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Sombor polynomial (function) of a graph

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Abstract

Let $G=(V,E)$ be a finite simple graph. The Sombor index $SO(G)$ of G is defined as $\sum_{uv\in E(G)}\sqrt{d_u^2+d_v^2}$, where d_u is the degree of vertex u in G. We consider the Sombor polynomial (function) as $S(G,x)=\sum_{uv\in E(G)}x^{\sqrt{d_u^2+d_v^2}}$ and study it for specific graphs. Also we investigate the location of the roots of this polynomial.

Subject Classification: *05C31.*

Keywords: *Sombor index, Sombor polynomial, Root.*

1. Introduction

A molecular graph has a model which is a simple graph in such away its vertices are the atoms and the edges are related to the bonds of a molecule. For a simple graph $G=(V,E)$, the degree of a vertex v in G is denoted by d_v. A topological index of graph G is a real number related to G which does not depend on the labeling or pictorial representation

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of a graph ([14]). For more information about the molecular topology, see [10]. As an example the Wiener index $W(G)$ is defined as $W(G) = \sum_{\{u,v\}\subseteq G} d(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$ with the summation runs over all pairs of vertices of G [27]. As another example related to topological indices, we refer to the Hosoya polynomial of a graph which is introduced by Hosoya [18] in 1988 and it is a generating function about distance distributing and for a connected graph G is defined as:

$$H(G,x) = \sum_{\{u,v\}\subseteq V(G)} x^{d(u,v)},$$

where $d(u,v)$ denotes the distance between vertices u and v, which is the minimum number of edges between u and v. The Hosoya polynomial has many chemical applications [11, 12, 15]. The two well-known and well-studied topological indices, i.e. Wiener index and hyper-Wiener index, can be directly obtained from the Hosoya polynomial. The first derivative of the Hosoya polynomial at $x = 1$ is equal to the Wiener index:

$$W(G) = (H(G,x))'|_{x=1}.$$

Gutman in [16] defined a new topological index which is vertex-degree-based molecular structure descriptor, the Sombor index, as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}.$$

Also he proposed the reduced Sombor index, as

$$SO_{red}(G) = \sum_{uv\in E(G)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2}.$$

In the same paper for a graph G of order n and size m the average Sombor index, is defined as

$$SO_{avr}(G) = \sum_{uv\in E(G)} \sqrt{(d_u - \frac{2m}{n})^2 + (d_v - \frac{2m}{n})^2}.$$

The graphs whose are extremal with respect to the Sombor index have characterized in [7]. More precisely, the chemical graphs, chemical trees and hexagon systems (see [13]) which are extremal, are characterized. The chemical importance of the Sombor index was investigated by Deng, Tang and Wu in [9]. This index is useful in predicting physico-chemical properties with high accuracy compared to some well-established and often used indices. They presented an upper bound for the Sombor index among all molecular trees with fixed numbers of vertices which is sharp bound and trees whose achieving the extremal value have characterized. More precisely, suppose that $\mathcal{CT}_n$ is the set of trees of order n and $\mathcal{T}_n$ is the set of molecular trees T of order n, where T has at most one vertex v with degree 2 or 3 and v is not incident with any pendant edge. Authors in [9] proved that if $T \in \mathcal{T}_n$ has at least two vertices of degree 3 or has at least two vertices of degree 2 or has at least one vertex of degree 2 and at least one vertex of degree 3, then it is not a maximal tree with respect to the Sombor index in $\mathcal{CT}_n$.

In [23] the predictive and discriminative potentials of Sombor index, reduced Sombor index and average Sombor index examined. Simple linear models that use one of these indices as the only predictor showed satisfactory predictive potential. The performance of these models was improved with the introduction of the first Zagreb index as a second predictor. The reduced Sombor index have shown the best performance among these three topological molecular descriptors [23].

In [8] some novel lower and upper bounds on the Sombor index of graphs has presented by using some graph parameters, especially, maximum and minimum degree. Moreover, several relations on Sombor index with the first and second Zagreb indices of graphs obtained. The mathematical relations between the Sombor index and some other well-known degree-based descriptors investigated in [26]. For example they have shown that if G is a connected graph of order n and size m, then $\frac{\sqrt{2}M_2(G)}{\Delta} \leq SO(G) \leq \frac{\sqrt{2}M_2(G)}{\delta}$, where $M_2(G)$ is the second Zagreb index (i.e., $\sum_{uv\in E(G)} d_u.d_v$) and these bounds are sharp for regular graphs. The Sombor index of polymers and certain graphs studied in [3, 13]. If G is a connected graph constructed from pairwise disjoint connected graphs $G_1,\ldots,G_k$ by selecting a vertex of G_1, a vertex of G_2 and identifying these two vertices. Then continue in this manner inductively. In [3] this graph G has called a polymer graph, obtained by point-attaching from monomer units $G_1,...,G_k$. Authors in [3] have been considered some particular cases of these graphs that are of importance in chemistry and studied their Sombor index.

The corona of two graphs G_1 and G_2, is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1)$ and

$v \in V(G_2)\}$. The Cartesian product $G\Box H$ of graphs G and H is a graph such that the vertex set of $G\Box H$ is the Cartesian product $V(G)\times V(H)$, and two vertices (u,u') and (v,v') are adjacent in $G\Box H$ if and only if either $u=v$ and u' is adjacent to v' in H, or $u'=v'$ and u is adjacent to v in G. The dutch windmill graph D_n^k is the graph obtained by taking n, $(n\geqslant 2)$ copies of the cycle graph C_k, $(k\geqslant 3)$ with a vertex in common.

Similar to Hosoya polynomial and Wiener index, we introduce the Sombor polynomial (function) of a graph and compute it for some specific graphs. Also we try to study the location of roots of this new polynomial.

2. Introduction to Sombor polynomial

In this section, first we introduce the Sombor polynomial of a graph and compute and study this polynomial for certain graphs. We begin with the following definition:

Definition 2.1 : Let G be a simple connected graph. The Sombor polynomial (function) of G is denoted by $S(G,x)$ is defined as:

$$S(G,x)=\sum_{uv\in E(G)} x^{\sqrt{d_u^2+d_v^2}},$$

where d_w denotes the degree of vertex w.

Remark 2.2 : The definition of the Sombor polynomial of G can be written as:

$$S(G,x)=\sum_r s_r x^r,$$

where $s_r=|\{\{u,v\}\in V(G), uv\in E(G) \mid \sqrt{d_u^2+d_v^2}=r\}|$. The terms of polynomials in the literature have natural powers and here in Sombor polynomial the power are almost irrational. Almost nothing there is in Literature about this kind of polynomials and their roots.

Remark 2.3 : Note that the first derivative of the Sombor polynomial at $x=1$ is equal to the Sombor index:

$$SO(G)=(S(G,x))'|_{x=1}.$$

First, we compute the Sombor polynomial of specific graphs:

Theorem 2.4 :

(i) *If* P_n *is the path graph of order* $n \geq 3$, *then*

$$S(P_n, x) = 2x^{\sqrt{5}} + (n-3)x^{2\sqrt{2}}.$$

(ii) *If* C_n *is the cycle graph of order n, then*

$$S(C_n, x) = nx^{\sqrt{8}}.$$

(iii) *If* $K_{1,n}$ *is the star graph of order n, then*

$$S(K_{1,n}, x) = nx^{\sqrt{n^2+1}}.$$

(iv) *If* F_n *is the friendship graph of order n (join of* K_1 *and* nK_2), *then*

$$S(F_n, x) = nx^{\sqrt{8}} + 2nx^{2\sqrt{n^2+1}}.$$

(v) *If* $D_n^{(m)}$ *is the Dutch windmill graph, then for every* $n \geq 3$ *and* $m \geq 2$,

$$S(D_n^{(m)}, x) = m(n-2)x^{\sqrt{8}} + 2mx^{2\sqrt{m^2+1}}.$$

(vi) *If* W_n *is the wheel graph of order n (join of* K_1 *and* C_{n-1}), *then*

$$S(W_n, x) = (n-1)x^{3\sqrt{2}} + (n-1)x^{\sqrt{9+(n-1)^2}}.$$

(vii) *If* L_n *is the ladder graph of order n. Then for every* $n \geq 3$,

$$S(L_n, x) = 2x^{2\sqrt{2}} + 4x^{\sqrt{13}} + (3n-8)x^{3\sqrt{2}}.$$

(viii) *If* B_n *is the Book graph (See Figure 1) of order n* $(B_n = K_{1,n} \square K_2)$, *then for every* $n \geq 3$,

$$S(B_n, x) = nx^{2\sqrt{2}} + 2nx^{\sqrt{4+(n-1)^2}} + x^{(n-1)\sqrt{2}}.$$

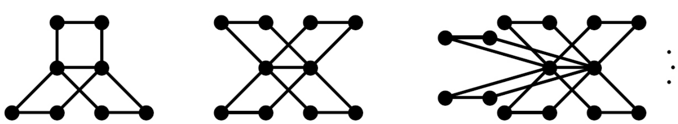

Figure 1

Book graph B_3, B_4 and B_n, respectively

Proof:

(i) There are two edges with endpoints of degree 1 and 2. So the coefficient of $x^{\sqrt{1+4}}$ in $S(P_n,x)$ is 2. Also there are $n-3$ edges with endpoints of degree 2. Therefore the coefficient of $x^{\sqrt{4+4}}$ in $S(P_n,x)$ is $n-3$, and we have the result.

(ii) There are n edges with endpoints of degree 2. Therefore the coefficient of $x^{\sqrt{4+4}}$ in $S(C_n,x)$ is n and we have the result.

(iii) There are n edges with endpoints of degree 1 and n. Therefore the coefficient of $x^{\sqrt{n^2+1}}$ in $S(K_{1,n},x)$ is n and we have the result.

(iv) There are n edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(F_n,x)$ is n. Also there are $2n$ edges with endpoints of degree 2 and $2n$. Therefore the coefficient of $x^{\sqrt{4n^2+4}}$ in $S(F_n,x)$ is $2n$ and we have the result.

(v) There are $m(n-2)$ edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(D_n^{(m)},x)$ is $m(n-2)$. Also there are $2m$ edges with endpoints of degree 2 and $2m$. Therefore the coefficient of $x^{\sqrt{4m^2+4}}$ in $S(D_n^{(m)},x)$ is $2m$, and we have the result.

(vi) There are $n-1$ edges with endpoints of degree 3. So the coefficient of $x^{\sqrt{9+9}}$ in $S(W_n,x)$ is $n-1$. Also there are $n-1$ edges with endpoints of degree 3 and $n-1$. Therefore the coefficient of $x^{\sqrt{9+(n-1)^2}}$ in $S(W_n,x)$ is $n-1$, and we have the result.

(vii) There are two edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(L_n,x)$ is 2. Also there are four edges with endpoints of degree 2 and 3 and there are $3n-8$ edges with endpoints of degree 3. Therefore the coefficient of $x^{\sqrt{4+9}}$ in $S(L_n,x)$ is 4 and the coefficient of $x^{\sqrt{9+9}}$ in $S(L_n,x)$ is $3n-8$, and we have the result.

(viii) There are n edges with endpoints of degree 2 and there is one edge with endpoints of degree $n-1$. So the coefficient of $x^{\sqrt{4+4}}$ in $S(B_n,x)$ is n and the coefficient of $x^{\sqrt{(n-1)^2+(n-1)^2}}$ in $S(B_n,x)$ is 1. Also there are $2n$ edges with endpoints of degree 2 and $n-1$. Therefore the coefficient of $x^{\sqrt{4+(n-1)^2}}$ in $S(B_n,x)$ is $2n$, and we have the result.

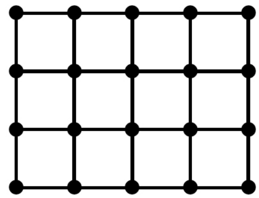

Figure 2

Grid graph $P_4 \times P_5$

Theorem 2.5 : *Let $P_m \square P_n$ be the gird graph (See $P_4 \times P_5$ in Figure 2). Then for every $n \geq 6$ and $m \geq 6$,*

$$S(P_m \square P_n, x) = 8x^{\sqrt{13}} + 2(m+n-6)x^{3\sqrt{2}} + (2m+2n-8)x^5 + (2nm-5n-5m+12)x^{4\sqrt{2}},$$

Proof: There are eight edges with endpoints of degree 2 and 3 and there are $2(m+n-6)$ edges with endpoints of degree 3. So the coefficient of $x^{\sqrt{4+9}}$ in $S(P_m \times P_n, x)$ is 8 and the coefficient of $x^{\sqrt{9+9}}$ in $S(P_m \times P_n, x)$ is $2(m+n-6)$. Also there are $2m+2n-8$ edges with endpoints of degree 3 and 4 and there are $2nm-5n-5m+12$ edges with endpoints of degree 4 . Therefore the coefficient of $x^{\sqrt{9+16}}$ in $S(P_m \times P_n, x)$ is $2m+2n-8$ and the coefficient of $x^{\sqrt{16+16}}$ in $S(P_m \times P_n, x)$ is $2nm-5n-5m+12$, and we have the result.

Cactus graphs were first known as Husimi tree appeared in the papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [17, 19, 24]. We refer the reader to papers [6, 22, 25] for more information about the parameters of cactus graphs.

Now we compute the Sombor polynomial of cactus chains:

Theorem 2.6 :

(i) *Let T_n be the chain triangular graph (see T_4 in Figure 3) of order n. Then for every $n \geq 2$,*

$$S(T_n, x) = 2x^{2\sqrt{2}} + 2nx^{2\sqrt{5}} + (n-2)x^{4\sqrt{2}}.$$

(ii) *Let Q_n be the para-chain square cactus graph (see Q_5 in Figure 3) of order n. Then for every $n \geq 2$,*

$$S(Q_n, x) = 4x^{2\sqrt{2}} + (4n-4)x^{2\sqrt{5}}.$$

(iii) *Let O_n be the para-chain square cactus (see O_4 in Figure 4) graph of order n. Then for every $n \geq 2$,*

$$S(O_n, x) = (n+2)x^{2\sqrt{2}} + 2nx^{2\sqrt{5}} + (n-2)x^{4\sqrt{2}}.$$

(iv) *Let O_n^h be the Ortho-chain graph (see O_4^h in Figure 4) of order n. Then for every $n \geq 2$,*

$$S(O_n^h, x) = (3n+2)x^{2\sqrt{2}} + 2nx^{2\sqrt{5}} + (n-2)x^{4\sqrt{2}}.$$

Proof:

(i) There are two edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(T_n, x)$ is 2. Also there are $2n$ edges with endpoints of degree 2 and 4 and there are $n-2$ edges with endpoints of degree 4. Therefore the coefficient of $x^{\sqrt{4+16}}$ in $S(T_n, x)$ is $2n$ and the coefficient of $x^{\sqrt{16+16}}$ in $S(T_n, x)$ is $n-2,$ and we have the result.

(ii) There are four edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(Q_n, x)$ is 4. Also there are $4n-4$ edges with endpoints of degree 2 and 4. Therefore the coefficient of $x^{\sqrt{4+16}}$ in $S(Q_n, x)$ is $4n-4,$ and we have the result.

(iii) There are $n+2$ edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(O_n, x)$ is $n+2.$ Also there are $2n$ edges with endpoints of degree 2 and 4 and there are $n-2$ edges with endpoints of degree 4. Therefore the coefficient of $x^{\sqrt{4+16}}$ in $S(O_n, x)$ is $2n$ and the coefficient of $x^{\sqrt{16+16}}$ in $S(O_n, x)$ is $n-2,$ and we have the result.

(iv) There are $3n+2$ edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(O_n^h, x)$ is $3n+2.$ Also there are $2n$ edges with endpoints of degree 2 and 4 and there are $n-2$ edges with endpoints of degree 4. Therefore the coefficient of $x^{\sqrt{4+16}}$ in $S(O_n^h, x)$ is $2n$ and the coefficient of $x^{\sqrt{16+16}}$ in $S(O_n^h, x)$ is $n-2,$ and we have the result.

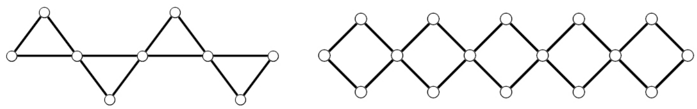

Figure 3

Chain triangular cactus T_4 and Para-chain square cactus Q_5

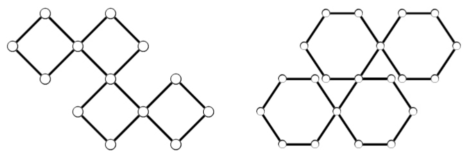

Figure 4

Para-chain square cactus O_4 and Ortho-chain graph O_4^h

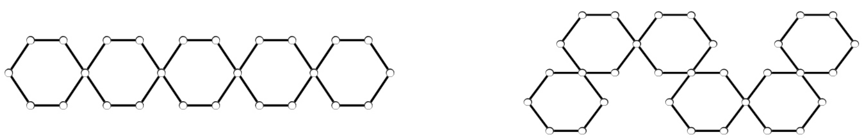

Figure 5

Para-chain L_5 and Meta-chain M_6

Theorem 2.7 : *Let L_n be the para-chain hexagonal cactus graph (See L_5 in Figure 5) of order n. Then for every $n \geq 2$,*

$$S(L_n, x) = (2n+4)x^{2\sqrt{2}} + (4n-4)x^{2\sqrt{5}}.$$

Proof: There are $2n+4$ edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(L_n, x)$ is $2n+4$. Also there are $4n-4$ edges with endpoints of degree 2 and 4. Therefore the coefficient of $x^{\sqrt{4+16}}$ in $S(L_n, x)$ is $4n-4$, and we have the result.

Theorem 2.8 : *Let M_n be the Meta-chain hexagonal cactus graph (See M_6 in Figure 5) of order n. Then for every $n \geq 2$,*

$$S(M_n, x) = (2n+4)x^{2\sqrt{2}} + (4n-4)x^{2\sqrt{5}}.$$

Proof: There are $2n+4$ edges with endpoints of degree 2. So the coefficient of $x^{\sqrt{4+4}}$ in $S(M_n, x)$ is $2n+4$. Also there are $4n-4$ edges with endpoints of degree 2 and 4. Therefore the coefficient of $x^{\sqrt{4+16}}$ in $S(M_n, x)$ is $4n-4$, and we have the result.

Corollary 2.9 : *Let* M_n *be the Meta-chain hexagonal cactus graph of order n. Then for every* $n \geq 2$,

$$SO(M_n) = (4n+8)\sqrt{2} + (8n-8)\sqrt{5}.$$

Corollary 2.10 : *Meta-chain hexagonal cactus graphs and para-chain hexagonal cactus graphs of the same order, have the same Sombor polynomial.*

Theorem 2.11 : *For the graph* $P_n \circ K_1$, *we have*

$$S(P_n \circ K_1, x) = (n-3)x^{3\sqrt{2}} + (n-2)x^{\sqrt{10}} + 2x^{\sqrt{5}} + 2x^{\sqrt{13}}.$$

Proof: There are two edges with endpoints of degree 1 and 2 and two edges with endpoints of degree 2 and 3. So the coefficient of $x^{\sqrt{4+9}}$ and $x^{\sqrt{1+4}}$ in $S(P_n \circ K_1, x)$ are 2. Also there are $n-2$ edges with endpoints of degree 1 and 3 and there are $n-3$ edges with endpoints of degree 3. Therefore the coefficient of $x^{\sqrt{1+9}}$ in $S(P_n \circ K_1, x)$ is $n-2$ and the coefficient of $x^{\sqrt{9+9}}$ in $S(P_n \circ K_1, x)$ is $n-3$, and we have the result.

Corollary 2.12: *For the graph* $P_n \circ K_1$, *we have*

$$SO(P_n \circ K_1) = (3n-9)\sqrt{2} + (n-2)\sqrt{10} + 2\sqrt{5} + 2\sqrt{13}.$$

Theorem 2.13 : *For the graph* $C_n \circ K_1$, *we have*

$$S(C_n \circ K_1, x) = nx^{\sqrt{10}} + nx^{3\sqrt{2}}.$$

Proof: There are n edges with endpoints of degree 1 and 3 and there are n edges with endpoints of degree 3. Therefore the coefficient of $x^{\sqrt{1+9}}$ and $x^{\sqrt{9+9}}$ in $S(C_n \circ K_1, x)$ are n and we have the result.

Corollary 2.14 : *For the graph* $C_n \circ K_1$, *we have*

$$SO(C_n \circ K_1) = 3n\sqrt{2} + n\sqrt{10}.$$

For any $k \in \mathbb{N}$, the k-subdivision of G is a simple graph $G^{\frac{1}{k}}$ which is constructed by replacing each edge of G with a path of length k. The following Theorem is about the Sombor polynomial and Sombor index of k-subdivision of graph G.

Theorem 2.15 : *Let $G=(V,E)$ be a graphs with $|V|=n$ and $|E|=m$. Also let d_u be the degree of vertex u in G. Then for every $k\geq 2$,*

$$S(G^{\frac{1}{k}},x)=m(k-2)x^{2\sqrt{2}}+\sum_{u\in V}d_u x^{\sqrt{d_u^2+4}}.$$

Proof: There are d_u edges around $u\in V$ with endpoints of degree d_u and 2 in $G^{\frac{1}{k}}$. Also there are $m(k-2)$ edges with endpoints of degree 2 in that. So

$$S(G^{\frac{1}{k}},x)=m(k-2)x^{\sqrt{4+4}}+\sum_{u\in V}d_u x^{\sqrt{d_u^2+4}},$$

and we have the result.

Corollary 2.16 : *Let $G=(V,E)$ be a graphs with $|V|=n$ and $|E|=m$. Also let d_u be the degree of vertex u in G. Then for every $k\geq 2$,*

$$SO(G^{\frac{1}{k}})=2m(k-2)\sqrt{2}+\sum_{u\in V}d_u\sqrt{d_u^2+4}.$$

As a result of the Corollary 2.16, we have:

Corollary 2.17 *: Let $G=(V,E)$ be a graphs with $|V|=n$ and $|E|=m$. Also let Δ be the maximum degree and δ be the minimum degree of vertices in G. Then for every $k\geq 2$,*

$$2m(k-2)\sqrt{2}+n\delta\sqrt{\delta^2+4}\leq SO(G^{\frac{1}{k}})\leq 2m(k-2)\sqrt{2}+n\Delta\sqrt{\Delta^2+4}.$$

Remark 2.18 : As in [13] stated the bounds in the Corollary 2.17 are sharp. It suffices to consider cycle graph or complete graph.

3. Roots of Sombor polynomial

Graph polynomials are a well-developed subject which are useful for analyzing properties of graphs. The roots of some graph polynomials have studied well. For example, Woodal [28] studied the roots and root-free regions of chromatic and flow polynomials. Also, the root distribution of chromatic polynomial, flow polynomials and characteristic polynomials

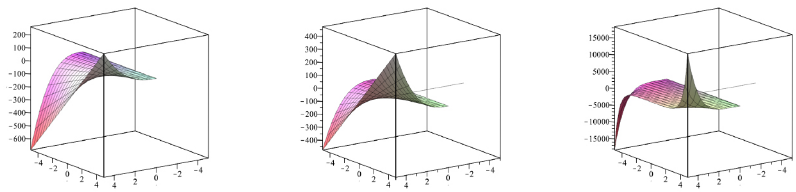

Figure 6

3D picture of Sombor polynomials of path, cycle and star, respectively.

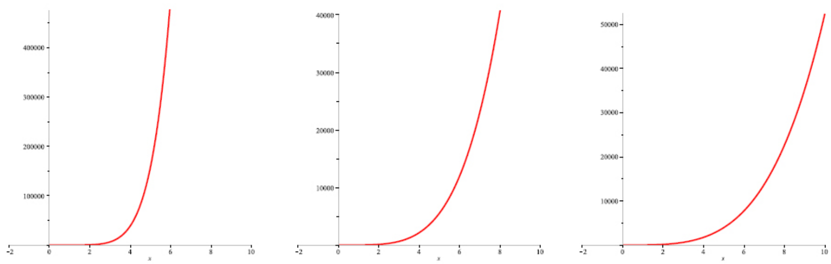

Figure 7

Picture of Sombor polynomials of F_3, W_4 and B_4, respectively.

of matroids have been examined by Jackson [20] and the roots of independence polynomials and domination polynomials have studied in [1, 4, 5].

Here, we want to study the location of roots of Sombor polynomial. The terms of polynomials in the literature have natural powers and in the Sombor polynomial the power are almost irrational. Almost nothing there is in the literature about this kind of polynomials and their roots. Since all coefficients of $S(G,x)$ are nonnegative, every real root of $S(G,x)$ is non-positive. The Sombor polynomial $S(G,x)$ is a function with two variables x and n, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$. The Figure 6 and 7 show the pictures of Sombor polynomials of paths, cycles and stars.

Here, we compute the roots of Sombor polynomial of some specific graphs. Let to denote the roots of Sombor polynomial of the graph G, by $Z(S(G,x))$. The proof of the following theorem is easy and follows from results in Section 1.

Theorem 3.1 :

(i) *Let* P_n *be the path graph of order* $n \geq 3$. *Then*

$$Z(S(P_n, x)) = \{0, e^{-\frac{\ln(\frac{-2}{n-3})}{\sqrt{5-2\sqrt{2}}}}\}.$$

(ii) *Let* C_n *be the cycle graph of order n. Then*

$$Z(S(C_n, x)) = \{0\}.$$

(iii) *Let* $K_{1,n}$ *be the star graph of order n. Then*

$$Z(S(K_{1,n}, x)) = \{0\}.$$

(iv) *Let* F_n *be the friendship graph of order n (join of* K_1 *and* nK_2*). Then*

$$Z(S(F_n, x)) = \{0, e^{\overline{e^{2(\sqrt{2}-\sqrt{n^2+1})}}^{-i\pi+\ln(2)}}\}.$$

(v) *Let* W_n *be the wheel graph of order n (join of* K_1 *and* C_{n-1}*). Then*

$$Z(S(W_n, x)) = \{0, e^{\overline{-\sqrt{n^2-2n+10}+3\sqrt{2}}^{-i\pi}}\}.$$

(vi) *Let* Q_n *be the para-chain square cactus graph (Figure 3) of order n. Then for every* $n \geq 2$

$$Z(S(Q_n, x)) = \{0, (-\frac{1}{n-1})^{\frac{1}{2(\sqrt{5}-\sqrt{2})}}\}.$$

(vii) *Let* L_n *be the para-chain hexagonal cactus graph (See Figure 5) of order n. Then for every* $n \geq 2$,

$$Z(S(L_n, x)) = \{0, e^{\frac{\ln(-\frac{n+2}{2n-2})}{2(\sqrt{5}-\sqrt{2})}}\}.$$

(viii) *Let* M_n *be the Meta-chain hexagonal cactus graph (See Figure 5) of order n. Then for every* $n \geq 2$,

$$Z(S(M_n, x)) = \{0, e^{\frac{\ln(-\frac{n+2}{2n-2})}{2(\sqrt{5}-\sqrt{2})}}\}.$$

(ix) *For the graph* $C_n \circ K_1$,

$$Z(S(C_n \circ K_1, x)) = \{0, e^{\frac{i\pi(3\sqrt{2}+\sqrt{10})}{8}}\}.$$

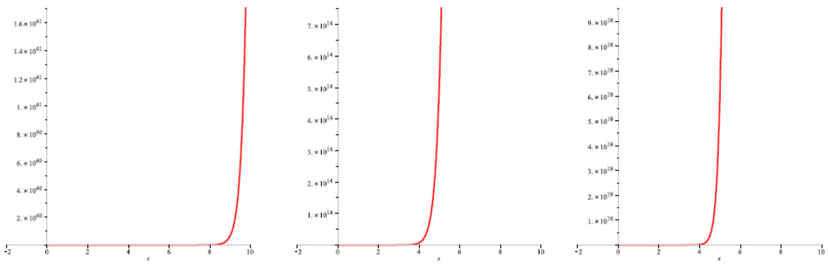

Figure 8

Picture of the Sombor polynomials of F_{20}, W_{20} and B_{20}, respectively.

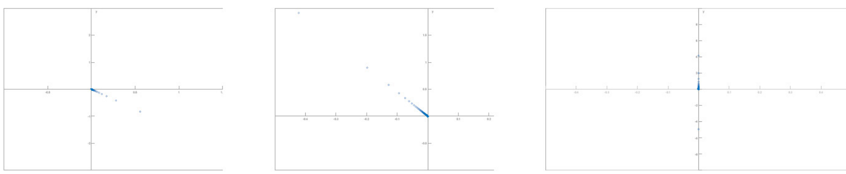

Figure 9

Complex roots of $S(P_n, x), S(F_n, x)$ and $S(W_n, x)$, respectively, for $n = 2, \ldots, 1000$.

We can consider the Sombor polynomial of graphs for fixed n. The pictures of the Sombor polynomials of F_{20}, W_{20} and B_{20}, drawn in Figure 8. From these figures, we think that the following conjecture is true:

Conjecture 3.2 : *The interval $(\frac{n}{5}, \infty)$ is zero-free interval of the Sombor polynomial for the family of all graphs of order n.*

In the end of this paper, we show the complex roots of $S(P_n, x), S(F_n, x)$ and $S(W_n, x)$, respectively, $(n = 2, 3, \ldots, 1000)$ in Figure 9.

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