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A characterization of some types of Cayley graphs and addition Cayley graphs and their total chromatic numbers

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Abstract

Let $\mathbb{Z}_n$ be the set of integer modulo n. The Cayley graph on $\mathbb{Z}_n$ is an undirected graph whose vertex set is $\mathbb{Z}_n$ and two vertices a, b are adjacent if and only if $a-b \in S \subseteq \mathbb{Z}_n \setminus \{0\}$. The addition Cayley graph on $\mathbb{Z}_n$ is a graph whose vertex set is $\mathbb{Z}_n$ and two vertices a, b are adjacent if and only if $a+b \in A \subseteq \mathbb{Z}_n$. In this paper , we characterize Cayley graphs and addition Cayley graphs of even orders. Their basic properties of them are investigated. We also give exact values for the total chromatic numbers of Cayley graphs and addition Cayley graphs where their orders are even integers. Moreover, we provide examples to illustrate these results.

Subject Classification: *05C15, 05C25.*

Keywords: *Cayley graph, Addition Cayley graph, Total coloring, Total chromatic number.*

1. Introduction

Throughout this paper, we consider only finite, simple, undirected graphs. For standard terminology and notation in graph theory we refer to Bondy [4] and for algebraic graph theory we follow the manuscript of Biggs [3]. A *path* of order n, denoted by $P_n = (v_1, v_2, \dots, v_n)$, is a finite sequence of adjacent vertices in which all vertices are distinct. A *cycle*

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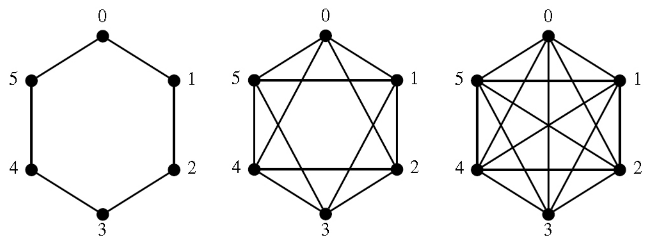

Figure 1

The $Cay(\mathbb{Z}_6, S_1)$, $Cay(\mathbb{Z}_6, S_2)$ and $Cay(\mathbb{Z}_6, S_3)$.

is a path with the same first and last vertices only. The cycle of order n can be written as $C_n = (v_1, v_2, \ldots, v_n, v_1)$. A *complete graph* of order n, denoted by K_n, is a graph whose all vertices are adjacent. The *union* of two vertex disjoint graphs G and H, denoted by $G \cup H$, has the vertex set $V(G \cup H) = V(G) \cup V(H)$ and the edge set $E(G \cup H) = E(G) \cup E(H)$. This operation is sometimes also known as the graph disjoint union.

For a positive integer n greater than 1. Let S be a subset of $\mathbb{Z}_n \setminus \{0\}$. Assume that $S^{-1} = \{s^{-1} \mid s \in S\} = S$. The *Cayley graph* $X = Cay(\mathbb{Z}_n, S)$ is the undirected graph having vertex set $V(X) = \mathbb{Z}_n$ and edge set $E(X) = \{(a, b) \mid a - b \in S\}$ where $a, b \in \mathbb{Z}_n$. The Cayley graph is a regular graph of degree $|S|$. Its connected components are the right cosets of the subgroup generated by S. Therefore, if S generates $\mathbb{Z}_n$, then the Cayley graph is a connected graph. Biggs [3] and Godsil & Royle [10] provide many information regarding Cayley graphs. For example, we consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. All possible sets of S are $S_1 = \{1, 5\}$, $S_2 = \{1, 2, 4, 5\}$ and $S_3 = \{1, 2, 3, 4, 5\}$. Some examples of Cayley graphs are displayed in Figure 1. Throughout this paper, we consider $n \geq 2$.

For a positive integer n greater than 1. Let A be a subset of $\mathbb{Z}_n$. The *addition Cayley graph*, denoted by $G = Cay^+(\mathbb{Z}_n, A)$, is the graph having vertex set $V(G) = \mathbb{Z}_n$ and edge set $E(G) = \{(a, b) \mid a + b \in A\}$ where $a, b \in \mathbb{Z}_n$. Several properties of addition Cayley graphs have been discussed in literature (see [8, 11, 12]). For example, we let $A_1 = \{1, 5\}$, $A_2 = \{1, 2, 4, 5\}$ and $A_3 = \{1, 2, 3, 4, 5\}$ where A_1, A_2 and A_3 be a subset of $\mathbb{Z}_6$. The addition Cayley graphs corresponding to the generating sets A_1, A_2 and A_3 are shown in Figure 2.

A *proper vertex coloring* of a graph G is an assignment of colors to the vertices such that no two adjacent vertices receive the same color. The *chromatic number* of the graph G, denoted by $\chi(G)$, is the minimum number of colors required for a proper vertex coloring. A *proper edge*

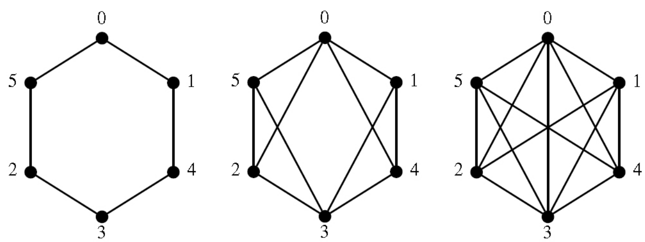

Figure 2

The $Cay^{+}(\mathbb{Z}_6, A_1)$, $Cay^{+}(\mathbb{Z}_6, A_2)$ and $Cay^{+}(\mathbb{Z}_6, A_3)$.

coloring of a graph G is an assignment of colors to the edges such that no two adjacent edges receive the same color. The *chromatic index*, denoted by $\chi'(G)$, is the minimum number of colors required for a proper edge coloring. König [14] showed that the chromatic index of any bipartite graph are equal to maximum degree. Brooks [6] claimed that $\chi(G) \leq \Delta + 1$, where Δ is maximum degree. Vizing [23] proved that $\chi'(G) = \Delta + 1$. The chromatic number of Sunlet and Bistar families of graphs are found by Rani and Parvathi [18]. A number of writers have generalized the concepts of chromatic number and chromatic index of a graph and further history of graph coloring can be found in [24].

A *proper total coloring* [13] of a graph G is a mapping $f : V(G) \cup E(G) \rightarrow C$ where C is a set of colors satisfying the following three conditions (1)-(3):

(1) $f(v_1) \neq f(v_2)$ for any two adjacent vertices $v_1, v_2 \in V(G)$.

(2) $f(e_1) \neq f(e_2)$ for any two adjacent edges $e_1, e_2 \in E(G)$.

(3) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and edge $e \in E(G)$ incident to v.

A *total independent set* [16] is a set of only vertices, only edges or both in G. Each element in the total independent set can use the same colors. A *k-total coloring* of a graph G is a total coloring of G from a set of k colors. A graph G is *k-total colorable* if there is a k-total coloring of G. The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum positive integer k for which G is k-total colorable. Let $\Delta(G)$ be the maximum degree of G. If f is a total coloring of G and degree of a vertex v is $\Delta(G)$, then f must assign distinct colors to the $\Delta(G)$ edges incident with v as well as to v itself. This implies that, for every graph G, $\chi''(G) \geq \Delta(G) + 1$. However, Behzad [2] and Vizing independently conjectured that the total chromatic number cannot exceed this lower bound by more than 1. This conjecture has become known

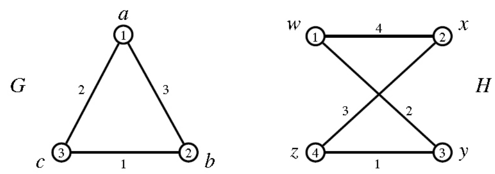

Figure 3

Some examples for total coloring of graphs.

as the *Total Coloring Conjecture* (TCC): For every graph G, $\chi''(G) \leq \Delta(G)+2$. This conjecture has been proved for several classes of graphs, for details see [7, 21]. Vijayaditya [22] proved the TCC for $\Delta(G)=3$. Borodin [5] verified the TCC for planar graphs. Kostochka [15] proved the TCC for $\Delta(G)=4$. Rosenfeld [19] solved the TCC for a complete 3-partite graph. The total dominator chromatic number of certain graphs is investigated by Ghanbari and Alikhani in [9]. Now if the TCC holds for a certain class of graphs, then it is said that G is *Type I* if $\chi''(G)=\Delta(G)+1$ and is *Type II* if $\chi''(G)=\Delta(G)+2$. These definitions are analogous to the definitions of Class I and Class II graphs in edge colorings of graphs. Some examples for total coloring of graphs are shown in Figure 3. The coloring of a graph G is 3-total coloring. Because the order of G is 3 and $\Delta(G)=2$, then we get that G has $3 \leq \chi''(G) \leq 4$. The total independent sets of G are $T_1(G)=\{a,(b,c)\}$, $T_2(G)=\{b,(a,c)\}$ and $T_3(G)=\{c,(a,b)\}$. Since G is 3-total colorable, then $\chi''(G) \leq 3$. Therefore, $\chi''(G)=3$. Next, H is 4-total colorable and there exists a partitioning of $V(H) \cup E(H)$ into 4 independent sets, as follow: $T_1(H)=\{w,(y,z)\}$, $T_2(H)=\{x,(w,y)\}$, $T_3(H)=\{y,(x,z)\}$, $T_4(H)=\{z,(w,x)\}$. Hence, $\chi''(H)=4$. Moreover, we can conclude that G is Type I and H is Type II.

In this paper, we focus on characterization of some types of Cayley graphs and addition Cayley graphs for even orders. Moreover, we also show total chromatic numbers of them and give examples to illustrate these results.

2. Preliminaries

We now give some results which are essentials and important in this study. First, the total chromatic numbers of complete graph and cycle are shown in the following theorems.

Theorem 2.1: [1] *For any complete graph* K_n, $\chi''(K_n) = \begin{cases} n & \text{if } n \text{ is odd}, \\ n+1 & \text{if } n \text{ is even}. \end{cases}$

Theorem 2.2: [25] *For any cycle* C_n, $\chi''(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \,(mod\, 3), \\ 4 & \text{otherwise}. \end{cases}$

Some properties of the addition Cayley graph used in this work are as follows. Let $\deg(v)$ be the degree of a vertex v. Degree of vertices in addition Cayley graphs are shown in the next theorem.

Theorem 2.3: *For any* $A \subseteq \mathbb{Z}_n$. *Let x be any vertex of the addition Cayley graph* $G = Cay^+(\mathbb{Z}_n, A)$. *Then,*

$$deg(x) = \begin{cases} |A|-1 & \text{if } 2x \in A, \\ |A| & \text{otherwise.} \end{cases}$$

From the previous theorem, the total number of edges in addition Cayley graphs can be computed.

Theorem 2.4: [17] *For any* $A \subseteq \mathbb{Z}_n$. *The total number of edges in the addition Cayley graph* $G = Cay^+(\mathbb{Z}_n, A)$ *is*

$$|E(G)| = \begin{cases} \frac{|A|(n-1)}{2} & \text{if } n \text{ is odd}, \\ \frac{n|A|}{2} - s & \text{if } n \text{ is even}, \end{cases}$$

where s is the number of even elements in A.

In what follows, let $\mathbb{U}_n$ denote the well known set of all units in $\mathbb{Z}_n$. The relation between Cayley graphs and addition Cayley graphs is established as follows.

Theorem 2.5: [20] *The Cayley graph* $X = Cay(\mathbb{Z}_n, \mathbb{U}_n)$ *is isomorphic to the addition Cayley graph* $G = Cay^+(\mathbb{Z}_n, \mathbb{U}_n)$ *if and only if n is even.*

In order to illustrate the previous result, let us consider the addition Cayley graph $Cay^+(\mathbb{Z}_6, A_2)$ in Figure 2. We have that $n = 6$ and $|A_2| = 4$. According to Theorem 2.3, we obtain $\deg(0) = \deg(3) = 4$ and $\deg(1) = \deg(2) = \deg(4) = \deg(5) = 3$. Since 2, 4 are even elements in A_2, then $s = 2$. Thus, the total number of edges in $Cay^+(\mathbb{Z}_6, A_2)$ is 10, by Theorem 2.4. The set of all units in $\mathbb{Z}_6$ is $\mathbb{U}_6 = \{1,5\}$. Due to Theorem 2.5, we get that $Cay(\mathbb{Z}_6, \mathbb{U}_6)$ is isomorphic to $Cay^+(\mathbb{Z}_6, \mathbb{U}_6)$ (see Figure 1, 2).

3. Characterization and total chromatic numbers of Cayley graphs

In this section, we present the characterization of Cayley graphs for even orders and their total chromatic numbers. We begin with single element of generating set S. By the definition of generating set S of $Cay(\mathbb{Z}_n, S)$ that if $s \in S$, then $-s \in S$. If $|S| = 1$, we have $s \equiv -s \pmod{n}$. Hence $s = \frac{n}{2}$ and $S = \left\{\frac{n}{2}\right\}$.

Lemma 3.1: *The Cayley graph* $X = Cay(\mathbb{Z}_n, \{\frac{n}{2}\})$, *$n$ is even, is* $\frac{n}{2}$ *copies of* P_2.

Proof: Suppose n is even. Let $X = Cay(\mathbb{Z}_n, \{\frac{n}{2}\})$. Since the cardinality of generating set is 1, then we get that all vertices of X have degree 1. This implies that X induces by path of length 1 with $\frac{n}{2}$ components. This complete the proof. □

Theorem 3.2: *Let n be an even integer. For any the Cayley graph* $X = Cay(\mathbb{Z}_n, \{\frac{n}{2}\})$. *Then total chromatic number of the Cayley graph X is* $\chi''(X) = 3$.

Proof: Suppose n is even. Let $X = Cay(\mathbb{Z}_n, \{\frac{n}{2}\})$. Due to Lemma 3.1, X is $\frac{n}{2}$ copies of P_2. Then $\chi''(X) = 3$. □

In Figure 4, the Cayley graph $Cay(\mathbb{Z}_6, \{3\})$ is 3 copies of P_2 and $Cay(\mathbb{Z}_8, \{4\})$ is 4 copies of P_2. So $\chi''(Cay(\mathbb{Z}_6, \{3\})) = 3 = \chi''(Cay(\mathbb{Z}_8, \{4\}))$.

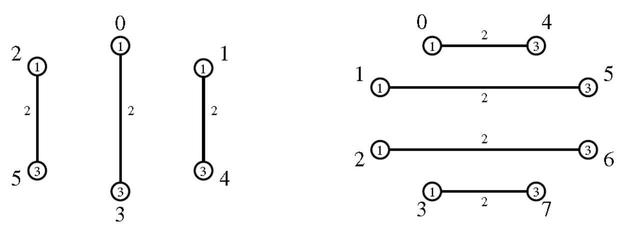

Figure 4

The total colorings of $Cay(\mathbb{Z}_6, \{3\})$ and $Cay(\mathbb{Z}_8, \{4\})$.

The following lemma used to explain a character of the Cayley graph corresponding to 2 elements generating set.

Lemma 3.3: *Let n be an even integer. If* $d = gcd(s, n)$, *then* $\frac{n}{d}s + i \not\equiv \left(\frac{n}{d} - 1\right)s + i \pmod{n}$, *for all* $1 \le i \le d - 1$ *and* $1 \le s \le \frac{n}{2} - 1$.

Proof: Assume $d = \gcd(s, n)$ and $\frac{n}{d}s + i \equiv \left(\frac{n}{d} - 1\right)s + i \pmod{n}$, for all $1 \le i \le d-1$ and $1 \le s \le \frac{n}{2} - 1$. Then we obtain $s \equiv 0 \pmod{n}$. That implies $n \mid s$. So, $s = n$. It is a contradiction to our assumption. □

Next, we present a characterization of the Cayley graph and their total chromatic numbers with 2 elements generating set.

Lemma 3.4: *The Cayley graph* $X = Cay(\mathbb{Z}_n, \{s, n-s\})$, *n is an even integer and* $1 \le s \le \frac{n}{2} - 1$, *is d copies of* $C_{\frac{n}{d}}$ *where* $d = gcd(s, n)$.

Proof: Suppose n is an even. Let $X = Cay(\mathbb{Z}_n, \{s, n-s\})$ where $1 \le s \le \frac{n}{2} - 1$, and $d = \gcd(s, n)$. Since the cardinality of generating set is 2, then X is a regular graph of degree 2. Then there exists d components of $C_{\frac{n}{d}}$, as follow:

$$\begin{aligned}
(1)\, C_{\frac{n}{d}} &= (s, 2s, 3s, \ldots, (\tfrac{n}{d} - 1)s, \tfrac{n}{d}s, s) \\
(2)\, C_{\frac{n}{d}} &= (s+1, 2s+1, 3s+1, \ldots, (\tfrac{n}{d} - 1)s+1, \tfrac{n}{d}s+1, s+1) \\
(3)\, C_{\frac{n}{d}} &= (s+2, 2s+2, 3s+2, \ldots, (\tfrac{n}{d} - 1)s+2, \tfrac{n}{d}s+2, s+2) \\
&\vdots \\
(d)\, C_{\frac{n}{d}} &= (s+d-1, 2s+d-1, \ldots, (\tfrac{n}{d} - 1)s+d-1, \tfrac{n}{d}s+d-1, s+d-1).
\end{aligned}$$

According to Lemma 3.3, the vertices in X are distinct and also adjacent for $\{s, n-s\}$. Hence $X = d\,C_{\frac{n}{d}}$. □

Theorem 3.5: *Let n be an even integer,* $1 \le s \le \frac{n}{2} - 1$ *and* $d = gcd(s, n)$. *Then the total chromatic number of the Cayley graph X is*

$$\chi''(X) = \begin{cases} 3 & \text{if } \frac{n}{d} \equiv 0 \ (mod\ 3), \\ 4 & \text{otherwise}. \end{cases}$$

Proof: Suppose n is even. Let $X = Cay(\mathbb{Z}_n, \{s, n-s\})$ where $1 \le s \le \frac{n}{2} - 1$. Due to Theorem 2.2 and Lemma 3.4, $\chi''(X) = 3$ if $n \equiv 0 \pmod{3}$. Otherwise, $\chi''(X) = 4$. □

Consider the Cayley graphs in Figure 5. Since $\gcd(1,10) = 1$, then $Cay(\mathbb{Z}_{10}, \{1, 9\})$ is C_{10}. Next, we obtain that $Cay(\mathbb{Z}_{10}, \{2, 8\}) = 2\,C_{10}$, because $\gcd(2,10) = 2$. Hence, $\chi''(Cay(\mathbb{Z}_{10}, \{2, 8\})) = 4$.

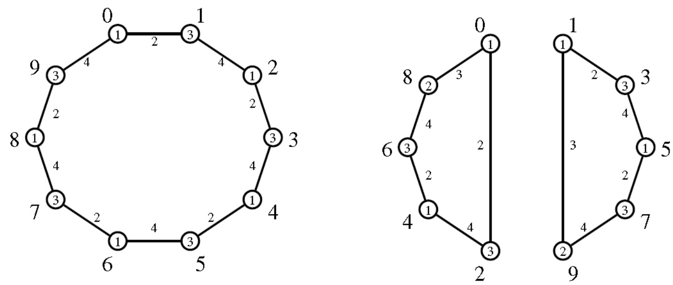

Figure 5

The total colorings of $Cay(\mathbb{Z}_{10},\{1,9\})$ and $Cay(\mathbb{Z}_{10},\{2,8\})$.

For $n=12$, we have $\gcd(4,12)=4$. So $Cay(\mathbb{Z}_{12},\{4,8\})=4C_3$, as Figure 6. Hence $\chi''(Cay(\mathbb{Z}_{12},\{4,8\}))=3$, by Theorem 3.5.

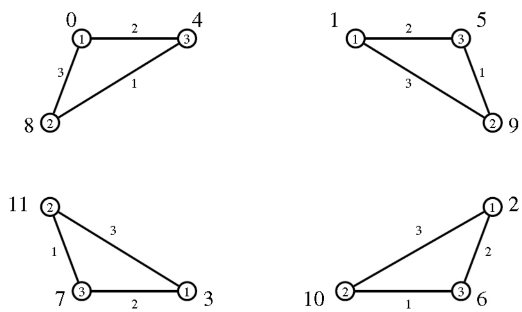

Figure 6

The total coloring of $Cay(\mathbb{Z}_{12},\{4,8\})$.

Next, we explain the character of Cayley graphs with generating set $\mathbb{Z}_n\setminus\{0\}$.

Lemma 3.6: *The Cayley graph $X=Cay(\mathbb{Z}_n,\mathbb{Z}_n\setminus\{0\})$, n is even, is K_n.*

Proof: Suppose that n is an even. Let $X=Cay(\mathbb{Z}_n,\mathbb{Z}_n\setminus\{0\})$. We have $|\mathbb{Z}_n|=n$, so $|\mathbb{Z}_n\setminus\{0\}|=n-1$. This implies that X is $(n-1)$-regular graph. Therefore, X is a complete graph of order n. □

Theorem 3.7: *Let n be an even integer. For any the Cayley graph $X=Cay(\mathbb{Z}_n,\mathbb{Z}_n\setminus\{0\})$. Then total chromatic number of the Cayley graph X is $\chi''(X)=n+1$.*

Proof: Suppose n is even. Let $X = Cay(\mathbb{Z}_n, \mathbb{Z}_n \setminus \{0\})$. Then $\chi''(X) = n+1$, by Theorem 2.1 and Lemma 3.6. □

For example, the Cayley graph $X = Cay(\mathbb{Z}_6, \{1,2,3,4,5\})$ in Figure 7 is K_6. Hence $\chi''(X) = 7$, by Theorem 3.7.

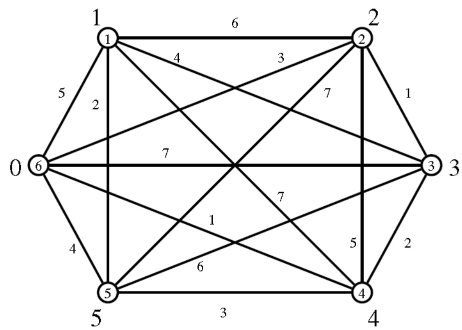

Figure 7

The total coloring of $\boldsymbol{Cay(\mathbb{Z}_6, \mathbb{Z}_6 \setminus \{0\})}$. .

Furthermore, we consider the Cayley graphs on the group $\mathbb{Z}_n$ corresponding to generating set S where n is an even integer. For any generating sets $\{\frac{n}{2}\}$ and $\mathbb{Z}_n \setminus \{0\}$, $Cay(\mathbb{Z}_n, \{\frac{n}{2}\})$ and $Cay(\mathbb{Z}_n, \mathbb{Z}_n \setminus \{0\})$ are Type II graphs. Next, for any generating set $\{s, n-s\}$, $1 \le s \le \frac{n}{2} - 1$. If $\frac{n}{d} \equiv 0 \pmod 3$ where $d = \gcd(s,n)$, then $Cay(\mathbb{Z}_n, \{s, n-s\})$ is Type I graph. Otherwise, $Cay(\mathbb{Z}_n, \{s, n-s\})$ is Type II graph.

4. Characterization and total chromatic numbers of addition Cayley graphs

In this section, we provide the characterization of addition Cayley graphs for even orders and their total chromatic numbers. We now give some results in case that generating set A is singleton set.

Lemma 4.1: *Let n be an even integer and a be an element in $\mathbb{Z}_n$. The addition Cayley graph $G = Cay^+(\mathbb{Z}_n, \{a\})$ is*

$$\begin{cases} \left(\frac{n}{2} - 1\right) P_2 \cup \left\{\frac{a}{2}, \frac{a+n}{2}\right\} & \text{if } a \text{ is even}, \\ \left(\frac{n}{2}\right) P_2 & \text{if } a \text{ is odd}. \end{cases}$$

Proof: Let $G = Cay^+(\mathbb{Z}_n, \{a\})$ where n is an even integer and $a \in \mathbb{Z}_n$. Suppose a is odd. We consider $2x \equiv a \pmod{n}$ for all $x \in \mathbb{Z}_n$. Since $\gcd(2,n) = 2$ and $2 \nmid a$, then that congruence has no solution. This implies that all vertices in G have degree one. Then we obtain $\frac{n}{2}$ paths of length one in G. Hence, $Cay^+(\mathbb{Z}_n, \{a\})$ is $\left(\frac{n}{2}\right)P_2$. Next case, suppose a is even. Since $2\left(\frac{a}{2}\right) \in \{a\}$ and $2\left(\frac{a+n}{2}\right) \in \{a\}$, we have that both vertices have degree 0. The other vertices have degree 1. Hence, $Cay^+(\mathbb{Z}_n, \{a\})$ is $\left(\frac{n}{2} - 1\right)P_2 \cup \left\{\frac{a}{2}, \frac{a+n}{2}\right\}$. □

Let $n = 4$, we have $\mathbb{Z}_4 = \{0,1,2,3\}$. By Lemma 4.1, the addition Cayley graphs $Cay^+(\mathbb{Z}_4, \{a\})$, $a \in \mathbb{Z}_4$, are displayed in Figure 8.

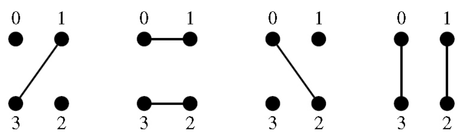

Figure 8

The $Cay^+(\mathbb{Z}_4, \{0\}), Cay^+(\mathbb{Z}_4, \{1\}), Cay^+(\mathbb{Z}_4, \{2\})$ and $Cay^+(\mathbb{Z}_4, \{3\})$.

Theorem 4.2: *Let n be an even integer and $a \in \mathbb{Z}_n$. For any the addition Cayley graph $G = Cay^+(\mathbb{Z}_n, \{a\})$. Then total chromatic number of the addition Cayley graph G is $\chi''(G) = 3$.*

Proof: Suppose n is even. Let $G = Cay^+(\mathbb{Z}_n, \{a\})$ where $a \in \mathbb{Z}_n$. From Lemma 4.1, if a is even then G is 2 single vertices and $\left(\frac{n}{2} - 1\right)$ copies of P_2, and if a is odd then G is $\left(\frac{n}{2}\right)$ copies of P_2. Then $\chi''(G) = 3$. □

For example, we consider the total colorings of $Cay^+(\mathbb{Z}_6, \{2\})$ and $Cay^+(\mathbb{Z}_6, \{3\})$ as Figure 9. The graph $Cay^+(\mathbb{Z}_6, \{2\})$ is 2 paths of length one union the vertices $\frac{2}{2} = 1$ and $\frac{2+6}{2} = 4$. The graph $Cay^+(\mathbb{Z}_6, \{3\})$ is 3 paths of length one. From Theorem 4.2, the total chromatic numbers of their graphs are 3.

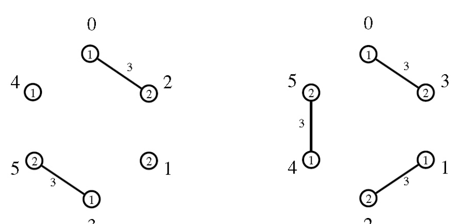

Figure 9

The total colorings of $Cay^+(\mathbb{Z}_6, \{2\})$ and $Cay^+(\mathbb{Z}_6, \{3\})$.

Next, we determine character of the addition Cayley graphs of even orders with generating set $A = \{a_1, a_2\}$. The following Lemma 4.3-4.6 will be used for proving the next theorem.

Lemma 4.3: *Let n be an even integer. For any distinct elements a, b in* $\mathbb{Z}_n$, *there exist integer k such that* $ka \equiv kb\,(mod\ n)$.

Proof: Let $a, b \in \mathbb{Z}_n$ be such that $a \not\equiv b$ (mod n). So $a - b \not\equiv 0$ (mod n) and $\gcd(a-b, n)$ exists. Since $\gcd(a-b, n) \mid 0$, we obtain that equation $(a-b)x \equiv 0$ (mod n) has roots, call k. Hence $ka \equiv kb$ (mod n). □

Lemma 4.4: *Let n be an even integer and* $j = 1, 2, \ldots, \frac{n}{2}$. *If a, b are odd elements in* $\mathbb{Z}_n$, *then* $ja - (j-1)b \not\equiv jb - ja\,(mod\ n)$.

Proof: Let $a, b \in \mathbb{Z}_n$ be odd integers. Assume that $ja - (j-1)b \equiv jb - ja$ (mod n). Then we get $b \equiv 2j(b-a)$ (mod n) for $j = 1, 2, \ldots, \frac{n}{2}$. Since a and b are odd, we have $b - a$ is even and also b is even. It contradicts to the assumption that b is odd. Hence the lemma proved. □

Lemma 4.5: *Let n be an even integer, a, b be even elements in* $\mathbb{Z}_n$ *and* $v \in \mathbb{Z}_n$, *such that* $2v \equiv a\,(mod\ n)$. *For each integer* $k \geq 1$ *such that* $ka \equiv kb\,(mod\ n)$. *Then*

$$\left(\frac{k}{2} - 1\right)a - \left[\left(\frac{k}{2} - 1\right)b - v\right] \not\equiv \left(\frac{k}{2}b - v\right) - \left(\frac{k}{2} - 1\right)a\,(mod\ n) \text{ if } k \text{ is even,}$$

$$\left[\left(\frac{k-1}{2}\right)b - v\right] - \left(\frac{k-3}{2}\right)a \not\equiv \left(\frac{k-1}{2}\right)a - \left[\left(\frac{k-1}{2}\right)b - v\right](mod\ n) \text{ if } k \text{ is odd.}$$

Proof: Let $k \geq 1$ is an integer such that $ka \equiv kb$ (mod n). There are two cases to be considered,

Case 1: Suppose that k is an even and

$$\left(\frac{k}{2} - 1\right)a - \left[\left(\frac{k}{2} - 1\right)b - v\right] \equiv \left(\frac{k}{2}b - v\right) - \left(\frac{k}{2} - 1\right)a\,(mod\ n).$$

Then we get $(k-1)b \equiv (k-2)a + 2v$ (mod n). Since $2v \equiv a$ (mod n) and $ka \equiv kb$ (mod n), so $a \equiv b$ (mod n). It's a contradiction.

Case 2: Suppose that k is an odd and

$$\left[\left(\frac{k-1}{2}\right)b - v\right] - \left(\frac{k-3}{2}\right)a \equiv \left(\frac{k-1}{2}\right)a - \left[\left(\frac{k-1}{2}\right)b - v\right](mod\ n).$$

We obtain that $(k-1)b \equiv (k-2)a + 2v \pmod{n}$. Similarly to case 1, we have that $a \equiv b \pmod{n}$. It's a contradiction. □

Lemma 4.6: *Let n be an even integer, a, b be even elements in $\mathbb{Z}_n$ and $u, v \in \mathbb{Z}_n$, such that $2u \equiv a \,(mod\, n)$ and $2v \equiv b \,(mod\, n)$. For each integer $k \geq 1$ such that $ka \equiv kb \,(mod\, n)$. Then*

$$u \equiv \left(\frac{k}{2}b - u\right) - \left(\frac{k}{2} - 1\right)a \,(mod\, n) \text{ if } k \text{ is even,}$$
$$v \equiv \left(\frac{k-1}{2}\right)a - \left[\left(\frac{k-1}{2}\right)b - u\right] (mod\, n) \text{ if } k \text{ is odd.}$$

Proof: Let $a, b \in \mathbb{Z}_n$ be even integers, $2u \equiv a \pmod{n}$ and $2v \equiv b \pmod{n}$ where $u, v \in \mathbb{Z}_n$. We assume $k \geq 1$ is an integer such that $ka \equiv kb \pmod{n}$. There are two cases to be considered,

Case 1: Suppose that k is an even and $u \not\equiv \left(\frac{k}{2}b - u\right) - \left(\frac{k}{2} - 1\right)a \pmod{n}$. We get that $2u \not\equiv \frac{1}{2}(ka - kb) + a \pmod{n}$. Since $ka \equiv kb \pmod{n}$, then $2u \not\equiv a \pmod{n}$. Hence, it's a contradiction.

Case 2: Suppose that k is an odd and $v \not\equiv \left(\frac{k-1}{2}\right)a - \left[\left(\frac{k-1}{2}\right)b - u\right] \pmod{n}$. We obtain that $v \not\equiv \frac{1}{2}(ka - kb) - \frac{1}{2}(a - b) + u \pmod{n}$. Since $a - b \equiv 2(u - v) \pmod{n}$ and $ka - kb \equiv 0 \pmod{n}$, then $v \not\equiv v \pmod{n}$. It's a contradiction. □

The following theorem describes the character of the addition Cayley graphs $G = Cay^{+}(\mathbb{Z}_n, \{a_1, a_2\})$ where n is an even integer.

Lemma 4.7: *For any the addition Cayley graph $G = Cay^{+}(\mathbb{Z}_n, \{a_1, a_2\})$ where n is even and a_1, a_2 are distinct elements in $\mathbb{Z}_n$. Let m be a smallest factor of n such that $ma_1 \equiv ma_2 \,(mod\, n)$. Then the addition Cayley graph G is*

$$\begin{cases} \left(\frac{n}{2m}\right)C_{2m} & \text{If } a_1, a_2 \text{ are both odd,} \\ P_m \cup \left(\frac{n-m}{2m}\right)C_{2m} & \text{If } a_1, a_2 \text{ not the same parity,} \\ 2\,P_m \cup \left(\frac{n}{2m} - 1\right)C_{2m} & \text{If } a_1, a_2 \text{ are both even.} \end{cases}$$

Proof: Let n be an even number, $a_1 < a_2$ be elements in $\mathbb{Z}_n$ and m be a smallest factor of n such that $ma_1 \equiv ma_2 \pmod{n}$. There are three cases to be considered,

Case 1: Suppose a_1, a_2 are both odd. Then we get that G is a cycle of order $2m$ with $\frac{n}{2m}$ components, as follow:

$$
\begin{aligned}
(1)\ C_{2m} &= (a_1, a_2 - a_1, 2a_1 - a_2, 2a_2 - 2a_1, \ldots, 2ma_1 - (2m-1)a_2, 2ma_2 - 2ma_1, a_1) \\
(2)\ C_{2m} &= (a_1 + 2, a_2 - (a_1 + 2), (2a_1 + 2) - a_2, 2a_2 - (2a_1 + 2), \ldots, \\
&\quad (2ma_1 + 2) - (2m-1)a_2, 2ma_2 - (2ma_1 + 2), a_1 + 2) \\
&\ \vdots \\
\left(\frac{n}{2m}\right) C_{2m} &= (a_1 + 2(i-1), a_2 - (a_1 + 2(i-1)), (2a_1 + 2(i-1)) - a_2, \\
&\quad 2a_2 - (2a_1 + 2(i-1)), \ldots, (2ma_1 + 2(i-1)) - (2m-1)a_2, \\
&\quad 2ma_2 - (2ma_1 + 2(i-1)), a_1 + 2(i-1))
\end{aligned}
$$

for all $i = 1, 2, \ldots, \frac{n}{2m}$. According to Lemma 4.4, the vertices in G are distinct and also adjacent for $\{a_1, a_2\}$. Hence $G = \left(\frac{n}{2m}\right) C_{2m}$.

Case 2: Suppose a_1 is even and a_2 is odd. Let $u, v \in \mathbb{Z}_n$ such that $2u \equiv 2v \equiv a_1$ (mod n) where $u < v$. Then we get that u and v are end-vertices of path of order m, as follow: if m is even, then either

$$
\begin{aligned}
P_m &= (u, a_2 - u, a_1 - (a_2 - u), (2a_2 - u) - a_1, 2a_1 - (2a_2 - u), \ldots, \\
&\quad \left(\frac{m}{2} - 1\right) a_1 - \left[\left(\frac{m}{2} - 1\right) a_2 - u\right], \left(\frac{m}{2} a_2 - u\right) - \left(\frac{m}{2} - 1\right) a_1) \text{ or} \\
P_m &= (v, a_2 - v, a_1 - (a_2 - v), (2a_2 - v) - a_1, 2a_1 - (2a_2 - v), \ldots, \\
&\quad \left(\frac{m}{2} - 1\right) a_1 - \left[\left(\frac{m}{2} - 1\right) a_2 - v\right], \left(\frac{m}{2} a_2 - v\right) - \left(\frac{m}{2} - 1\right) a_1).
\end{aligned}
$$

If m is odd, then either

$$
\begin{aligned}
P_m &= (u, a_2 - u, a_1 - (a_2 - u), (2a_2 - u) - a_1, 2a_1 - (2a_2 - u), \ldots, \\
&\quad \left[\left(\frac{m-1}{2}\right) a_2 - u\right] - \left(\frac{m-3}{2}\right) a_1, \left(\frac{m-1}{2}\right) a_1 - \left[\left(\frac{m-1}{2}\right) a_2 - u\right]) \text{ or} \\
P_m &= (v, a_2 - v, a_1 - (a_2 - v), (2a_2 - v) - a_1, 2a_1 - (2a_2 - v), \ldots, \\
&\quad \left[\left(\frac{m-1}{2}\right) a_2 - v\right] - \left(\frac{m-3}{2}\right) a_1, \left(\frac{m-1}{2}\right) a_1 - \left[\left(\frac{m-1}{2}\right) a_2 - v\right]).
\end{aligned}
$$

Due to Lemma 4.5 and 4.6, we can conclude that the vertices in P_m are distinct. Also, for any $j = 1, 2, \ldots, \frac{n-m}{2m}$, G is contained a cycle

$$\left(\frac{n-m}{2m}\right)C_{2m} = (a_1+2(j-1), a_2-(a_1+2(j-1)), (2a_1+2(j-1))-a_2,$$
$$2a_2-(2a_1+2(j-1)), \ldots, (2ma_1+2(j-1))-(2m-1)a_2,$$
$$2ma_2-(2ma_1+2(j-1)), a_1+2(j-1))$$

of length $2m$ with $\frac{n-m}{2m}$ components. Moreover, if $m=n$, then G not have a cycle. Hence $G = P_m \cup \left(\frac{n-m}{2m}\right)C_{2m}$.

Case 3: Suppose a_1, a_2 are both even. Let $x_1, x_2, y_1, y_2 \in \mathbb{Z}_n$ such that $2x_1 \equiv 2x_2 \equiv a_1$ and $2y_1 \equiv 2y_2 \equiv a_2 \pmod{n}$ where $x_1 < x_2$ and $y_1 < y_2$. Then we obtain G has two path of order m, as follow: if m is even, then

(1) $P_m = (x_1, a_2-x_1, a_1-(a_2-x_1), (2a_2-x_1)-a_1, 2a_1-(2a_2-x_1), \ldots,$
$\left(\frac{m}{2}-1\right)a_1 - \left[\left(\frac{m}{2}-1\right)a_2 - x_1\right], x_2)$ and

(2) $P_m = (y_1, a_2-y_1, a_1-(a_2-y_1), (2a_2-y_1)-a_1, 2a_1-(2a_2-y_1), \ldots,$
$\left(\frac{m}{2}-1\right)a_1 - \left[\left(\frac{m}{2}-1\right)a_2 - y_1\right], y_2)$.

If m is odd, then

(1) $P_m = (x_1, a_2-x_1, a_1-(a_2-x_1), (2a_2-x_1)-a_1, 2a_1-(2a_2-x_1), \ldots,$
$\left[\left(\frac{m-1}{2}\right)a_2 - x_1\right] - \left(\frac{m-3}{2}\right)a_1, y_2)$ and

(2) $P_m = (y_1, a_2-y_1, a_1-(a_2-y_1), (2a_2-y_1)-a_1, 2a_1-(2a_2-y_1), \ldots,$
$\left[\left(\frac{m-1}{2}\right)a_2 - y_1\right] - \left(\frac{m-3}{2}\right)a_1, x_2)$.

So the two path have distinct adjacent vertices, by Lemma 4.5, 4.6. Also G is contained a cycle of order $2m$ with $\frac{n}{2m}-1$ components, as follow:

$$\left(\frac{n}{2m}-1\right)C_{2m} = (a_1+2(k-1), a_2-(a_1+2(k-1)), (2a_1+2(k-1))-a_2,$$
$$2a_2-(2a_1+2(k-1)), \ldots, (2ma_1+2(k-1))-(2m-1)a_2,$$
$$2ma_2-(2ma_1+2(k-1)), a_1+2(k-1))$$

for all $k = 1, 2, \ldots, \frac{n}{2m}-1$. Moreover, if $m = \frac{n}{2}$, then G not have a cycle. Hence $G = 2P_m \cup \left(\frac{n}{2m}-1\right)C_{2m}$. This complete the proof. □

We let $n=10$. First case, $Cay^+(\mathbb{Z}_{10}, \{1,3\})$ and $m_1 \equiv 3m_1 \pmod{10}$. So, $m_1 = 5$ and $Cay^+(\mathbb{Z}_{10}, \{1,3\}) = C_{10}$. Next case, $Cay^+(\mathbb{Z}_{10}, \{2,7\})$ and

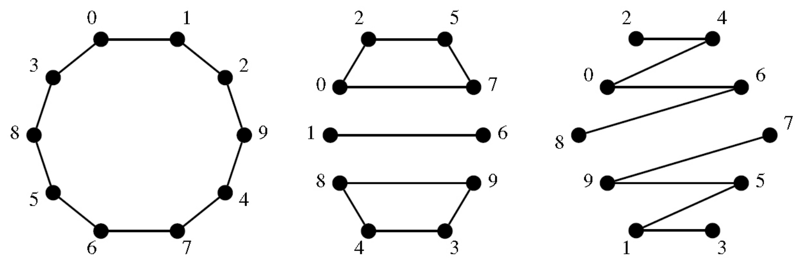

Figure 10

The $Cay^+(\mathbb{Z}_{10}, \{1,3\})$, $Cay^+(\mathbb{Z}_{10}, \{2,7\})$ and $Cay^+(\mathbb{Z}_{10}, \{4,6\})$.

$2m_2 \equiv 7m_2 \pmod{10}$. Then $m_2 = 2$ and $Cay^+(\mathbb{Z}_{10}, \{2,7\}) = P_2 \cup 2C_4$. Last case, $Cay^+(\mathbb{Z}_{10}, \{4,6\})$ and $4m_3 \equiv 6m_3 \pmod{10}$. Thus, $m_3 = 5$ and $Cay^+(\mathbb{Z}_{10}, \{4,6\}) = 2P_5$. The character of above graphs are shown in Figure 10.

The total chromatic numbers of the addition Cayley graphs corresponding to 2 elements generating set are shown in the next theorem.

Theorem 4.8: *For any the addition Cayley graph $G = Cay^+(\mathbb{Z}_n, \{a_1, a_2\})$ where n is even and a_1, a_2 are distinct elements in $\mathbb{Z}_n$. Let m be a smallest factor of n such that $ma_1 \equiv ma_2 \pmod{n}$. Then total chromatic number of the addition Cayley graph G is*

$$\chi''(G) = \begin{cases} 3 & \text{if } m \equiv 0 \pmod{3}, \\ 3 & \text{if } m \geq \frac{n}{2} \text{ and } a_1, a_2 \text{ are both even}, \\ 4 & \text{otherwise}. \end{cases}$$

Proof: Let n is an even number, $a_1 < a_2$ are elements in $\mathbb{Z}_n$ and $G = Cay^+(\mathbb{Z}_n, \{a_1, a_2\})$. Assume that m is a smallest factor of n such that $ma_1 \equiv ma_2 \pmod{n}$. Then there are three cases to be considered. First, $m \equiv 0 \pmod{3}$. Due to Theorem 2.2 and Lemma 4.7, $\chi''(G) = 3$. Next case, $m \geq \frac{n}{2}$ and a_1, a_2 are both even. We obtain that G is two paths of order m without cycle, by Lemma 4.7. So $\chi''(G) = 3$. In the last case, according to Lemma 4.7 we obtain G is contained either path or cycle. By Theorem 2.2, $\chi''(G) = 4$.

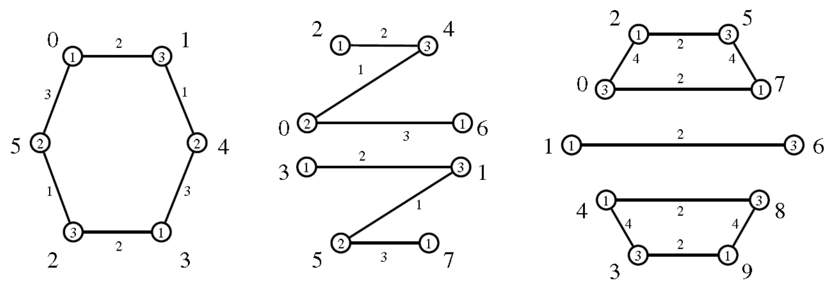

Figure 11

The total colorings of $Cay^{+}(\mathbb{Z}_6,\{1,5\})$, $Cay^{+}(\mathbb{Z}_8,\{4,6\})$ and $Cay^{+}(\mathbb{Z}_{10},\{2,7\})$.

Determine $Cay^{+}(\mathbb{Z}_6,\{1,5\})$, $Cay^{+}(\mathbb{Z}_8,\{4,6\})$ and $Cay^{+}(\mathbb{Z}_{10},\{2,7\})$, as Figure 11. We have the congruence $m_1 \equiv 5m_1 \pmod 6$ has a solution $m_1 = 3$. Since $m_1 \equiv 0 \pmod 3$, then $\chi''(Cay^{+}(\mathbb{Z}_6,\{1,5\})) = 3$. Next, a solution of the congruence $4m_2 \equiv 6m_2 \pmod 8$ is $m_2 = 4$. Since $m_2 \geq \frac{8}{2}$ and 4, 6 are both even, then $\chi''(Cay^{+}(\mathbb{Z}_8,\{4,6\})) = 3$. Next, we consider $Cay^{+}(\mathbb{Z}_{10},\{2,7\})$. From previous example, we obtain $m_3 = 2$. So $\chi''(Cay^{+}(\mathbb{Z}_{10},\{2,7\})) = 4$, by Theorem 4.8.

For the generating set is $\mathbb{Z}_n$, the character and total chromatic numbers of Cayley graphs are shown next theorems.

Lemma 4.9: *A characterization of the addition Cayley graph $G = Cay^{+}(\mathbb{Z}_n,\mathbb{Z}_n)$, n is even, is K_n.*

Proof: Let $G = Cay^{+}(\mathbb{Z}_n,\mathbb{Z}_n)$ where *n* is an even integer. Since the generating set is $\mathbb{Z}_n$ and by Theorem 2.3, then *G* is a $(n-1)$-regular graph. Thus, the graph *G* in this case is a complete graph of order *n*. □

Theorem 4.10: *Let n be an even integer. For any the addition Cayley graph $G = Cay^{+}(\mathbb{Z}_n,\mathbb{Z}_n)$. Then total chromatic number of the addition Cayley graph G is $\chi''(G) = n+1$.*

Proof: Suppose *n* is even. Let $G = Cay^{+}(\mathbb{Z}_n,\mathbb{Z}_n)$. Then $\chi''(G) = n+1$, by Theorem 2.1 and Lemma 4.9.

Let $n = 8$, we obtain that $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$. According to Lemma 4.9 and Theorem 4.10, we can conclude that $Cay^{+}(\mathbb{Z}_8,\mathbb{Z}_8) = K_8$ and $\chi''(Cay^{+}(\mathbb{Z}_8,\mathbb{Z}_8)) = 9$. The total coloring of $Cay^{+}(\mathbb{Z}_8,\mathbb{Z}_8)$ are displayed in Figure 12.

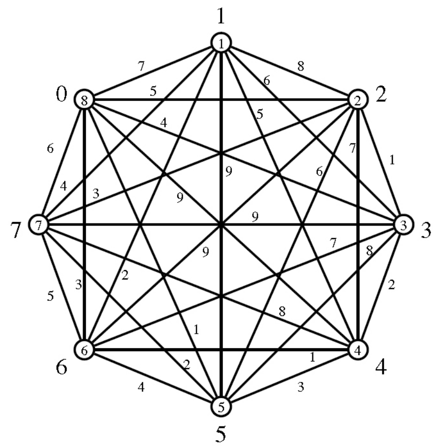

Figure 12

The total coloring of $Cay^{+}(\mathbb{Z}_8, \mathbb{Z}_8)$.

From Theorem 4.2, 4.8 and 4.10, we can summarize it as follows. The addition Cayley graphs $Cay^{+}(\mathbb{Z}_n, \{a\})$, $a \in \mathbb{Z}_n$, and $Cay^{+}(\mathbb{Z}_n, \mathbb{Z}_n)$ are Type II graphs. Moreover, we let m be a smallest factor of n such that $ma_1 \equiv ma_2 \pmod{n}$. For any 2 elements generating set $\{a_1, a_2\}$, $a_1, a_2 \in \mathbb{Z}_n$, $Cay^{+}(\mathbb{Z}_n, \{a_1, a_2\})$ is Type I graph, if $m \equiv 0 \pmod{3}$ or $m \geq \frac{n}{2}$ and a_1, a_2 are both even. Otherwise, $Cay^{+}(\mathbb{Z}_n, \{a_1, a_2\})$ is Type II graph.

5. Conclusions

In this paper, we have studied the characteristics of some types of Cayley graphs and addition Cayley graphs. We characterize the Cayley graph and the addition Cayley graph corresponding to 1 or 2 elements generating set. We also find conditions for generating sets to make the Cayley graph and addition Cayley a complete graph. Moreover, we show the total chromatic numbers of them and also give examples to illustrate these results.

Now, we have only considered the Cayley graph and the addition Cayley graph for even orders. Of course, the result is not an exhaustive list. Some results are not given, some exact values can be further improved, and the total chromatic numbers of their graphs can be added, such as the number of vertices being odd, or the size of generating sets being more than two. The author expects that it will be developed, enriched, and elaborated further in the future.

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