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Hyperoperations on directed graphs

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Abstract

We analyze three hyper-operations derived from directed graphs, demonstrate the influence of the underlying graph structure on these hyper-operations and study the algebraic properties of the related classes of graphs. The paper also illustrates the relationship between directed graphs and hyper-structures using graph hyper-operations. In addition, we investigate the property of commutativity for an appropriate type of hyper-groupoids and its relationship to properties of directed graphs. Therefore, the paper consists of three sections, section one introduces and investigates three distinct hyper-groupoids derived from hyper-

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operations on directed graphs, section two discusses the fundamental ideas of directed graphs, and section three elaborates on path, simple path, and ancestry hyper-operations.

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1. Introduction

Binary connections and directed graphs appear naturally in order to encode complex information and describe graphical structures. C-hyper-groupoids, introduced in [3]-[6] are hyper-groupoids derived from binary relations, see also [1], [7], [19], [22]-[24]. Additional studies investigated the relationship between hyper-structures and binary relations in general [8]-[10], [25]-[27]. Directed graphs are a natural and efficient technique for describing binary relations while capturing more complicated characteristics [13]-[15], [18] and hence the natural symmetry arising between directed graphs and hyper-structures was further investigated [2], [11]-[12], [16], [17], [20], [21]. In this paper, we construct and investigate the attributes of three hyper-operations derived from directed graphs, namely: the path hyper-operation, the simple path hyper-operation, and the ancestry hyper-operation. We show that the features of these hyper-operations are dictated by the structure of their underlying graphs and that the attributes of these hyper-operations characterize the associated classes of graphs. This important link between hyper-operations and graph properties demonstrates that we can discover and recognize graphs by examining the properties of the related hyper-operations and inversely.

For any directed graph *G*, we define the path hyper-operation $\star_G$ as a mapping that associates a given pair of graph nodes (vertices) to the set composed by all graph nodes which lie on a directed path between the given nodes. This definition extends a well-known hyper-operation introduced by Corsini in the sense that given a relation *R* on *V* and *u*, *v* elements of *V*, then their product with respect to the Corsini operation is properly included in $u \star_G v$.

Moreover, given *G* as before, the simple path hyper-operation $\star_G^s$ is a mapping that associates a pair of nodes of the graph *G* to the set that includes all nodes which lie on a directed simple path between the two given nodes. It is clear that the latter is always included in the former of the two for fixed nodes $u, v \in V$. The third is *ancestry hyper-operation* which, given graph *G*, assigns two nodes *u* and *v* to the subset that is composed of all the nodes ω with the property that there exists a path from ω to u and from ω to v.

In Section 2, we introduce some fundamental notions on directed graphs in relation to hyper-groupoids. In the following section we intriduce three hyperoperations, namely path, simple path and ancestry. We present the unique properties of these hyperoperations due to their connection with the associated underlying graph structure. It is also illustrated how the underlying graph structure is associates to the properties of the corresponding path hyperoperation, by identifying unique path hyperoperation properties which produce discrete graphs, strongly connected graphs and acyclic graphs. Moreover the connection of these hyperoperations with graphs defines decisively their own properties. Indeed we prove that all path hyperoperations are associative, that non-partiality implies commutativity, but not the opposite for the path hyperoperation and that commutative path hyperoperations structure the set of the nodes of the underlying graph into an equivalence. On the other hand the ancestry hyperoperation has a much different type of connection with the underlying graph and we prove that it is commutative and associative for any given graph.

2. Preliminaries

A hyper-groupoid is defined as a couple $(U, *)$, consisting of a non-empty subset U and a hyper-operation $*$

$$\star : U \times U \to \mathcal{P}(U),\ (v, w) \mapsto v \star w.$$

where by $\mathcal{P}(U)$ we denote the powerset of U. If $X, Y \in \mathcal{P}(U) - \{\varnothing\}$, then

$$X \star Y = \bigcup_{x \in X, y \in Y} x \star y.$$

We indicate the hyperproduct $X \star Y$ by $x \star Y$ if the set X is just the singleton set $\{x\}$ and similarly for $X \star y$. Furthermore, $(U, *)$ is referred to as a nonpartial hyper-groupoid if $v, w \neq \varnothing$ for all elements v, w of the set U. Moreover, a hyper-groupoid is termed degenerative if for all elements v, w of the set U, the product $v \star w \neq \varnothing$, and similarly for a total hyper-groupoid we demand $v \star w = U$.

Assuming $R \subseteq U \times U$ is a binary relation, then the hyper-operation defined by Corsini

$$*_R : U \times U \to \mathcal{P}(U)$$

is described by

$$(v,w) \mapsto v *_R w = \{z \in U \mid (v,z), (z,w) \in R\}.$$

The Corsini partial hyper-groupoid (simply partial C-hyper-groupoid denoted U_R) is the hyper-structure $(U, *_R)$ that derives in the above way from the binary relation R [5, 13-14].

If $v *_R w \neq \emptyset$, for every element $v, w \in U$, then the pair $(U, *_R)$ is a C-hyper-groupoid. A C-hyper-groupoid U_R is called partial if and only if $R \circ R = U \times U$, where we denote by $\circ$ the composition of two relations. If we have a relation $R \subseteq U \times U$ with $U = \{v_1, v_2, \dots, v_k\}$ then the Boolean $k \times k$ matrix of R is $M_R = s_{i,jk \times k}$ with $s_{i,j} = 1$ for $(v_i, v_j) \in R$ and $s_{i,j} = 0$, otherwise. A concrete directed graph G is defined by a couple (V, R) consisting of:

- A set V of elements termed vertices or nodes
- A set of edges R represented as pairs of nodes: $R \subseteq V \times V$

An node or vertex is merely displayed as an arbitrary location in this case, while an edge is represented as a directed link between two nodes, the outgoing node is called the head and the incoming node is called the tail.

If for two graphs G and G' with nodes V, V' and edges R, R' we have $V' \subseteq V$ and $R' \subseteq R$, then we call G' subgraph of G. Contrary to that, given a graph G its supergraph is any graph that contains G. An isomorphism between two graphs respecting the edges is called graph isomorphism. We call the set of all graphs that are isomorphically equal abstract graph or just graph. Graph properties that remain unaffected by isomorphisms are called invariant.

For any given node $v \in V$ of a graph G, the *degreeout* is the number of outgoing edges of v while the number of incoming edges of a node is called similarly the *degreein*. In addition, we set *degreeout*(G) to be a set consisting of the *degreeout* of all nodes of G, *degreein*(G) is defined analogously. The cardinality of the set of nodes of a graph G is termed order of G, while the cardinality of the set of edges is termed the size of G.

An edge that has the same node as head and tail is referred to as a loop. An edge is termed multiple if there exists a second edge having identical head and tail. A graph that doesn't have multiple edges is termed simple. If a node doesn't have any edge connected to it either as a head or as a tail is called isolated. In the subsequent sections we will investigate graphs that are simple and don't have nodes that are isolated.

We now introduce paths inside a graph in the following manner. Given a graph G with set of nodes V and set of edges R, then for any two nodes w_1, w_n a path between w_1 and w_n is a sequence of nodes $w_2, \ldots, w_{n-1}$ of G, such that $(w_i, w_{i+1}) \in R$, for $i \in 2, n-2$. This sequence $w_1, \ldots, w_n$ stands for the path between w_1 and w_n. Notice that by this definition a node may be repeated inside a path. If the path is such that no node is repeated then the path is called simple. A node v belongs to (or lies on) such a path if $v = w_i$, for some $i \in 1, n$. We define

$$path(w_1, w_n) = \{v \mid v \text{ lies on a path from } w_1 \text{ to } w_n\}.$$

The length of $path(w_1, \ldots, w_n)$ is by definition $n-1$, a number identical to the number of edges we traverse getting from w_1 to w_n. Hence a path is simple if $w_i \neq w_j$ for all $i \neq j$.

A strongly connected graph G possess the following property: *there always exist at least one path between any two nodes of G in both directions*. In this respect, a *strongly connected component* of a graph G is defined accordingly.

In what follows, we will discuss and demonstrate the properties of three different forms of hyper-groupoids that result from three different hyper-operations: *Path hyper-operation*, *Simple path hyper-operation* and *Ancestry path hyper-operation*.

3. Path, Simple Path and Ancestry hyperoperations

In this section we present three different hypergroupoids deriving from hyperoperations on directed graphs and study their basic properties. The results in the rest of this section concern arbitrary directed graphs $G = (V, R)$.

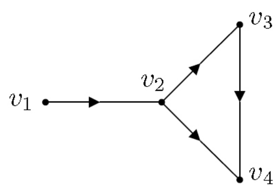
The path product or path hyper-operation is a mapping $\star_G$: $V \times V \to \mathcal{P}(V)$, given by:

$$v \star_G w = \{u \in V \mid u \text{ lies on a path between } v \text{ and } w\},$$

for all nodes v, w of G.

From this we obtain a hyper-structure $(V, \star_G)$, termed the path hyper-groupoid corresponding to the given graph. If, for every $v, w \in V$, the product $v \star_G w$ is not empty we say that the hyper-opeartion is nonpartial and the related path hyper-groupoid is termed nonpartial. We identify the so obtained family by $npPH$. At this point we can also introduce commutative path hyper-groupoids, we denote the associated class by cPH.

Example 1: We examine the following graph



The corresponding multiplicative table of $\star_G$ is

$\star_G$	v_1	v_2	v_3	v_4
v_1	$\{v_1\}$	$\{v_1,v_2\}$	$\{v_1,v_2,v_3\}$	$\{v_1,v_2,v_3,v_4\}$
v_2	$\varnothing$	$\{v_2\}$	$\{v_2,v_3\}$	$\{v_2,v_3,v_4\}$
v_3	$\varnothing$	$\varnothing$	$\{v_3\}$	$\{v_3,v_4\}$
v_4	$\varnothing$	$\varnothing$	$\varnothing$	$\{v_4\}$

Hence for a given graph $G=(V,R)$ and two nodes v, w of V, the Corsini product $v *_R w$ is included in the path product, which means that it holds

$$v *_R w \subseteq v \star_G w.$$

The graph property of strong connectivity introduced earlier can be described by virtue of the path hyper-operation as follows.

Proposition 1: For any graph G and nodes $v,w \in V$, the below conditions are equivalent

i) $v \star_G w \neq \varnothing$ and $w \star_G v \neq \varnothing$.

ii) There is a strongly connected component of G that includes the nodes v,w.

We prove that strong connectivity, nonpartiality and totality of the path hyper-operation are equivalent.

Theorem 1: *Given a graph G, the below properties are equivalent.*

i) *The graph G is strongly connected.*

ii) *The associated path hyper-operation $\star_G$ organizes the set of nodes of G in a nonpartial hyper-groupoid.*

iii) *The associated path hyper-operation $\star_G$ is total.*

Proof: **i)** $\Leftrightarrow$ **ii)**. By definition, a graph is strongly connected if equivalently there exists inside G a path between any two nodes, if equivalenty $\star_G$ is nonpartial.

i) $\Leftrightarrow$ **iii)** Let $v, w \in V$, we shall show that $v \star_G w = V$. Since G is strongly connected graph, for any given node x there exists a path between nodes v and x and between nodes x and w, which implies that there is a path between v and w passing through the node x and thus $x \in v \star_G w$, as asserted.

iii) $\Rightarrow$ **ii)**. Straightforward.

The path hyper-operation also determines the presence of cycles inside a graph as it it illustrated below.

Theorem 2: *Given a graph G and two nodes v,w of G, the following conditions are equivalent.*

i) *It holds* $v \star_G w = w \star_G v \neq \emptyset$.

ii) *There exists a cycle in G that includes both nodes v and w.*

From the above theorem we obtain a corollary which relates commutativity of the path hyper-operation with the existence of cycles inside the underlying graph.

Corollary 1: *Necessary and sufficient condition for any given graph G to be acyclic is that for every pair of nodes v,w of G, either* $w \star_G v \neq v \star_G w$, *or* $w \star_G v = v \star_G w = \emptyset$.

The path hyper-operation is has some unique properties due to its underlying relation with a directed graph. For this reason we can see that for the path hyper-operation nonpartiality implies commutativity.

Theorem 3: *For any graph G and nodes* $v, w \in V$, *if* $v \star_G w \neq \emptyset$ *and* $w \star_G v \neq \emptyset$, *then*

$$v \star_G w = w \star_G v.$$

Combining Theorems 1 and 3 we can see the way that the properties of the path hyper-operation are determined by the underlying directed graph.

Corollary 2: *If a path hyper-groupoid is nonpartial then it is also commutative.*

Hence we obtained the following relation for the two corresponding classes.

Theorem 4: *The set of nonpartial path hyper-groupoids is properly included into the set of commutative path hyper-groupoids*

$$npPH \subsetneq cPH.$$

The relation of path hyper-operations with directed graphs gives them properties which are not found in usual hyper-operations.

Theorem 5: *If the path hyper-operation* $\star_G$, *derived from a graph G, is commutative then* $v \star_G w \neq \varnothing$ *and* $w \star_G u \neq \varnothing$, *implies* $v \star_G w = w \star_G u$, *for all nodes* v, w, u *of G.*

It turns out that in order to obtain commutative path hyper-groupoids we only have to form the union of a set of disjoint nonpartial hyper-groupoids.

To construct this partition we define an appropriate equivalence. For every given graph *G*, we introduce a corresponding path relation as follows

$$v \equiv_G u \Leftrightarrow v \star_G u \neq \varnothing$$

for all nodes v, u of *G*.

Proposition 2: If the path hyper-groupoid associated to a graph $G = (V, R)$ is commutative then the path relation is an equivalence on V.

Theorem 6: *The below conditions are equivalent for a given graph* $G = (V, R)$.

i) *The path hyper-operation* $\star_G$ *is commutative.*
ii) *G can obtained as the union of disjoint strongly connected graphs.*

Hence, by combining Theorems 6 and 1, we obtain

Theorem 7: *Given a path hyper-groupoid* $(V, \star_G)$, *the following conditions are equivalent*

i) *The path hyper-operation* $(V, \star_G)$ *is commutative.*
ii) *V is obtained as the union of disjoint nonpartial path hyper-groupoids*

$$(V_1, \star_{G_1}), \ldots, (V_n, \star_{G_n}).$$

The following proposition characterizes discrete graphs in relation to the structure of the associated hyper-groupoid.

Proposition 3: The below conditions are equivalent for a graph $G = (V, R)$.

i) The path hyper-operation $\star_G$ is weakly degenerative.
ii) The graph *G* is discrete.

Proof: The graph is discrete if equivalently for all nodes $v, w \in V, v \neq w$, there is no path between them, if and only if $\star_G$ is weakly degenerative.

The simple path hyper-operation (or simple path product) is the second hyper-operation we will investigate. Considering a graph G with nodes V, the simple path hyper-operation is a mapping $\star_G^s : V \times V \to \mathcal{P}(V)$ defined by:

$$v \star_G^s w = \{u \in V \mid u \text{ lies on a simple path from } v \text{ to } w\}$$

for all nodes v, w of V.

The hyper-structure $(V, \star_G)$ that results in this way is named the simple path hyper-groupoid corresponding to the graph. Nonpartial and commutative simple path hyper-groupoids are defined analogously and their classes are denoted by $npSPH$ and $cSPH$ respectively. Simple paths are a special case of paths inside a graph and moreover we get

Proposition 4: For a graph G and nodes $v, w \in V$, there exists a path from v to w if and only if there exists a simple path from v to w.

Proposition 5: For a graph G and nodes v,w, their Corsini product $v *_R w$ is included in their simple path product $v \star_G^s w$ which is included in the path product, hence we have:

$$v *_R w \subseteq v \star_G^s w \subseteq v \star_G w.$$

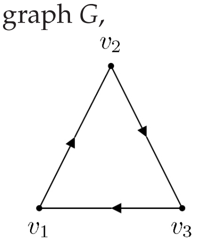
The combination of Theorem 1 and Proposition 4 yields

Theorem 8: *Given a graph G, the conditions below are equivalent*

i) *The graph G is strongly connected.*
ii) *The path hyper-operation $\star_G$ organizes V into a nonpartial hyper-groupoid.*
iii) *The simple path hyper-operation $\star_G^s$ organizes V into a nonpartial hypergroupoid.*

As shown in Theorem 1, nonpartiality and totality are equivalent for any path hyper-operation $\star_G$. This is not the case for the simple path hyper-operation as it is demonstrated in the next example.

Example 2: For the below graph G,



the corresponding multiplicative table of the simple path hyper-operation is

$\star_G^s$	v_1	v_2	v_3
v_1	$\{v_1,v_2,v_3\}$	$\{v_1,v_2\}$	$\{v_1,v_2,v_3\}$
v_2	$\{v_1,v_2,v_3\}$	$\{v_1,v_2,v_3\}$	$\{v_2,v_3\}$
v_3	$\{v_1,v_3\}$	$\{v_1,v_2,v_3\}$	$\{v_1,v_2,v_3\}$

By inspecting the above we see that $\star_G^s$ is nonpartial but not total.

From the above Example, we also see that Theorem 2 does not hold for the simple path hyper-operation. Indeed, the nodes a and b of G belong to the same cycle while it holds

$$a \star_G^s b \neq b \star_G^s a.$$

As an outcome, neither Theorems 3 and 4 nor Corollaries 1 and 2 hold; by examining the multiplicative table of Example 2, we can see that commutativity is not implied by nonpartiality for the simple path hyper-operation. Furthermore, the characterization of Theorem 6 is not valid for the simple path hyper-operation.

In the case of discrete graphs, however, we obtain the following, through the combination of Propositions 3 and 4.

Proposition 6: For any given graph G, the conditions below are equivalent

- i) The graph G is discrete.
- ii) The path hyper-operation $\star_G$ organizes V into a weakly degenerative hyper-groupoid.
- iii) The simple path hyper-operation $\star_G^s$ organizes V into of a weakly degenerative hyper-groupoid.

Theorem 9: *For all graphs G with node set V and all $v, w, u \in V$ it holds*

$$(v \star_G w) \star_G u = v \star_G (w \star_G u) = v \star_G w \star_G u.$$

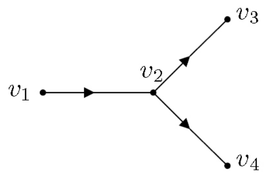
The previous Theorem yields subsequently,

Corollary 3: *The path hyper-operation is associative*

In what follows, we construct the ancestry hyper-operation. For a graph G with nodes V, the ancestry hyper-operation (or ancestry product) $\star_G^c : V \times V \to \mathcal{P}(V)$ can be introduced for nodes $v, w \in V$ by:

$$v \star_G^c w = \{u \in V \mid \text{path}(u,v) \neq \varnothing \text{ and path } (u,w) \neq \varnothing\}.$$

Example 3: Examine the graph G below



The corresponding multiplicative table for the path hyper-operation $\star_G$ is

$\star_G$	v_1	v_2	v_3	v_4
v_1	$\{v_1\}$	$\{v_1, v_2\}$	$\{v_1, v_2, v_3\}$	$\{v_1, v_2, v_4\}$
v_2	$\varnothing$	$\{v_2\}$	$\{v_2, v_3\}$	$\{v_2, v_4\}$
v_3	$\varnothing$	$\varnothing$	$\{v_3\}$	$\varnothing$
v_4	$\varnothing$	$\varnothing$	$\varnothing$	$\{v_4\}$

and the multiplicative table of the ancestry hyper-operation $\star_G^c$ is

$\star_G^c$	v_1	v_2	v_3	v_4
v_1	$\{v_1\}$	$\{v_1\}$	$\{v_1\}$	$\{v_1\}$
v_2	$\{v_1\}$	$\{v_1, v_2\}$	$\{v_1, v_2\}$	$\{v_1, v_2\}$
v_3	$\{v_1\}$	$\{v_1, v_2\}$	$\{v_1, v_2, v_3\}$	$\{v_1, v_2\}$
v_4	$\{v_1\}$	$\{v_1, v_2\}$	$\{v_1, v_2\}$	$\{v_1, v_2, v_4\}$

As an outcome of this we get that the ancestry product is incomparable to the path, the simple path, and the Corsini product. Indeed, for G as stated above, we get:

$$v_3 \star_G v_4 = v_3 \star_G^s v_4 = v_3 *_R v_4 = \varnothing \quad \text{and} \quad v_3 \star_G^c v_4 = \{v_1, v_2\}$$

and

$$v_1 \star_G v_3 = v_1 \star_G^s v_3 = \{v_1, v_2, v_3\}, \quad v_1 *_R v_3 = \{v_2\} \quad \text{and} \quad v_1 \star_G^c v_3 = \{v_1\}.$$

Note that a simple ancestry hyper-operation $\star_G^{sc} : V \times V \to \mathcal{P}(V)$ can be established via:

$$v \star_G^{sc} w = \{u \in V \mid s\ \mathrm{path}(u, v) \neq \varnothing \text{ and } s\ \mathrm{path}(u, w) \neq \varnothing\},$$

for all of $v, w \in V$. As it turns out, due to Proposition 4, such a hyper-operation would be equivalent to the introduced ancestry hyper-operation.

As the following Theorem illustrates, the ancestry hyper-operation has substantially distinct properties than the path and simple path hyper-operations.

Theorem 10: *The ancestry hyper-operation* $\star_G^c$ *is commutative and associative.*

Proof: Commutativity of the given hyper-operation is straightforwardly obtained from the definition.

In order to prove that $\star_G^c$ is associative we have to show that for any graph $G=(V,R)$ and for all $v,w,u \in V$, it holds

$$v \star_G^c (w \star_G^c u) = (v \star_G^c w) \star_G^c u.$$

For this we set

$$v \star_G^c w \star_G^c u = \{x \in V \mid \text{path}(x,v) \neq \varnothing,\ \text{path}(x,w) \neq \varnothing \text{ and path } (x,u) \neq \varnothing\}$$

i.e., it is the set consisting of all nodes x of G such that there are paths from x to the nodes v,w,u. We are going to prove that

$$v \star_G^c w \star_G^c u = v \star_G^c (w \star_G^c u).$$

Let $x \in v \star_G^c w \star_G^c u$, then, by definition, $path(x,v) \neq \varnothing$, $path(x,w) \neq \varnothing$ and $path(x,u) \neq \varnothing$. From the last two we get $x \in (w \star_G^c u)$ and since $path(x,x) \neq \varnothing$ and $path(x,v) \neq \varnothing$ we have $x \in v \star_G^c (w \star_G^c u)$. Therefore

$$v \star_G^c w \star_G^c u \subseteq v \star_G^c (w \star_G^c u).$$

Conversely, let $x \in v \star_G^c (w \star_G^c u)$, this means that there exists $z \in (w \star_G^c u)$ such that: A: there exists a path from z to w, B: there exists a path from z to u, C: there exists a path from x to v and D: there exists a path from x to z. By *A*, *B* and *D* we get that there exists a path from x to w and there exists a path from x to u. Hence it holds that $x \in v \star_G^c w \star_G^c u$ and so

$$v \star_G^c (w \star_G^c u) \subseteq v \star_G^c w \star_G^c u$$

as wanted.

Similarly, we can prove that

$$v \star_G^c w \star_G^c u = (v \star_G^c w) \star_G^c u.$$

which concludes the proof.

4. Conclusion

The three types of graph hyper-operations we examined illustrate a rich relationship between Graph Theory and Hyper-structures pertaining to both a purely mathematical as well as an applications oriented point of view. Several research directions arise from the presented results, including the identification of various classes of directed graphs related with more specific properties of the three graph hyper-operations, as well as the implications of extending the above hyper-operations to different types of graphs such as hypergraphs. On the other hand it is also interesting to investigate the repercussions of limiting the class of directed graphs that generate these hyper-operations to more specific ones such as, for instance the Hamiltonian and the Eulerian graphs.

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