ ; } j = 1, \dots, \mathfrak{r}\right\}$

Definition 2.4 [6]: The complement of Binary Relation-FS-set $\mathfrak{B}$ is symbolized by $\mathfrak{B}^{c}$ and is defined as the following; $(f_{\mathfrak{B}})^{c}: \mathfrak{E}_1 \times \mathfrak{E}_2 \to \mathbb{I} = [0,1]$, is a mapping given by $(f_{\mathfrak{B}})^{c}(\mathfrak{c}_i, \mathfrak{c}_j) = 1 - f_{\mathfrak{B}}(\mathfrak{c}_i, \mathfrak{c}_j)$, where, $i = 1, \dots, \mathfrak{s}$; $j = 1, \dots, \mathfrak{r}$.

Definition 2.5 [6]: Union two Binary Relation-FS-sets $\mathfrak{B}$ and $\mathfrak{Y}$ on the common universe $\mathfrak{X}$ is the Binary Relation-FS-set $\mathfrak{N}$ and where $\mathfrak{N} = (\mathfrak{B} \sqcup \mathfrak{Y})$, then the member-ship mapping; $f_{\mathfrak{N}}(\mathfrak{c}_i, \mathfrak{c}_j) = \max\{f_{\mathfrak{B}}(\mathfrak{c}_i, \mathfrak{c}_j), f_{\mathfrak{Y}}(\mathfrak{c}_i, \mathfrak{c}_j)\}$, where $\mathfrak{c}_i \in \mathfrak{E}_1$ and $\mathfrak{c}_j \in \mathfrak{E}_2$, $i = 1, \dots, \mathfrak{s}$; $j = 1, \dots, \mathfrak{r}$.

Definition 2.6 [6]: Intersection two Binary Relation-FS-sets $\mathfrak{B}$ and $\mathfrak{Y}$ on the common universe $\mathfrak{X}$ is the Binary Relation-FS-set $\mathfrak{N}$ and where $\mathfrak{N} = (\mathfrak{B} \sqcap \mathfrak{Y})$. Then the member-ship mapping; $f_{\mathfrak{N}}(\mathfrak{h}_i, \mathfrak{h}_j) = \min\{f_{\mathfrak{B}}(\mathfrak{c}_i, \mathfrak{c}_j), f_{\mathfrak{Y}}(\mathfrak{c}_i, \mathfrak{c}_j)\}$, where $\mathfrak{c}_i \in \mathfrak{E}_1$ and $\mathfrak{c}_j \in \mathfrak{E}_2$, $i = 1, \dots, \mathfrak{s}$; $j = 1, \dots, \mathfrak{r}$.

3. The Binary Relation-FS-Topology

In this section we will define the novel type of FS- topology based on the Binary Relation-FS-set which was initially proposed in [6], with useful applied examples to make it clear to the reader.

Definition 3.1: $\mathfrak{X}$ is initial universe, $\mathfrak{E}_1, \mathfrak{E}_2$ are two sets of parameters, $\mathfrak{B}$ and $\mathfrak{Y}$ represent the Binary Relation-FS-sets on $\mathfrak{X}$, a Binary Relation-FS-topology on $\mathfrak{X}$ is symbolized by $\mathcal{T}_{BRFS}$, which must satisfy the following:

1- $0(\mathfrak{c}_i, \mathfrak{c}_j)$, $1(\mathfrak{c}_i, \mathfrak{c}_j) \in \mathcal{T}_{BRFS}$
2- If $\mathcal{O}$ and $\mathcal{P} \in \mathcal{T}_{BRFS} \to \mathcal{O} \sqcap \mathcal{P} \in \mathcal{T}_{BRFS}$
3- For any index ζ and for all $\eta \in \zeta$, $\mathcal{O}_\eta \in \mathcal{T}_{BRFS} \Rightarrow \sqcup_{\eta \in \zeta} \mathcal{O}_\eta \in \mathcal{T}_{BRFS}$

And $(\mathfrak{X}, \mathcal{T}_{BRFS})$ is called Binary Relation-FS-topological space. The elements of $(\mathfrak{X}, \mathcal{T}_{BRFS})$ are called Binary Relation-FS-open-set.

Definition 3.2: Let $(\mathfrak{X}, \mathcal{T}_{BRFS})$ be a Binary Relation-FS-topological space on $\mathfrak{X}$ and let $\mathcal{P}$ be a Binary Relation-FS-set on $\mathfrak{X}$. Then $\mathcal{P}$ is called a Binary Relation-FS-closed if the complement of $\mathcal{P}$ is Binary Relation-FS-open-set.

Example 3.3: Suppose that $\mathfrak{E}_1$ represent the parameter set which indicates student levels $\mathfrak{E}_1 = \{\mathfrak{s}_1 = fail, \mathfrak{s}_2 = successful\}$, $\mathfrak{E}_2$ be the parameter set which indicates student cases of living level such as $\mathfrak{E}_2 = \{\mathfrak{r}_1 = poor, \mathfrak{r}_2 = rich\}$, let $\mathfrak{X}$ be the universal set consisting of nine students from each stage of study (suppose that every stage has nine students) such as $\mathfrak{X} = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5, \mathfrak{l}_6, \mathfrak{l}_7, \mathfrak{l}_8, \mathfrak{l}_9\}$, And $\mathfrak{F}(A) = \mathfrak{w}$; $A \in P(\mathfrak{X})$, $\mathfrak{w} \in \mathbb{I} = [0,1]$ indicates the magnitude of financial donations proportional to the tuition fees at any or certain stage of the study in the private college in which they study, then let $\mathfrak{B}$ represents the Binary Relation-FS-set on $\mathfrak{X}$ for the first stage, s.t.;

$$\mathfrak{S}(\mathfrak{s}_1 = fail, \mathfrak{r}_1 = poor) = \{\mathfrak{l}_5\}, \mathfrak{F}(\{\mathfrak{l}_5\}) = 0.5, \text{ then } f_{\mathfrak{B}}(\mathfrak{s}_1, \mathfrak{r}_1) = 0.5$$

$\mathfrak{S}(\mathfrak{s}_2 = successful, \mathfrak{r}_1 = poor) = \{\daleth_1, \daleth_2, \daleth_3\}$, $\mathfrak{F}(\{\daleth_1, \daleth_2, \daleth_3\}) = 0.9$,

then $f_{\mathfrak{B}}(\mathfrak{s}_2, \mathfrak{r}_1) = 0.9$

$\mathfrak{S}(\mathfrak{s}_1 = fail, \mathfrak{r}_2 = rich) = \{\daleth_7, \daleth_8, \daleth_9\}$, $\mathfrak{F}(\{\daleth_7, \daleth_8, \daleth_9\}) = 0.3$,

then $f_{\mathfrak{B}}(\mathfrak{s}_1, \mathfrak{r}_2) = 0.3$

$\mathfrak{S}(\mathfrak{s}_2 = successful, \mathfrak{r}_2 = rich) = \{\daleth_4, \daleth_6\}$, $\mathfrak{F}(\{\daleth_4, \daleth_6\}) = 0.7$,

then $f_{\mathfrak{B}}(\mathfrak{s}_2, \mathfrak{r}_2) = 0.7$, then

$$\mathfrak{B} = \{((\mathfrak{s}_1, \mathfrak{r}_1), 0.5), \big((\mathfrak{s}_2, \mathfrak{r}_1), 0.9\big), \big((\mathfrak{s}_1, \mathfrak{r}_2), 0.3\big), ((\mathfrak{s}_2, \mathfrak{r}_2), 0.7)\},$$

then let $\mathfrak{D}$ represents the Binary Relation-FS-set on $\mathfrak{T}$ for the second stage, such that;

$$\mathfrak{D} = \{((\mathfrak{s}_1, \mathfrak{r}_1), 0.5), \big((\mathfrak{s}_2, \mathfrak{r}_1), 0\big), \big((\mathfrak{s}_1, \mathfrak{r}_2), 0.3\big), ((\mathfrak{s}_2, \mathfrak{r}_2), 0)\},$$

then let $\mathfrak{W}$ represent the Binary Relation-FS-set on $\mathfrak{T}$ for the third stage, such that;

$$\mathfrak{W} = \{((\mathfrak{s}_1, \mathfrak{r}_1), 0), \big((\mathfrak{s}_2, \mathfrak{r}_1), 0.9\big), \big((\mathfrak{s}_1, \mathfrak{r}_2), 0\big), ((\mathfrak{s}_2, \mathfrak{r}_2), 0.7)\},$$

and let $1(\mathfrak{c}_i, \mathfrak{c}_j)$ for the fourth stage, and $0(\mathfrak{c}_i, \mathfrak{c}_j)$ for the final stage.

Thus, it is clear that $\mathcal{T}_{BRFS} = \{\mathfrak{B}, \mathfrak{D}, \mathfrak{W}, 0(\mathfrak{c}_i, \mathfrak{c}_j), 1(\mathfrak{c}_i, \mathfrak{c}_j)\}$ is a Binary Relation -FS-topological space; $\mathfrak{B}, \mathfrak{D}, \mathfrak{W}, 0(\mathfrak{c}_i, \mathfrak{c}_j), 1(\mathfrak{c}_i, \mathfrak{c}_j)$ are called Binary Relation-FS-open sets.

Definition 3.4: The space $(\mathfrak{T}, \mathcal{T}_{BRFS})$ is a Binary Relation-FS-topological space on $\mathfrak{T}$ and $\mathfrak{B}$ is a Binary Relation-FS-set on $\mathfrak{T}$, then the Binary Relation-FS-closure of $\mathfrak{B}$, symbolized by $BRFS$-cl($\mathfrak{B}$), is defined as the intersection of all Binary Relation-FS-closed supersets of $\mathfrak{B}$. It is clear that, $BRFS$-cl($\mathfrak{B}$) represent the smallest Binary Relation-FS-closed set which contains $\mathfrak{B}$.

Example 3.5: In the example 3.3 above, let A be a Binary Relation-FS-set, $A = \{((\mathfrak{s}_1, \mathfrak{r}_1), 0.2), \big((\mathfrak{s}_2, \mathfrak{r}_1), 0.1\big), \big((\mathfrak{s}_1, \mathfrak{r}_2), 0.2\big), ((\mathfrak{s}_2, \mathfrak{r}_2), 0.2)\}$, then the smallest Binary Relation-FS-closed-set contains A is $\mathfrak{B}^C$, i.e. $BRFS$-cl(A) $= \mathfrak{B}^C$

Theorem 3.6: *Let* $(\mathfrak{T}, \mathcal{T}_{BRFS})$ *be a Binary Relation-FS-topological space on* $\mathfrak{T}$*,* $\mathfrak{B}$ *and* $\mathfrak{D}$ *are Binary Relation-FS-sets on* $\mathfrak{T}$ *. Then:*

(1) $BRFS\text{-cl}\big(0(\mathfrak{c}_i, \mathfrak{c}_j)\big) = 0(\mathfrak{c}_i, \mathfrak{c}_j)$ *and* $BRFS\text{-cl}\big(1(\mathfrak{c}_i, \mathfrak{c}_j)\big) = 1(\mathfrak{c}_i, \mathfrak{c}_j)$

(2) $\mathfrak{B} \sqsubseteq BRFS\text{-cl}(\mathfrak{B})$

(3) $\mathfrak{B}$ is a Biary Relation -FS-closed set iff $\mathfrak{B} = BRFS\text{-cl}(\mathfrak{B})$

(4) $BRFS\text{-cl}\ (BRFS\text{-cl}(\mathfrak{B})) = BRFS\text{-cl}(\mathfrak{B})$

(5) $\mathfrak{B} \sqsubseteq \mathfrak{D} \Rightarrow BRFS\text{-cl}(\mathfrak{B}) \sqsubseteq BRFS\text{-cl}(\mathfrak{D})$

(6) $BRFS - \text{cl}\ (\mathfrak{B} \sqcup \mathfrak{D}) = BRFS\text{-cl}(\mathfrak{B}) \sqcup BRFS\text{-cl}(\mathfrak{D})$

(7) $BRFS - \text{cl}\ (\mathfrak{B} \sqcap \mathfrak{D}) \sqsubseteq BRFS\text{-cl}(\mathfrak{B}) \sqcap BRFS\text{-cl}(\mathfrak{D})$

Proof: Straightforward manner via the definition 3.4.

Definition 3.7: Let $(\mathfrak{T}, \mathcal{T}_{BRFS})$ be a Binary Relation-FS-topological space on $\mathfrak{T}$ and $\mathfrak{B}$ be a Binary Relation-fuzzy-soft-set on $\mathfrak{T}$, then the Binary Relation-FS-interior of $\mathfrak{B}$, symbolized by $BRFS$-int($\mathfrak{B}$), is representing a union of every BRFS-open subsets of $\mathfrak{B}$. It is clear that, $BRFS$-int($\mathfrak{B}$) represents the largest BRFS-open which contained in $\mathfrak{B}$.

Example 3.8: Also, in the example 3.3 above, let G be a Binary Relation-FS-set, $G=\{((\mathfrak{s}_1,\mathfrak{r}_1),0.5),((\mathfrak{s}_2,\mathfrak{r}_1),0.4),((\mathfrak{s}_1,\mathfrak{r}_2),0.3),((\mathfrak{s}_2,\mathfrak{r}_2),0.2)\}$, then the largest Binary Relation-FS-open-set containing G is $\mathfrak{D}$, i.e. $BRFS$-int(G)= $\mathfrak{D}$.

Theorem 3.9: *Let $(\mathfrak{T}, \mathcal{T}_{BRFS})$ be a Binary Relation-FS-topological space on $\mathfrak{T}$ and $\mathfrak{B}$ and $\mathfrak{D}$ are Binary Relation-FS-sets on $\mathfrak{T}$. Then:*

(1) $BRFS\text{-int}(0(\mathfrak{c}_i,\mathfrak{c}_j)) = 0(\mathfrak{c}_i,\mathfrak{c}_j)$ *and* $BRFS\text{-int}(1(\mathfrak{c}_i,\mathfrak{c}_j)) = 1(\mathfrak{c}_i,\mathfrak{c}_j)$

(2) $\mathfrak{B} \sqsubseteq BRFS\text{-int}(\mathfrak{B})$

(3) $\mathfrak{B}$ is a Binary Relation-FS-open set iff $\mathfrak{B} = BRFS\text{-int}(\mathfrak{B})$

(4) $BRFS\text{-int}\,(BRFS\text{-int}(\mathfrak{B})) = BRFS\text{-int}(\mathfrak{B})$

(5) $\mathfrak{B} \sqsubseteq \mathfrak{D} \Rightarrow BRFS\text{-int}(\mathfrak{B}) \sqsubseteq BRFS\text{-int}(\mathfrak{D})$

(6) $BRFS - \text{int}\,(\mathfrak{B}\sqcap\mathfrak{D}) = BRFS\text{-int}(\mathfrak{B}) \sqcap BRFS\text{-int}(\mathfrak{D})$

(7) $BRFS\text{-int}(\mathfrak{B}) \sqcup BRFS\text{-int}(\mathfrak{D}) \sqsubseteq BRFS - \text{int}(\mathfrak{B}\sqcup\mathfrak{D})$

Proof: Straightforward manner via the definition 3.7.

Discussions and Conclusions

Binary Relation-FS-topology is a novel and promising area that implies the development of many and various recent mathematical ideas and models that will significantly introduce to basic applications in some natural sciences like biomathematics, information systems, and decision making. Topological structures of the Binary Relation-FS- open (closed) set are proposed in this work with useful applied example. Some important definitions and notions have been studied. The main aim of this work is to create foundational concepts. There are a lot of frameworks for the scientific researchers to introduce their investigations on this topic, which means that, this is the start of several new topological ideas, structures, and concepts such as the Binary Relation-FS-separation axioms, Binary Relation-FS-continuous functions, and other necessary concepts that can be studied in this novel type of fuzzy soft topology.

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