

Square difference labeling and co-secure domination in middle graph of certain graphs

G. Sarmitha [§]

S. Vidyanandini [†]

Department of Mathematics

Faculty of College of Engineering and Technology

SRM Institute of Science and Technology

Kattankulathur 603203

Chengalpattu

Tamil Nadu

India

Soumya Ranjan Nayak ^{*}

School of Computer Engineering

KIIT Deemed to be University

Bhubaneswar 751024

Odisha

India

Abstract

This paper focus on the dynamic and quickly developing subject of graph Domination study. In this paper, Co-Secure Domination and Square difference Labeling for Middle graph of Paths and Regular Spider Graphs has been determined. The findings can be applied on protection strategy for a various communication and interconnection networks, where the routing servers as the network's vertices and the connection between them as its edges, protecting each node in addition to the guard nodes. This ensures that the guard nodes remain secure as well, and the network as a whole is protected.

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[§] E-mail: sg7675@srmist.edu.in

[†] E-mail: vidyanas@srmist.edu.in

^{*} E-mail: nayak.soumya17@gmail.com (Corresponding Author)

1. Introduction

The graphs considered in this paper are simple, finite, undirected and connected graphs of order $|V(G)|$ and size $|E(G)|$ for a graph $G = (V, E)$, let $|E(G)| \equiv m$ and $|V(G)| = n$. Claude Berge [5] invented the concept of domination in graphs in 1958. The concept of placing guards at every vertex of subset S of G resulted in study of several concepts in protection strategies. The theory of domination in graphs finds extensive applications in the field of defensive networks, protection strategies, communication networks etc. Co-secure domination is one of the protection strategies introduced by S.Arumugam *et.al* [1] in the year 2014. For further study in domination and co-secure domination reader can refer to [1, 2, 3, 4, 6, 10, 16]. Ajitha. V [15] first implemented square difference labeling in graphs in 2014. A reader can refer to dynamic survey by Gallian [8] for a brief learning about graph labeling and a reader can refer to [15, 12, 13] for a brief learning about square difference labeling. A reader can refer to [9,11] For further studies about Graphs. In this paper co-secure domination number and square difference labeling for middle graph of paths and regular spider graphs has determined.

Definition 1.1 [15]: A function of a graph $G = (V, E)$, $|V(G)| = n$, $|E(G)| = m$ is said to be a square difference graph if it admits a bijective function $f: V \rightarrow \{0, 1, 2, \dots, n-1\}$ that the induced function $f^*: E(G) \rightarrow \mathbb{N}$ given by are all distinct, for every $xy \in E(G)$.

Definition 1.2 [14]: The Middle graph $M(G)$ of a graph G is the graph in which the vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent edges of G or one is vertex of G and the other is an edge incident with it.

Definition 1.3 [1]: The open neighborhood of a vertex $v \in V$ is denoted by $N(v) = \{u \in V : uv \in E\}$ and its closed neighborhood is $N[v] = N(v) \cup v$. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V \setminus S$ is an S -external private neighbor of v if $N(u) \cap S = \{v\}$. The set of all S -epns of v is denoted by $EPN(v, S)$.

Definition 1.4 [9]: A path P_n of n vertices is a walk with no repeated vertex.

Definition 1.6 [7]: A spider graph is a tree with atmost one vertex of degree greater than 2 and the vertex is called branch vertex v_0 . Leg of spider is the path from brach of vertex to a leaf of tree. Regular Spider graph denoted as $S_M(n)$ Where m denotes number of legs and n denotes length of legs.

2. Results and Analysis

Lemma 2.1: Let S be a dominating set of graph G . Then S is γ_{cs} set G iff for each $u \in S$, $|EPN(u, S)| \leq 1$.

Proof: Let S be a dominating set of G . Then $\forall v \in V \setminus S$, v belongs to neighbourhood of one or more elements belongs to S . When S is a co-secure

dominating set $\forall u \in S, \exists v \in V \setminus S$ such that $v \cup Su$ is also a co-secure dominating set, Thus it can be concluded that there exist at most one neighbour of u which does not get dominated by more than one element which belongs to S . thus $|EPN(u, S)| \leq 1 \forall u \in S$.

Lemma 2.2 For a graph $G, M(G)$ is middle graph of G . Then for every three vertices e_i, e_{i+1}, e_{i-1} there exists two vertices belongs to γ_{cs} -set of $M(G)$.

Proof: For a graph G , Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$, $n, m \in \mathbb{N}$. Such that $V(M(G)) = v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n$ and $E(M(G)) = \{v_i e_j, e_i e_k, e_i e_j \mid i = 1, 2, \dots, n, k = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, m\}$. Let us consider the elements $\{e_i, e_{i-1}, e_{i+1}\}$, $i = 1, 2, \dots, m$. $N[e_i] = \{e_i, e_{i+1}, e_{i-1}, v_i, v_{i+1}\}$, $N[e_{i-1}] = \{e_{i-1}, e_{i-2}, e_i, v_{i-1}, v_i\}$, $N[e_{i+1}] = \{e_{i+1}, e_i, e_{i+2}, v_{i+1}, v_{i+2}\}$. If there exist only one element in e_i, e_{i-1}, e_{i+1} belongs to γ_{cs} -set of $M(G)$, say $e_i \in S$ we get $EPN(e_i, S) = \{v_i, v_{i+1}\}$ that is $|EPN(e_i, S)| \geq 1$ which contradicts Lemma 2.1. Similarly if we have only $e_{i-1} \in S$ or $e_{i+1} \in S$ we get $EPN(e_{i-1}, S) = \{v_i, e_i\}$, $EPN(e_{i+1}, S) = \{e_i, v_i, v_{i+1}\}$ that is $|EPN(e_{i-1}, S)| \geq 1$, $|EPN(e_{i+1}, S)| \geq 1$ which contradicts Lemma 2.1. Thus for every three vertices e_i, e_{i+1}, e_{i-1} there exists two vertices belongs to γ_{cs} -set of $M(G)$.

Lemma 2.3 [1]: For complete graph K_n , $\gamma_{cs}(K_n) = 1$.

Theorem 2.1: For a graph $G = P_n$, $n \geq 3$, middle graph of G admits square difference labeling and

$$\gamma_{cs}(G) = \begin{cases} \frac{2n}{3} & \text{when } n = 3 + 3k, k = 1, 2, \dots \\ \frac{2n+1}{3} & \text{when } n = 1 + 3k, k = 1, 2, \dots \\ \frac{2n+2}{3} & \text{when } n = 2 + 3k, k = 1, 2, \dots \end{cases}$$

Proof: For graph $G = P_n$, We have $V(G) = v_1, v_2, \dots, v_n, |V(G)| = n$ and $E(G) = e_1, e_2, \dots, e_{n-1}$, $|E(G)| = n - 1$. Thus we get $V(M(G)) = v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_{n-1}, |V(M(G))| = 2n - 1$ and edge set of $M(G)$ is $E(M(G)) = \{e_i e_{i+1}, v_i e_i, e_i v_{i+1} \mid i = 1, 2, \dots, n - 1\}$. Considering the following cases:

Case 1: When $n = 3 + 3k$, We have $N(e_i) = \{e_i, e_{i+1}, e_{i-1}, v_i, v_{i+1}\}$, $N(v_i) = \{e_i, e_{i-1}\}$. As $|N(e_i)| \geq |N(v_i)|$ elements in CSDS are of e_i 's. Now for $e_i \in S$, $N[e_i] = \{e_i, v_i, e_{i-1}, v_{i+1}, e_{i+1}\}$ from Lemma 2.2 either e_{i-1} or e_{i+1} belongs to γ_{cs} -set of $M(G)$. Thus for every e_i, e_{i+1}, e_{i-1} there exists two elements that belongs to γ_{cs} -set of $M(G)$. Such that we get $|S| \geq \frac{2n}{3}$. If $|S| > \frac{2n}{3}$ then

all v_i, v_{i+1}, v_{i-1} belongs to γ_{cs} -set of $M(G)$ which contradicts Lemma 2.2. Thus $|S| = \frac{2n}{3}, S = e_1, e_2, e_4, e_5, \dots, e_{n-2}, e_{n-1}$ and $\gamma_{cs}(M(G)) = \frac{2n}{3}$.

Case 2: When $n = 1 + 3k$, We have $n \equiv 1(mod 3)$. Thus $N(e_i) = \{e_{i+1}, e_{i-1}, v_i, v_i + 1\}, N(v_i) = e_i, e_{i-1}, N(v_n) = e_n$. As $|N(e_i)| > |N(v_i)|$ elements in CSDS are of e_i 's. Now, For $e_i \in S$ we have $N[e_i] = e_i, v_i, e_{i-1}, e_{i+1}, v_{i+1}$. From Lemma 2.2 either e_{i-1} or e_{i+1} belongs to CSDS, thus $\frac{2n}{3}$ vertices belongs to CSDS of $M(G)$. Such that the vertices $e_1, e_2, e_4, e_5, \dots, e_{n-3}, e_{n-2}$ belongs to CSDS of $M(G)$. Since $n \equiv 1(mod 3)$ element e_{n-1} is not been dominated in graph $M(G)$, Assume $e_{n-1} \in S$, $N[e_{n-1}] = \{e_{n-1}, e_{n-2}, v_{n-1}, v_n\}, EPN(e_{n-1}, S) = \{v_n\}, |EPN(e_{n-1}, S)| = 1$ from Lemma 2.1 it's true that $e_{n-1} \in S$. Considering three vertices e_{n-2}, e_{n-1}, v_n , The vertex e_{n-1} belongs to γ_{cs} -set of $M(G)$. $\frac{2n+1}{3}$ vertices belongs to γ_{cs} -set of $M(G)$. Thus $|S| \geq \frac{2n+1}{3}$. If $|S| > \frac{2n+1}{3}$ then all e_i, e_{i+1}, e_{i-1} belongs to CSDS of $M(G)$ which contradicts lemma 2.2. Therefore $|S| = \frac{2n+1}{3}, S = e_1, e_2, e_4, e_5, \dots, e_{n-3}, e_{n-2}, e_{n-1}$ and $\gamma_{cs}(M(G)) = \frac{2n+1}{3}$.

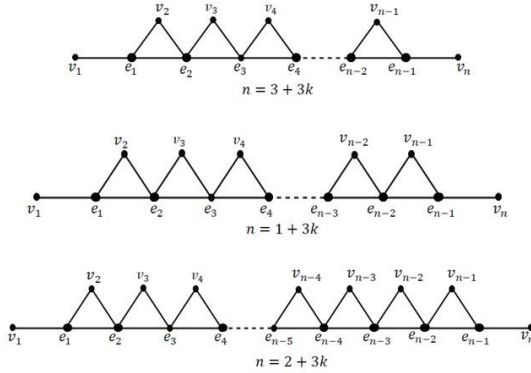


Figure 1

Graph $G = P_n$, Shaded vetices denote γ_{cs} set of G

Case 3: When $n \equiv 2 + 3k, k = 1, 2, \dots$, We have $n \equiv 2(mod 3)$. Considering $N(v_i) = \{e_i, e_{i-1}\}, N(e_i) = e_{i-1}, e_{i+1}, v_i, v_{i+1}$ as $|N(e_i)| > |N(v_i)|$ elements in CSDS are of e_i 's. Now, For $e_i \in S$ we have $N[e_i] = e_i, v_i, e_{i-1}, e_{i+1}, v_{i+1}$ as shown in Figure 1. From Lemma 2.2 either e_{i-1} or e_{i+1} belongs to CSDS, thus $\frac{2n}{3}$ vertices belongs to S , that is $e_1, e_2, e_4, e_5, \dots, e_{n-4}, e_{n-3}$ belongs to CSDS of $M(G)$. For $n \equiv 2(mod 3)$, the elements e_{n-1}, e_{n-2} are not been dominated in $M(G)$. Assume only e_{n-1} or e_{n-2} belongs to γ_{cs} set. When e_{n-1} belongs to γ_{cs} -set, We get $|EPN(e_{n-1}, S)| = v_{n-1}, v_n$ which is a contradiction from Lemma 2.1. when e_{n-2} belongs to γ_{cs} -set, $EPN(e_{n-2}, S) = v_{n-1}, e_{n-1}$ which is

a contradiction from Lemma 2.1. Thus both e_{n-2}, e_{n-1} belongs to γ_{cs} -set of $M(G)$. Such that $\frac{2n+2}{3}$ elements belongs to γ_{cs} -set of $M(G)$, $|S| \geq \frac{2n+2}{3}$. If $|S| > \frac{2n+2}{3}$ then all e_i, e_{i+1}, e_{i-1} belongs to γ_{cs} -set of $M(G)$ which contradicts

Lemma 2.2: Thus $|S| = \frac{2n+2}{3}$, $S = e_1, e_2, e_4, e_5, \dots, e_{n-3}, e_{n-2}, e_{n-1}$ and $\gamma_{cs}(M(G)) = \frac{2n+2}{3}$.

Now introducing a bijection $f: V(M(G)) \rightarrow 0, 1, 2, \dots, 2n-2$ defined by: $f(v_1) = 0, f(e_1) = 1, f(v_2) = 2, f(e_2) = 3, \dots, f(v_i) = 2(i-1), f(e_i) = 2i-1, \dots, f(e_{n-1}) = 2(n-1)-1, f(v_n) = 2(n-1)$. Thus the edge function f^* of $M(G)$ is defined as $f^*(v_i v_{i+1}) = 4i-3, f^*(e_i v_{i+1}) = 4i-1, f^*(e_i e_{i+1}) = 8i$.

Theorem 2.2: For $G = S_m(n)$, $M(G)$ admits square difference labeling, and

$$\gamma_{cs}(G) = \begin{cases} \frac{2n}{3} & \text{when } n = 3 + 3k, k = 1, 2, \dots \\ \frac{2n+1}{3} & \text{when } n = 1 + 3k, k = 1, 2, \dots \\ \frac{2n+2}{3} & \text{when } n = 2 + 3k, k = 1, 2, \dots \end{cases}$$

Proof: For graph $G = S_m(n)$, $V(G) = \{v, v_{11}, v_{12}, \dots, v_{1n}, v_{21}, v_{22}, \dots, v_{2n}, \dots, v_{j1}, v_{j2}, \dots, v_{ji}, \dots, v_{j(n-1)}, v_{jn}, \dots, v_{m1}, v_{m2}, \dots, v_{mn}\}$, $|V(G)| = mn + 1$, $E(G) = \{vv_{j1}, v_{ji}v_{j(i+1)}, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$, $|E(G)| = mn$. Thus vertex and edge set of $M(G)$ can be defined as: $V(M(G)) = v, v_{11}, v_{12}, \dots, v_{1n}, v_{21}, v_{22}, \dots, v_{2n}, \dots, v_{j1}, v_{j2}, \dots, v_{ji}, \dots, v_{j(n-1)}, v_{jn}, \dots, v_{m1}, v_{m2}, \dots, v_{mn}, e_{11}, e_{12}, \dots, e_{1n}, e_{21}, e_{22}, \dots, e_{2n}, \dots, e_{m1}, e_{m2}, \dots, e_{mn}$, $|V(M(G))| = 2mn + 1$, $E(M(G)) = ve_{j1}, e_{ji}e_{j(i+1)}, e_{ji}v_{ji}, v_{ji}e_{j(i+1)}, e_{ji}e_{ki}, i = 1, 2, \dots, n, j = 1, 2, \dots, m, k = 1, 2, \dots, m, j \neq k$. Now considering the following cases:

Case 1: $n \equiv 0 \pmod{3}$, Let $M(G)$ is divided into Subgraphs $M_1(G), M_2(G), \dots, M_j(G), \dots, M_m(G), M_v(G)$ Where,

$M_1(G)$ induced by vertices $e_{1i}, v_{1i}, i = 1, 2, \dots, n$

$M_2(G)$ induced by vertices $e_{2i}, v_{2i}, i = 1, 2, \dots, n$

...

$M_j(G)$ induced by vertices $e_{ji}, v_{ji}, i = 1, 2, \dots, n$

...

$M_n(G)$ induced by vertices $e_{ni}, v_{vi}, i = 1, 2, \dots, n$

$M_v(G)$ induced by vertices $v, v_{11}, v_{21}, \dots, v_{j1}, \dots, v_{n1}$

In subgraph $M_1(G)$, $M_1(G) = M(P_n)$, $n \geq 3$ Where e_{11} is connected with v . Thus we get $\gamma_{cs}(M_1(G)) = \gamma_{cs}(P_n)$, $n \equiv 0(mod 3)$,

$$S_{M_1} = e_{1n}, e_{1(n-1)}, e_{1(n-3)}, e_{1(n-4)}, \dots, e_{13}, e_{12}.$$

$$\gamma_{cs}(M_2(G)) = \gamma_{cs}(M(P_n)), n \equiv 0(mod 3),$$

$$S_{M_2} = e_{2n}, e_{2(n-1)}, e_{2(n-3)}, e_{2(n-4)}, \dots, e_{23}, e_{22}$$

$$\gamma_{cs}(M_3(G)) = \gamma_{cs}(M(P_n)), n \equiv 0(mod 3),$$

$$S_{M_3} = e_{3n}, e_{3(n-1)}, e_{3(n-3)}, e_{3(n-4)}, \dots, e_{33}, e_{32}$$

...

$$\gamma_{cs}(M_j(G)) = \gamma_{cs}(M(P_n)), n \equiv 0(mod 3),$$

$$S_{M_j} = e_{jn}, e_{j(n-1)}, e_{j(n-3)}, e_{j(n-4)}, \dots, e_{j3}, e_{j2}$$

...

$$\gamma_{cs}(M_n(G)) = \gamma_{cs}(M(P_n)), n \equiv 0(mod 3),$$

$$S_{M_n} = e_{nn}, e_{n(n-1)}, e_{n(n-3)}, e_{n(n-4)}, \dots, e_{n3}, e_{n2}$$

Considering sub graph $M_v(G)$, From Lemma 2.3 $\gamma_{cs}(M_v(G)) = 1$, $S_{M_v} = v$. $\gamma_{cs}(M(G)) = \gamma_{cs}(M_1(G)) + \gamma_{cs}(M_2(G)) + \dots + \gamma_{cs}(M_n(G)) + \gamma_{cs}(M_v(G)) = m\gamma_{cs}(M(P_n)) + 1$.

Case 2: $n \not\equiv 0(mod 3)$, Either $n \equiv 1(mod 3)$ or $n \equiv 2(mod 3)$. Thus we have the following subcases:

Subcase 1: $n \equiv 1(mod 3)$, Let $M(G)$ is divided into Subgraphs $M_1(G), M_2(G), \dots, M_j(G), \dots, M_m(G), M_v(G)$, Where,

$$M_1(G) \text{ induced by vertices } e_{1i}, v_{1i}, i = 1, 2, \dots, n$$

$$M_2(G) \text{ induced by vertices } e_{2i}, v_{2i}, i = 1, 2, \dots, n$$

...

$$M_j(G) \text{ induced by vertices } e_{ji}, v_{ji}, i = 1, 2, \dots, n$$

...

$$M_n(G) \text{ induced by vertices } e_{ni}, v_{vi}, i = 1, 2, \dots, n$$

$$M_v(G) \text{ induced by vertices } v, v_{11}, v_{21}, \dots, v_{j1}, \dots, v_{n1}$$

In subgraph $M_1(G)$, $M_1(G) = M(P_n)$, $n \equiv 1(mod 3)$ Where e_{11} is connected with v . Thus

$$\gamma_{cs}(M_1(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_1} = e_{1n}, e_{1(n-1)}, e_{1(n-3)}, e_{1(n-4)}, \dots, e_{14}, e_{13}, e_{11}.$$

$$\gamma_{cs}(M_2(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_2} =$$

$$e_{2n}, e_{2(n-1)}, e_{2(n-3)}, e_{2(n-4)}, \dots, e_{24}, e_{23}, e_{21}$$

$$\gamma_{cs}(M_3(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_3} = e_{3n}, e_{3(n-1)}, e_{3(n-3)}, e_{3(n-4)}, \dots, e_{34}, e_{33}, e_{31}$$

...

$$\gamma_{cs}(M_j(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_j} = e_{jn}, e_{j(n-1)}, e_{j(n-3)}, e_{j(n-4)}, \dots, e_{j4}, e_{j3}, e_{j1}$$

...

$$\gamma_{cs}(M_n(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_n} = e_{nn}, e_{n(n-1)}, e_{n(n-3)}, e_{n(n-4)}, \dots, e_{n4}, e_{n3}, e_{n1}.$$

Considering sub graph $M_v(G)$, $M_v(G)$ is already been co-secure dominated by vertices $e_{11}, e_{21}, \dots, e_{n1}$. Thus $\gamma_{cs}(M(G)) = \gamma_{cs}(M_1(G)) + \gamma_{cs}(M_2(G)) + \dots + \gamma_{cs}(M_n(G)) + \gamma_{cs}(M_v(G)) = m\gamma_{cs}(M(P_n))$.

Subcase 2: $n \equiv 2(mod 3)$, Let $M(G)$ is divided into Subgraphs $M_1(G), M_2(G), \dots, M_j(G), \dots, M_m(G), M_v(G)$, Where,

$M_1(G)$ induced by vertices $e_{1i}, v_{1i}, i = 1, 2, \dots, n$

$M_2(G)$ induced by vertices $e_{2i}, v_{2i}, i = 1, 2, \dots, n$

...

$M_j(G)$ induced by vertices $e_{ji}, v_{ji}, i = 1, 2, \dots, n$

...

$M_n(G)$ induced by vertices $e_{ni}, v_{vi}, i = 1, 2, \dots, n$

$M_v(G)$ induced by vertices $v, v_{11}, v_{21}, \dots, v_{j1}, \dots, v_{n1}$

In subgraph $M_1(G)$, $M_1(G) = M(P_n), n \equiv 1(mod 3)$ Where e_{11} is connected with v . Thus

$$\gamma_{cs}(M_1(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_1} = e_{1n}, e_{1(n-1)}, e_{1(n-3)}, e_{1(n-4)}, \dots, e_{15}, e_{14}, e_{12}, e_{11}. \text{ Similarly,}$$

$$\gamma_{cs}(M_2(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_2} = e_{2n}, e_{2(n-1)}, e_{2(n-3)}, e_{2(n-4)}, \dots, e_{25}, e_{24}, e_{22}, e_{21}$$

$$\gamma_{cs}(M_3(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_3} = e_{3n}, e_{3(n-1)}, e_{3(n-3)}, e_{3(n-4)}, \dots, e_{35}, e_{34}, e_{32}, e_{31}$$

...

$$\gamma_{cs}(M_j(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_j} = e_{jn}, e_{j(n-1)}, e_{j(n-3)}, e_{j(n-4)}, \dots, e_{j5}, e_{j4}, e_{j2}, e_{j1}$$

...

$$\gamma_{cs}(M_n(G)) = \gamma_{cs}(M(P_n)), n \equiv 1(mod 3), S_{M_n} = e_{nn}, e_{n(n-1)}, e_{n(n-3)}, e_{n(n-4)}, \dots, e_{n5}, e_{n4}, e_{n2}, e_{n1}.$$

Considering sub graph $M_v(G)$, $M_v(G)$ is already ben co-secure dominated by vertices $e_{11}, e_{21}, \dots, e_{n1}$. $\gamma_{cs}(M(G)) = \gamma_{cs}(M_1(G)) + \gamma_{cs}(M_2(G)) + \dots + \gamma_{cs}(M_n(G)) + \gamma_{cs}(M_v(G)) = mn$ as illustraeted in Figure 2, and 3, respectively.

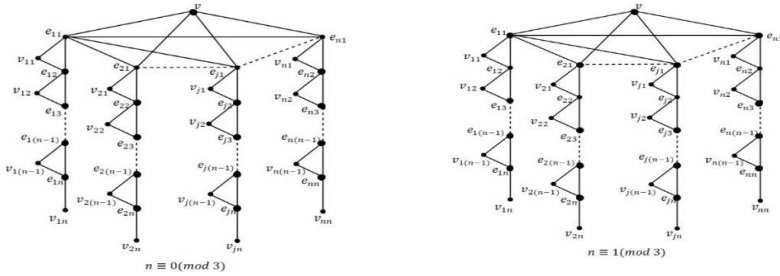


Figure 2

Graph $M(G) = M(S_m(n))$, Shaded vertices denote γ_{cs} set of $M(G)$.

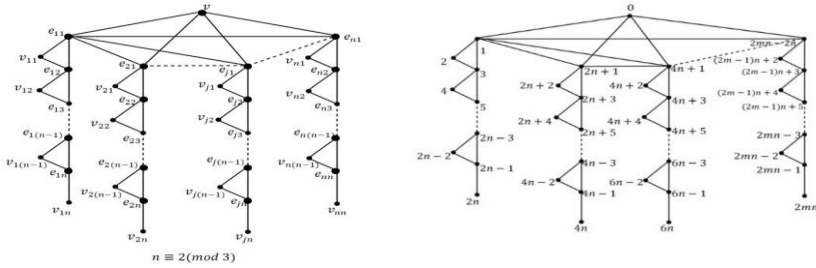


Figure 3

Graph $M(G) = M(S_m(n))$, $M(G)$ has been Square Difference Labeled and Shaded vertices denote γ_{cs} set of $M(G)$.

Now introducing bijection $f: V(G) \rightarrow 0, 1, 2, \dots, 2mn$ defined by: $f(v_{1i}) = 2i, f(e_{1i}) = 2i - 1, f(v) = 0, f(v_{ji}) = f(v_{1i}) + 2n(j - 1), f(e_{ji}) = f(e_{1i}) + 2n(j - 1)$. Edge function is defined as:

$$f^*(e_{ji}e_{j(i+1)}) = 4f(e_{ji}) + 4$$

$$f^*(e_{ji}v_{ji}) = 2f(e_{ji}) + 1$$

$$f^*(v_{ji}e_{j(i+1)}) = 2f(e_{ji}) + 3$$

$$f^*(ve_{j1}) = f(e_{j1})^2$$

$$f^*(e_{j1}e_{k1}) = f(e_{ji})^2 - f(e_{k1})^2, j = 1, 2, \dots, m, k = 1, 2, \dots, m, j \neq k.$$

3. Application

Interconnection networks are established to employ at numerous levels both within and between computer systems in order to meet the operational requirements of various application fields. Within the network, In order to

defend against an attack at a vertex without a guard, a guard moves along an edge, the resulting guard placement may make the system more prone to failure. To solve this issue and safeguard the network as a whole, an innovative method for determining the minimum number of protectors to protect all vertices together with the guards has been created.

3.1 *Application idea*

A network can be analyzed into a graph such that the co-secure guards and square difference labeling parameters are been marked thus every node is protected by the guards in its neighborhood in order to make the entire network is secured.

- Graph representation: picturing a network as a graph, the entire network is secured by deploying guards in such a manner that every node along with the guards is also protected by the guards in its neighborhood.
- Co-Secure Domination: The vertex subset S of $V(G)$ is introduced in order to protect the network, Where each $u \in V \setminus S$ there exists $v \in S$ such that v is adjacent to u and $(S \setminus v) \cup u$ is a dominating set of G .
- Square difference Labeling: For a network with P nodes as vertices Such that we are defining vertex functions $f: V(G) \rightarrow 0, 1, \dots, p-1$ for which an edge valued injective function f^* can be defined on G as function of squares of vertex values.
- Guard protection: interference in network connection may occur if the guards were not protected which may lead to failure of the network. Thus placing guards in such a way that a guard needs to be substituted by another.
- Network protection: In order to secure networks, protection strategies have been extensively investigated. Thus every guard v can be replaced by another guard and the resulting set of guards continues to protect the network.
- Network enumeration: nodes of a network have been labeled in such a way that the nodes are labeled from 0 to $p-1$ for p nodes. The connection between the set of nodes in the network are labeled as difference between the square values of labels of the corresponding nodes which is also uniquely defined.
- Adaptive learning: A network system can utilize a variety of application in the field of communication networks, security programs, services, and tools.

3.2 *Illustration*

Benes Network: Representing the communication network as an undirected graph with communication links represented by edges and processors represented by nodes. One of the most potent and adaptable interconnection networks is the r -dimension Benes network. For r dimension Benes network, There exist $2r + 1$ levels in the network. Since this allows safe message flow to

all other processors, calculating the co-secure dominant set for Benes is a substantial problem. These findings can be used to ensure secure transmission of messages in interconnected networks. Co-secure domination and square difference labeling has been obtained for 2 dimension benes network in the Figure 4.

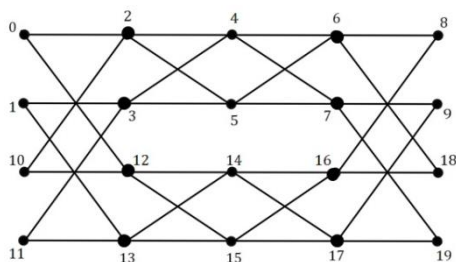


Figure 4

A 2-Dimensional Benes Network, Shaded vertices denote γ_{cs} set of $M(G)$ and $M(G)$ has been Square Difference Labeled.

4. Conclusion

The concept of Co-Secure Domination in graph theory can be used to address a variety of real-world practical problems, including facility location problem, communication network monitoring problem, protection of space or a region, etc. Square Difference Labeling has an ability to identify each and every node and communication network in a distinct manner allowing the network to effectively carry out its intended functions. In this paper co-secure domination number of the middle graph of paths and regular spider graphs has determined. Additionally it is showed that these graphs are square difference graphs. Application based on our observations and our findings has been discussed.

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