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Certain domination numbers for Cartesian product of graphs

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Abstract

Let G be a network, any node $v \in V(G) \setminus D$ is adjacent to atleast one node in D, then the set D is named a dominating set, represented by $\gamma(G)$. A set D of nodes in a network G is called an independent dominating set if D is an independent set. A total dominating set of a network G is a set D of nodes such that every nodes in G is adjacent to a node in D. A dominating set D of G is called paired dominating set whose induced graph of D contains a perfect matching. A 2-dominating set of a network G is a dominating set, in which any node not in D has minimum two neighbours in D. The minimal size of a 2-dominating set of G denoted by $\gamma_2(G)$, is the 2-domination number of G. In this paper, we obtain certain domination parameters, such as domination, independent domination, total domination and paired domination numbers for the class of Cartesian product of cycle C_m, and

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circulant graph $G(6n;\pm\{1,2\})$ and $G(8n;\pm\{1,2,3\})$, where $m \equiv 0 \,(\mathrm{mod}\, 4)$, $n \geq 1$ and there by proving that the cardinality of all parameters are same for the above graphs. And also we obtain 2-domination number for the Cartesian product of cycle C_m, and circulant graph $G(2^n;\pm\{1,2\})$, with $m \equiv 0 \,(\mathrm{mod}\, 4)$, and Cartesian product of cycle C_m, and circulant graph $G(2^n;\pm\{1,3\})$, with $m \equiv 0,2 \,(\mathrm{mod}\, 4)$, $n \geq 3$, Furthermore, we demonstrating that the lower bound found in [1], [2], and [3] are quite precise for the results obtained here for domination and independent domination, total and paired domination, and 2-domination respectively.

Subject Classification: *05C69, 05C76, 05C90.*

Keywords: *Domination, Independent domination, Total domination, Paired domination, 2-domination, Cartesian product, Cycle, Circulant graph.*

1. Introduction

A graph is the most widely used and effective systematic and methodical tool for representational modelling, analysis, and solutions to complicated real-world issues. The extensive research of dominating sets began closely around 1960. In the year 1977, Cockayane, and Hedetniemi [4], completed a survey of graphs and their dominant sets. Many research scholars were inspired to work in the topic of graph dominance as a result of the following applications. In the field of chemistry, domination aids in comprehending the properties of numerous chemical formations [5], and in computer science it helps in the analysis of wireless communication networks as well as design theory [6]. These sets can forecast the positioning of restricted processors, allowing each node to access the processors either locally or through an adjacent node. A minimal dominance set affords this access to processors with low cost. The adequate results is to avert replication of access to processors. For an undirected simple network G, the nodes and lines of G are referred to as $V(G)$ and $E(G)$, respectively. Let $N(a) = \{b \in V(G) \mid ab \in E(G)\}$ is the open neighbourhood of a node $a \in V(G)$ and $N[a] = N(a) \cup \{a\}$ is the closed neighbourhood of a vertex $a \in V(G)$. Let $\delta(G) = min\{deg(a) \mid a \in V(G)\}$ and $max\{deg(a) \mid a \in V(G)\} = \Delta(G)$ is the least and greatest degree of G, respectively, where $deg(a)$ is the degree of a vertex 'a'. If the set D dominates each vertex of G, then the set $D \subseteq V(G)$ is a dominating set, and the number of dominators $\gamma(G)$ is the minimum number of elements of such a set D. For more results on domination, see [7].

An independent set in a network G is a set of pairwise isolated vertices of G. A set D of nodes in a network G is named an independent dominating set, if D is an independent as well as a dominating set of G. The independent domination value $i(G)$ of a network G is the smallest number of elements of an independent dominating set in G. A whole dominating set is a dominating set D with no isolated nodes in its induced subgraph, represented by $G[D]$. In 1862, the theory of an independent dominating set arose in chessboard problems [8]. Berge and Ore defined the concept of independent dominance in 1962. The independent domination value and the symbol $i(G)$ were hosted by Cockayne and Hedetniemi in [9]. For more results on independent domination, see [10]. A total dominating set $D \subseteq V(G)$ introduced by Cockayne *et. al*. in [11], is a set such that every vertex $v \in V(D)$ is adjacent to a vertex in D. Clearly, any total dominating set is also a dominating set, and there is a total dominating set in a graph if and only if it does not have isolated vertices. A minimum total dominating set is defined as an inclusion minimal total dominating set. The upper total domination number, or the greatest size of a minimum total dominating set, is indicated by $\gamma_t(G)$. The paper provides a survey on total dominance that includes several recent results by Henning in [12]. A paired dominating set is a dominating set with a perfect matched induced subgraph. The paired domination number is the minimum size of a paired dominating set, denoted by $\gamma_{pr}(G)$. An inclusion wise minimal paired dominating set is said to be minimal. The upper paired domination number, $\Gamma_{pr}(G)$, denotes the maximal size of a minimum paired dominating set. Paired domination was hosted by Haynes *et. al.* in 1998 and has obtained much concentration in the literature [13].

Fault tolerance is one of the most important characteristics that allows a system to continue to function even when one or more components fail (faults). Fink *et. al.* introduced the concept of n-domination in general graphs [3]. Further, Bujtás *et. al*. in [14]. proved that 2-domination number is strictly inferior to $\frac{n}{2}$, if minimum degree of $G \geq 6$. In this paper, we obtain certain domination numbers for special classes of Cartesian product of cycles and circulant graphs. Further, we prove that the equality of the lower bound obtained in [1–3] is attained.

2. Circulant Graphs

Highest node degree, and incremental extendibility (number of nodes added) are two essential characteristics in network architecture. Circulant graphs is a class of Cayley graph that allows for extendibility in stages while keeping the networks they simulate, the number of connections to each node remains constant. All circulant graph is a vertex

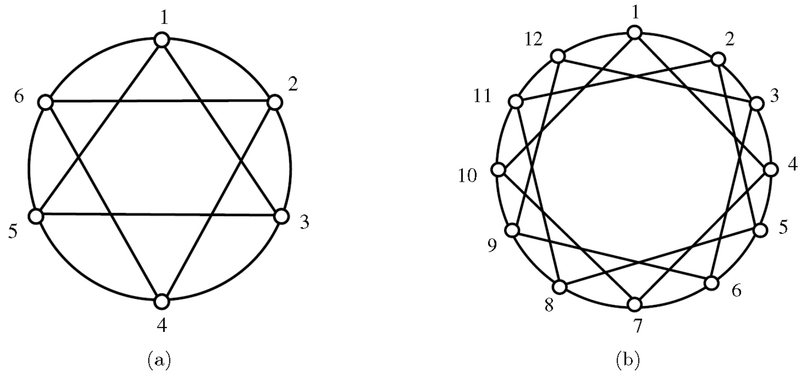

Figure 1

The circulant graphs: (a) $G(6,\pm\{1,2\})$ **and (b)** $G(12,\pm\{1,3\})$

transitive graph and it is a Cayley graph [15]. Cayley graphs enable for the creation of efficient routing, since they are vertex transitive. The topological characteristics of circulant graph have been studied widely and examined by Bermond *et. al.* in [16]. Mathematically, it can be defined as follows. The circulant graph $G(m;B)$, $B\subseteq\pm\{1,2,\ldots,l\}$, $1\le l\le\lfloor m/2\rfloor$ is a graph with node set $V=\{0,1,\ldots,m-1\}$ and the edge set $E=\{(p,q):|q-p|\equiv b\,(\mathrm{mod}\,m),$ $b\in B\}$. Example of the circulant graphs are displayed in Figure 1. Obradović *et. al.* categorised circulant graphs with efficient dominant sets of degree 2, 3, and 4 in [17].

Let C and D be any two graphs and the Cartesian product of C and D denoted by $C\,\square\,D$ and it is defined as the graph whose node set $V(C\,\square\,D)=V(C)\times V(D)$ and edge set $E(C\,\square\,D)$ = $\{(c_1,d_1),(c_2,d_2):(c_1,c_2)\in$ $E(C)$ and $d_1=d_2$ or $c_1=c_2$ and $(d_1,d_2)\in E(D)\}$.

The Cartesian product of cycle C_2 and circulant graph $G(8;\pm\{1,2\})$ and $G(8;\pm\{1,3\})$ are displayed in Figures 2(a) and 2(b) respectively. Note that the graph in Figure 2(a) has 2-disc (copies) of circulant graph $G(8;\pm\{1,2\})$. In general, $C_m\,\square\,G(2^n;\pm\{1,2\})$ contains m-disc of circulant graph $G(2^n;\pm\{1,2\})$, say $G_1(2^n;\pm\{1,2\})$, $G_2(2^n;\pm\{1,2\})$, ..., $G_m(2^n;\pm\{1,2\})$. For brevity, we call it as $G_1,G_2,\ldots,G_m$. In the year 1963, Vizing [18], given a conjecture that $\gamma(G\,\square\,H)\ge\gamma(G)\gamma(H)$. In [19] Hartnell *et. al.* drew attention to the presence of many classes of networks G and H which satisfies $\gamma(G\,\square\,H)=\gamma(G)\gamma(H)$. In [20] Zahar *et. al.* proved the relation $\gamma(G\,\square\,H)=\gamma(G)\gamma(H)$ for non-trivial networks in the case of cycle C_m, where $m\equiv 0\,(\mathrm{mod}\,3)$. Hou *et. al.* discussed the k-domination value of Cartesian product of graphs[21], there exists a new inferior bound for

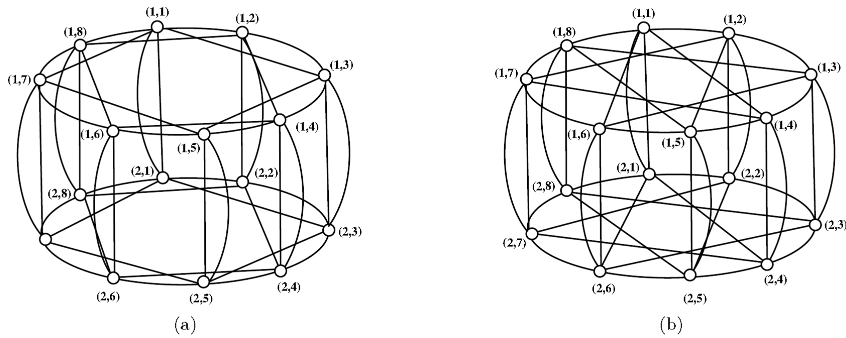

Figure 2

The Cartesian product of (a) $C_2 \square G(8;\pm\{1,2\})$ **: (b)** $C_2 \square G(8;\pm\{1,3\})$

$\gamma^{\{k\}}(G \square H)$ in terms of packing and $\{k\}$-domination numbers of G and H. For more informations on variants of domination in Cartesian product of graphs, the readers may refer to [22–24].

Many researchers have come up with 2-domination number in many graphs. For instance, trees [25], cactus graphs [26], toroidal grid graphs [27], grid graphs [28], trees with equal 2-independence and 2-domination value [29], and bipartite graphs [30]. In this paper, we have estimated the domination numbers for Cartesian product of cycle C_m, and circulant graph $G(6n;\pm\{1,2\})$ and $G(8n;\pm\{1,2,3\})$, where $m \equiv 0 \,(\text{mod}\, 4)$, $n \geq 1$. Also we obtain the 2-domination number for the Cartesian product of cycle C_m, and circulant graph $G(2^n;\pm\{1,2\})$ and $G(2^n;\pm\{1,3\})$, where $m \equiv 0, 2 \,(\text{mod}\, 4)$, $n \geq 3$

3. Main Results

In this section, we find certain classes of Cartesian product of cycles and circulant graphs to show that the lower bound obtained in [1-3] is sharp.

Theorem 1: [1] *For any network G with n nodes and highest degree* Δ, $i(G) \geq \lceil \frac{n}{1+\Delta} \rceil$.

Theorem 2: [2] *If G is a network of order n, then* $\gamma_t(G) \geq \lceil \frac{n}{\Delta(G)} \rceil$.

Theorem 3: [3] *If G is a network of order n with highest degree* Δ, *then for every non-negative integer k,* $\gamma_k(G) \geq \lceil \frac{kn}{k+\Delta} \rceil$.

Theorem 4: *Let G be a Cartesian product of cycle C_m, and circulant graph $G(6n;\pm\{1,2\})$ denoted by $C_m \square G(6n;\pm\{1,2\})$. Then $\gamma(G) = mn$, where $m \equiv 0 \pmod 4$ and $n \geq 1$.*

Proof: Since $\Delta = 6$ and by Theorem 1, we have $\gamma(G) \geq \lceil \frac{6mn}{7} \rceil \geq mn$. To prove the reverse inequality, we exhibit the dominating set $D \subseteq V(G)$ with mn vertices. The label of vertices of the Cartesian product of $C_m \square G(6n;\pm\{1,2\})$ is in the form of ordered pairs. We assign the labels of vertices for the i^{th} disc G_i in $C_m \square G(2^n;\pm\{1,2\})$ as $(i,1),(i,2),\ldots,(i,2^n)$ for all i, $1 \leq i \leq m$. For illustration, the labeling of $C_4 \square G(6;\pm\{1,2\})$ is given in Figure 3. Note that, in the labeling (x,y), x and y will be taken modulo m and modulo $6n$ respectively. For any j, $0 \leq j \leq \frac{n}{6} - 1$, let $D = \{(x,6j+1)\}$ if x is odd, and $D = \{(x,6j+4)\}$ if x is even, for all $1 \leq x \leq m$ be the dominating set of $C_m \square G(6n;\pm\{1,2\})$.

Since it is vertex transitive, with out loss of generality, choose any vertex with label $(x,y) \in V \setminus D$, then $N[(x,y)] = \{(x,y+1),\ (x,y+2),\ (x,y+6n-2),\ (x,y+6n-1),\ (x+1,y),\ (x+m-1,y)\}$. Further, if x is odd

$$N[(x,y)] \cap D = \begin{cases} \{(x,6j+1)\} & ; \quad \text{if } y \equiv 0,2,3,5 \pmod 6 \\ \{(x+1,y)\} & ; \quad \text{if } y \equiv 4 \pmod 6 \end{cases}$$

and if x is even

$$N[(x,y)] \cap D = \begin{cases} \{(x,6j+4)\} & ; \quad \text{if } y \equiv 0,2,3,5 \pmod 6 \\ \{(x-1,y)\} & ; \quad \text{if } y \equiv 1 \pmod 6 \end{cases}$$

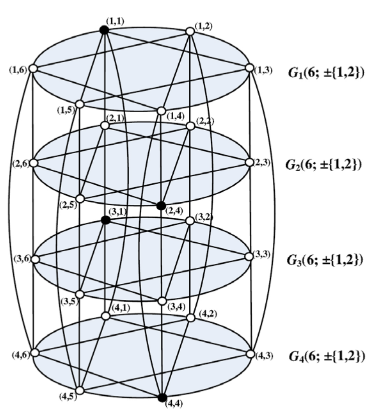

Figure 3

The dominating vertices in $C_4 \square G(6;\pm\{1,2\})$

and hence $|N[(x,y)]\cap D|=1$. Thus the set D is a dominating set and $|D|=mn$. For illustration, the dominating vertices for the graph $C_4\,\square\,G(6;\pm\{1,2\})$ are marked in dark black colour, see Figure 3.

Proceeding along the same lines of Theorem 4, we have the following result.

Theorem 5: *Let G be a Cartesian product of cycle C_m, and circulant graph $G(6n;\pm\{1,2\})$ denoted by $C_m\,\square\,G(6n;\pm\{1,2\})$. Then $i(G)=mn$, where $m\equiv 0\,(\mathrm{mod}\,4)$ and $n\geq 1$.*

Now use the same logic to label the graph $C_m\,\square\,G(6n;\pm\{1,2\})$, where $m\equiv 0\,(\mathrm{mod}\,4)$, $n\geq 1$ and For any j, $0\leq j\leq \frac{n}{6}-1$, $1\leq x\leq m$, then the total dominating set $D=\{(x,6j+1)\}$ if $x\equiv 1,2\,(\mathrm{mod}\,4)$, and $D=\{(x,6j+4)\}$ if $x\equiv 0,3\,(\mathrm{mod}\,4)$. For any vertex $(x,y)\in V\setminus D$, $N[(x,y)]=\{(x,y+1),$ $(x,y+2)$, $(x,y+6n-2)$, $(x,y+6n-1)$, $(x+1,y)$, $(x+m-1,y)\}..$ Then proceeding along the same lines of Theorem 4, we have the following result.

Theorem 6: *Let G be a Cartesian product of cycle C_m, and circulant graph $G(6n;\pm\{1,2\})$ denoted by $C_m\,\square\,G(6n;\pm\{1,2\})$. Then $\gamma_t(G)=mn$, where $m\equiv 0\,(\mathrm{mod}\,4)$ and $n\geq 1$.*

Theorem 7: *Let G be a Cartesian product of cycle C_m, and circulant graph $G(6n;\pm\{1,2\})$ denoted by $C_m\,\square\,G(6n;\pm\{1,2\})$. Then $\gamma_{pr}(G)=mn$, where $m\equiv 0\,(\mathrm{mod}\,4)$ and $n\geq 1$.*

Now use the same logic to label the graph $C_m\,\square\,G(8n;\pm\{1,2,3\})$, where $m\equiv 0\,(\mathrm{mod}\,4)$, and $n\geq 1$. For any j, $0\leq j\leq \frac{n}{8}-1$, $1\leq x\leq m$, then the dominating set and independent dominating set $D=\{(x,8j+1)\}$ if x is odd, and $D=\{(x,8j+5)\}$ if x is even. For the total and paired dominating set $D=\{(x,8j+1)\}$ if $x\equiv 1,2\,(\mathrm{mod}\,4)$, and $D=\{(x,8j+5)\}$ if $x\equiv 0,3\,(\mathrm{mod}\,4)$. For any vertex $(x,y)\in V\setminus D$, $N[(x,y)]=\{(x,y+1)$, $(x,y+2)$, $(x,y+3)$, $(x,y+8n-3)$, $(x,y+8n-2)$, $(x,y+8n-1)$, $(x+1,y)$, $(x+m-1,y)\}$. Then proceeding along the same lines of Theorem 4, we have the following result.

Theorem 8: *Let G be a Cartesian product of cycle C_m, and circulant graph $G(8n;\pm\{1,2,3\})$ denoted by $C_m\,\square\,G(8n;\pm\{1,2,3\})$. Then $\gamma(G)=i(G)=\gamma_t(G)=\gamma_{pr}(G)=mn$, where $m\equiv 0\,(\mathrm{mod}\,4)$ and $n\geq 1$.*

Theorem 9: *Let G be a Cartesian product of cycle C_m, and circulant graph $G(2^n;\pm\{1,2\})$ denoted by $C_m\,\square\,G(2^n;\pm\{1,2\})$. Then $\gamma_2(G)=m\,2^{n-2}$, where $m\equiv 0\,(\mathrm{mod}\,4)$ and $n\geq 3$.*

Proof: Since $\Delta = 6$ and by Theorem 3, we have $\gamma_2(G) \geq m\,2^{n-2}$. To prove the reverse inequality, we exhibit the 2-dominating set $D \subseteq V(G)$ with $m\,2^{n-2}$ vertices. The label of vertices of the Cartesian product of $C_m \square G(2^n;\pm\{1,2\})$ is in the form of ordered pairs. We assign the labels of vertices for the i^{th} disc G_i in $C_m \square G(2^n;\pm\{1,2\})$ as $(i,1),(i,2),\ldots,(i,2^n)$ for all i, $1 \leq i \leq m$. For illustration, the labeling of $C_4 \square G(8;\pm\{1,2\})$ is given in Figure 4. Note that, in the labeling (x,y), x and y will be taken modulo m and modulo 2^n respectively. For any j, $0 \leq j \leq 2^{n-2}-1$, let $D = \{(x,x+4j) : 1 \leq x \leq m\}$ be the dominating set of $C_m \square G(2^n;\pm\{1,2\})$.

Since it is vertex transitive, with out loss of generality, choose any vertex with label $(x,y) \in V \setminus D$, then $N[(x,y)] = \{(x,y+1),\ (x,y+2),\ (x,y+2^n-2),\ (x,y+2^n-1),\ (x+1,y),\ (x+m-1,y)\}$. Further,

$$N[(x,y)] \cap D = \begin{cases} \{(x,x+4j),(x+1,x+4j+1)\} & ; \text{ if } y = x+4j+1 \\ \{(x,x+4j),(x,x+4j+4)\} & ; \text{ if } y = x+4j+2 \\ \{(x,x+4j+4),(x+m-1,x+4j+3)\} & ; \text{ if } y = x+4j+3 \end{cases}$$

and hence $|N[(x,y)] \cap D| = 2$. Thus the set D is a 2-dominating set and $|D| = m\,2^{n-2}$. For illustration, the 2-dominating vertices for the graph $C_4 \square G(8;\pm\{1,2\})$ are marked in dark black colour, see Figure 4.

Remark: Since the graph is vertex-transitive, for any j, $0 \leq j \leq 2^{n-2}-1$, consider any other 2-dominating set $D' = \{(x,x+4j+l) : 1 \leq l \leq 2^{n-1}-1;$

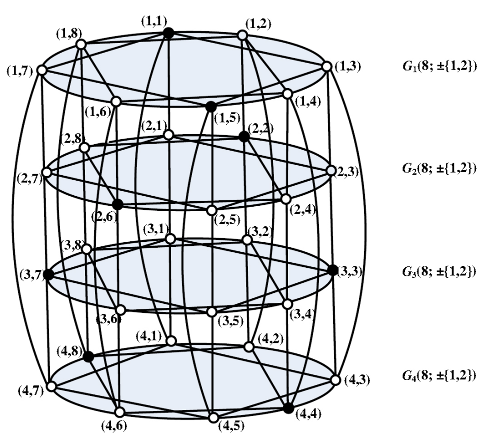

Figure 4

The 2-dominating vertices in $C_4 \square G(8;\pm\{1,2\})$

$1 \le x \le m\}$, rotationally symmetry to the vertex set of D, then D' is also a 2-dominating set with size $m2^{n-2}$.

Now use the same logic to label the graph $C_m \square G(2^n;\pm\{1,3\})$, where $n \ge 3$ and $m \equiv 0,2 \pmod 4$. For any j, $0 \le j \le 2^{n-2}-1$, $1 \le x \le m$, then the 2-dominating set $D = \{(x, 4j+1)\}$ if x is odd, and $D = \{(x, 4j+3)\}$ if x is even. For any vertex $(x, y) \in V \setminus D$, $N[(x, y)] = \{(x, y+1)$, $(x, y+3)$, $(x, y+2^n-3)$, $(x, y+2^n-1)$, $(x+1, y)$, $(x+m-1, y)\}$. Then proceeding along the same lines of Theorem 9, we have the following result.

Theorem 10: *Let G be a Cartesian product of cycle* C_m, *and circulant graph* $G(2^n;\pm\{1,3\})$ *denoted by* $C_m \square G(2^n;\pm\{1,3\})$. *Then* $\gamma_2(G) = m\,2^{n-2}$, *where* $m \equiv 0,2 \pmod 4$ *and* $n \ge 3$.

4. Implementation and Experiments

In this section, we look at some of our 2-domination variant's in real-world applications, as well as the comparison of star and bistar graphs with our suggested network Cartesian product of cycles and circulant graphs. In this case, the 2-dominating set can be seen as totally disconnected vertices or subsets. The disjoint sets are utilised to perform energy-saving routing and organizing. Multiple disjoint sets, each of which has no nodes in common, can guarantee a variety of message carrying paths and node structures in mobile wireless networks.

In a wireless sensor system, if the connection is critical, then the associated disjoint set separates each node into a collection of disjoint sets, each of which entirely covers all of the target locations. The problem of finding the largest collections of sets is commonly formulated using the connected disjoint set problem (CDS). In ad-hoc networks, particularly wireless sensor networks, the CDS problem is critical.

The disconnected set covers (DSC) issue is a classical problem in which the goal is to find the greatest collection of disconnected covers, each of which is a collection of sensors that jointly observe all of the specified locations. It can be expressed as a network and solved using the mixed integer programming combinatorial optimization method. Besides, this type of disconnected set issue could be used for energy-saving node arranging, fault-tolerance topology control, and routing strategy.

The problem of collecting data has become an important application in the field of sensor networks [14], where our results can be applied. The two main components of sensors are: it will either calculate and report data or it can receive and collect data, where only one process can be done at a particular moment. Following deployment, the organization phase decides which sensors in the provided network should provide measuring and collecting functions. Since it is natural for any measurement to be stored in more than two separate devices, the collection of gatherer sensors in the network should build a 2-dominating set.

We briefly specify there are several other types of applications. For instance, in a facility location issue, may necessitate that every place is circulated by its personal facility or have more than two neighbouring regions that do. In this case, facility location could also refer to the placement of security cameras or emergency ambulance aid centers [31].

Another important applications of network theory is fault-tolerance. Informally, it is defined as a system's capacity to provide the promised service even when there are some faults. A widespread misperception concerning real-time finding is that fault-tolerance and real-time needs are mutually exclusive. It is commonly considered that a system's availability and reliability criteria may be meet regardless of its time limitations. This assumption, however, ignores a key feature of real-time computing a system's correctness is determined not just by the accuracy of its findings, but also by its ability to meet tight timing criteria.

In other words, a real-time network may fail to work properly due to hardware or software problems, or Â it fails to respond in time to meet the timing requirements because ofÂ its "environment". As a result, a real-time system is one that should provide the promised service in a timely manner even when there are errors. When a system specification calls for a specific service to be provided in a given amount of time, the system's failure to meet the time constraint can be considered a failure.

However, characterizing a missed deadline as a timing issue using traditional faultâ€"tolerant system design methodologies does not adequately fulfil the needs of realâ€"time applications. The main distinction is that real-time network should be predictable even when mistakes occur. As a result, when developing such systems, fault tolerance and real-time needs should be taken into account simultaneously. The difficulty is to incorporate timing and fault-tolerance criteria into the system's specification at each stage of abstraction, as well as to use a design technique that considers system predictability even throughout fault detection, separation, system reconfiguration, and revival. The formal specification of dependability criteria, as well as their impact on achieving timing limitations, is an area that needs to be explored further.

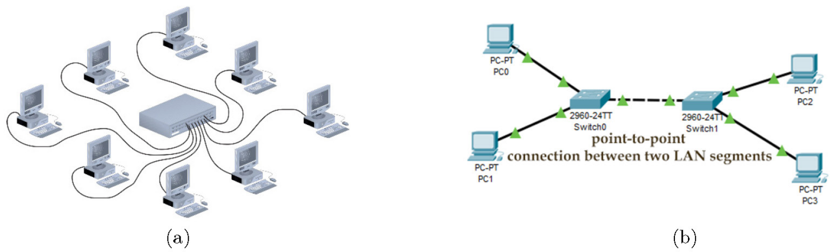

Figure 5

(a) Star topology: (b) Bistar topology

Star topology: A Local Area Network (LAN) with a star topology has each node associated to a main connecting point, such as a hub or switch. A star needs extra wire than a linear bus topology, but it has the benefit of only affecting one node if one wire breaks. A hub, which is a main device, it associated to each device in the system. If a device relocates data to any other device, it should first send the data to the hub, the hub will send the data to the desired device. The star and bistar topologies are shown in Figure 5.

Features of Star Topology: The star graph is extremely reliable, if one cable or device breaks, the remaining will continue to work, and it is also exceptionally high-performing since no data collisions may occur. Because every device needs only one input or output port and wishes to be associated to the hub through only one way, this topology is less expensive,Â because the link is continuously identified.Â This network is easy to discover faults. When connecting or disconnecting devices, there are no network interruptions. Each device required only one port to associate to the hub. If k devices are associated with the star, the collection of cables needed to joint them is also k.

Disadvantages of Star Topology: This sort of network uses more wire than a linear bus, and if the associated system device Â fails, the nodes linked will be disabled and unable to interconnect with the remaining network. Because of the worth of the associated devices, this network topology is costlier than a linear bus topology. When the hub device fails, all other devices fail as well,Â none of the devices can work without the hub, and it needs greater resources and regular care because it is the star topology’s central system. In terms of cost and maintenance, this is more costly.

In [32], Chellali studied the concept that each 2-dominating set of a network G holds every leaf. Let $\gamma_2(G) = n-1$ iff G is star. For the bistar graph the 2-domination value is $n-2$, and for the caterpillar with dimension n, the 2-domination value is $n-k$. When comparing the above-mentioned networks with our suggested network, our network will produce the smallest 2-dominant set. It is understood that it will require the least amount of cost and maintenance in comparison to the star and bistar networks.

5. Concluding Remarks

In this paper, we have identified certain classes of graphs, such as $C_m \square G(6n;\pm\{1,2\})$ and $C_m \square G(8n;\pm\{1,2,3\})$, where $m \equiv 0 \pmod 4$, $n \geq 1$ in which domination, independent domination, total domination and paired domination numbers are same, which is quite precise lower bound derived from [1] and [2]. Further, we have obtained 2-domination number for $C_m \square G(6n;\pm\{1,2\})$, with $m \equiv 0 \pmod 4$, and $C_m \square G(6n;\pm\{1,3\})$, with $m \equiv 0,2 \pmod 4$, $n \geq 3$, which is quite precise lower bound derived in [3]. Finding other domination parameters for certain classes of Cartesian product of graphs are under investigation.

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