

Some applications of trace of powers of matrices to graph theory, algebra and combinatorics

Kailash M. Patil

Department of Mathematics

Faculty of Technology

Dharmsinh Desai University

Nadiad 387001

Gujarat

India

Abstract

In this paper, using Cayley Hamilton Theorem, we exhibit a combinatorial formula for $\text{Tr}(A^n)$, ($n \in \mathbb{Z}$, $A \in M_3(\mathbb{K})$), over the field $\mathbb{K}(= \mathbb{R} \text{ or } \mathbb{C})$ using $\text{Tr}(A)$, $\det(A)$ and sum of principal minors of A . Further, a formula for $\sum_{k=0}^n \text{Tr}(A^k)$ is derived. For specific values of $\text{Tr}(A)$ and $\det(A)$, the formula of $\text{Tr}(A^n)$ is highly simplified. We exhibit some applications of our results in Graph Theory, Algebra and Combinatorics. We have verified our results on various matrices using Scilab and found that Scilab fails tremendously because of the truncation. But our formulae give exact result.

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1. Introduction

Sum and Trace of powers of a matrix play crucial role in different branches of mathematics [2]. The total number of cliques of a connected simple undirected graph G in a complex network is $\frac{\text{Tr}(A(G)^3)}{6}[1]$. The following holds for Hermitian matrices $A, B \in M_m$ given by

$$\text{Tr}([AB]^{2^k}) \leq \text{Tr}(A^{2^k} B^{2^k}),$$

where k is an integer, is studied by Chang [5]. In continuation of the work of [5], Yang [19] proved the following inequalities in 2000.

E-mail: kailashpatil2111.maths@ddu.ac.in; kailashpatil2111@gmail.com

$$0 \leq \text{Tr}(AB)^{2n} \leq (\text{Tr}(A))^2 (\text{Tr}(A^2))^{n-1} (\text{Tr}(B^2))^n;$$

and

$$0 \leq \text{Tr}(AB)^{2n+1} \leq (\text{Tr}(A))(\text{Tr}(B))(\text{Tr}(A^2))^n (\text{Tr}(B^2))^n,$$

where $A, B \geq 0$ and $n \in \mathbb{N}$. For a prime number p and $r \in \mathbb{Z}$, Zarelua [20] obtained the following congruence for $A \in M_m(\mathbb{Z})$.

$$\text{Tr}(A^{p^r}) \equiv \text{Tr}(A^{p^{r-1}}) \pmod{p^r}.$$

Pugačev [18] gave the formula to obtain the eigenvalue of $A \in M_m$ having maximum modulus using $\text{Tr}(A^n)$.

The study of the element sum of powers of a matrix had been initiated by Marcus and Newman [14] in 1962. They conjectured the inequality

$$n \, \text{su}(A^3) \geq \text{su}(A) \, \text{su}(A^2) \quad (1)$$

for a nonnegative symmetric matrix $A \in M_n(\mathbb{R})$. Later in 1966, London [12] and [13] showed that the conjecture holds for $A \in M_n(\mathbb{R})$ for $1 \leq n \leq 3$ and also gave a counter example for $n = 4$.

Merikoski [15] surveyed extensively the analogy between the trace and the sum. In 1985, Kankaanpää and Merikoski [10], obtained a stronger result establishing the equivalence of the Marcus-Newmann inequality (1) with

$$m \, \text{su}(A^{n+1}) \geq \text{su}(A) \, \text{su}(A^n),$$

for nonnegative symmetric matrix $A \in M_m$, $m \leq 3$, and $n \in \mathbb{N} \cup \{0\}$. Explicit formula for the sum of powers of a matrix has been obtained in [11].

For more results on trace identities for polynomials using CHT, refer [8] and [7]. Domokos [6] obtained a trace identity for a 2×2 matrix over a Grassmann algebra E , generated by an infinite sequence of anticommutative indeterminates. A Cayley Hamilton trace identity for 2×2 matrices over Lie solvable rings has been studied by Meyer et al [16]. This fact, together with several other applications of sum and trace of powers of a matrix, provided the motivation for [17], which is in the sequel of Karia et al [11]. Characteristic equation in max plus algebra has been discussed by Siswanto and Nurhayati in [19-24]

In this paper, using CHT for matrices, we obtain a simple closed form for trace of arbitrary positive powers of $A \in M_3(\mathbb{K})$, negative powers of $A \in GL_3(\mathbb{K})$, and its applications, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1: For $A = [a_{ij}] \in M_3(\mathbb{K})$,

$$\begin{aligned} S_1 &= \text{Tr}(A) = a_{11} + a_{22} + a_{33}. \\ S_2 &= \sum_{i_1=1}^3 \sum_{i_2=1+1}^3 M_{i_1, i_2}. \\ S_3 &= \det(A). \end{aligned}$$

Here M_{i_1, i_2} denotes a principal minor of A . Thus S_2 is the sum of all the three principal minors of A of order 2.

Throughout this paper, $\mathbb{K}$ will stand for $\mathbb{R}$ or $\mathbb{C}$. However, while discussing any inequality, we restrict $\mathbb{K}$ to be $\mathbb{R}$. Also, for $m \in \mathbb{N}$, M_m will stand for the set $M_m(\mathbb{K})$ of all $m \times m$ matrices over $\mathbb{K}$.

2. Main Results

For this section, we fix $A \in M_3$.

$$f_A(x) = \det(A - xI_3) = x^3 - \text{Tr}(A)x^2 + S_2x - \det(A) \in \mathbb{K}[x].$$

The equation $f_A(\lambda) = 0 \Leftrightarrow \lambda^3 - \text{Tr}(A)\lambda^2 + S_2\lambda - \det(A) = 0$ is called the *characteristic equation of A*. The solutions $\lambda_1, \lambda_2, \lambda_3$ of the characteristic equation are the eigenvalues of A . Note that

$$\begin{aligned} \text{Tr}(A^2) &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 - 2\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ &= \text{Tr}(A)^2 - 2S_2 \end{aligned} \quad (2)$$

By the Cayley Hamilton Theorem, we have $A^3 - \text{Tr}(A)A^2 + S_2A - \det(A)I = 0$. Multiplying this by A^n , for $n \in \mathbb{N}$, we get

$$A^{n+3} - \text{Tr}(A)A^{n+2} + S_2A^{n+1} - \det(A)\text{Tr}(A^n) = 0. \quad (3)$$

Using (3), we get

$$\text{Tr}(A^{n+3}) = \text{Tr}(A)\text{Tr}(A^{n+2}) - S_2\text{Tr}(A^{n+1}) + \det(A)\text{Tr}(A^n), \quad (4)$$

which is central to this section. We shall need the following in the proof of Theorem 2.1.

$$A^3 = \text{Tr}(A)^3 - 3\text{Tr}(A)S_2 + 3\det(A). \quad (5)$$

We formulate (6) based on analysis of the pattern in computation of $\text{Tr}(A^n)$ for various $n \in \mathbb{N}$.

Theorem 2.1: For $A = [a_{ij}] \in M_3$,

$$\text{Tr}(A^n) = \sum_{i \geq j} \sum_{2i+j \leq n} (-1)^{i-j} \frac{n}{n-i-j} \binom{n-i-j}{i} \binom{i}{j} \text{Tr}(A)^{n-2i-j} S_2^{i-j} \det(A)^j. \quad (6)$$

Proof: Taking $n = 1$ forces $i = j = 0$ in (6), making it trivially true. Also, taking $n = 2, 3$ reduces (6) to (2), (5) respectively. Using (4) and mathematical induction,

$$\begin{aligned} &\text{Tr}(A^n) \\ &= \text{Tr}(A)\text{Tr}(A^{n-1}) - S_2\text{Tr}(A^{n-2}) + \det(A)\text{Tr}(A^{n-3}) \\ &= \text{Tr}(A) \sum_{i \geq j} \sum_{2i+j \leq n-1} (-1)^{i-j} \frac{n-1}{n-1-i-j} \binom{n-1-i-j}{i} \binom{i}{j} \text{Tr}(A)^{n-1-2i-j} S_2^{i-j} \det(A)^j \\ &\quad - S_2 \sum_{i \geq j} \sum_{2i+j \leq n-2} (-1)^{i-j} \frac{n-2}{n-2-i-j} \binom{n-2-i-j}{i} \binom{i}{j} \text{Tr}(A)^{n-2-2i-j} S_2^{i-j} \det(A)^j \\ &\quad + \det(A) \sum_{i \geq j} \sum_{2i+j \leq n-3} (-1)^{i-j} \frac{n-3}{n-3-i-j} \binom{n-3-i-j}{i} \binom{i}{j} \text{Tr}(A)^{n-3-2i-j} S_2^{i-j} \det(A)^j \\ &= \sum_{i \geq j} \sum_{2i+j \leq n-1} (-1)^{i-j} \frac{n-1}{n-1-i-j} \binom{n-1-i-j}{i} \binom{i}{j} \text{Tr}(A)^{n-2i-j} S_2^{i-j} \det(A)^j \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{2i+j \leq n \\ i-1 \geq j}} (-1)^{i-j} \frac{n-2}{n-1-i-j} \binom{n-1-i-j}{i-1} \binom{i-1}{j} \text{Tr}(A)^{n-2i-j} S_2^{i-j} \det(A)^j \\
& + \sum_{\substack{2i+j \leq n \\ i \geq j}} (-1)^{i-j} \frac{n-3}{n-1-i-j} \binom{n-1-i-j}{i-1} \binom{i-1}{j-1} \text{Tr}(A)^{n-2i-j} S_2^{i-j} \det(A)^j.
\end{aligned}$$

Further, using the Pascal identity $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$ and clubbing of summations, we arrive at (6).

The following theorem is easily proved by the principle of mathematical induction.

Theorem 2.2: For $A \in M_3(\mathbb{K})$ if $\text{Tr}(A) - S_2 + \det(A) - 1 \neq 0$, then

$$\sum_{k=0}^n \text{Tr}(A^k) = \frac{\text{Tr}(A^{n+2}) - (\text{Tr}(A) - 1) \text{Tr}(A^{n+1}) + \det(A) \text{Tr}(A^n) - 3 + 2 \text{Tr}(A) - S_2}{\text{Tr}(A) - S_2 + \det(A) - 1}.$$

Next, we prove an identity that reduces determinant of a symmetric matrix with entries containing traces of higher powers.

Theorem 2.3: Let $A \in GL_3$ and $n \in \mathbb{Z}$. Then

$$\begin{vmatrix} \text{Tr}(A^{n+2}) & \text{Tr}(A^{n+1}) & \text{Tr}(A^n) \\ \text{Tr}(A^{n+1}) & \text{Tr}(A^n) & \text{Tr}(A^{n-1}) \\ \text{Tr}(A^n) & \text{Tr}(A^{n-1}) & \text{Tr}(A^{n-2}) \end{vmatrix} = \det(A^n) \begin{vmatrix} \text{Tr}(A^2) & \text{Tr}(A) & 3 \\ \text{Tr}(A) & 3 & \text{Tr}(A^{-1}) \\ 3 & \text{Tr}(A^{-1}) & \text{Tr}(A^{-2}) \end{vmatrix}. \quad (7)$$

Proof: We prove this for positive powers by the principle of mathematical induction and the proof for the negative powers is similar. For $n = 0$, (7) is obvious. So, we assume that the result holds for $n = k$ and prove the same for $n = k + 1$. In favour of space, we adopt the notation of block matrix $B =$

$$\begin{bmatrix} \text{Tr}(A^{k+2}) & \text{Tr}(A^{k+1}) \\ \text{Tr}(A^{k+1}) & \text{Tr}(A^k) \\ \text{Tr}(A^k) & \text{Tr}(A^{k-1}) \end{bmatrix}. \text{ Now,}$$

$$\begin{aligned}
\begin{vmatrix} \text{Tr}(A^{k+3}) \\ \text{Tr}(A^{k+2}) & B \\ \text{Tr}(A^{k+1}) \end{vmatrix} &= \begin{vmatrix} \text{Tr}(A) \text{Tr}(A^{k+2}) - S_2 \text{Tr}(A^{k+1}) + \det(A) \text{Tr}(A^k) \\ \text{Tr}(A) \text{Tr}(A^{k+1}) - S_2 \text{Tr}(A^k) + \det(A) \text{Tr}(A^{k-1}) & B \\ \text{Tr}(A) \text{Tr}(A^k) - S_2 \text{Tr}(A^{k-1}) + \det(A) \text{Tr}(A^{k-2}) \end{vmatrix} \\
&= \det(A) \begin{vmatrix} \text{Tr}(A^k) \\ \text{Tr}(A^{k-1}) & B \\ \text{Tr}(A^{k-2}) \end{vmatrix} \\
&= \det(A) \begin{vmatrix} & \text{Tr}(A^k) \\ B & \text{Tr}(A^{k-1}) \\ & \text{Tr}(A^{k-2}) \end{vmatrix} \\
&= \det(A)^{k+1} \begin{vmatrix} \text{Tr}(A^2) & \text{Tr}(A) & 3 \\ \text{Tr}(A) & 3 & \text{Tr}(A^{-1}) \\ 3 & \text{Tr}(A^{-1}) & \text{Tr}(A^{-2}) \end{vmatrix}.
\end{aligned}$$

This proves (7) for $n \geq 0$. Similar proof can be given for $n < 0$.

The following corollary gives a simplified version of the above that actually could be used for singular matrices too. It essentially uses the relation of trace, determinant and eigenvalues of a 3×3 matrix. The proof is a simple verification by expanding the right hand side of the (7).

Corollary 2.4: *Let $A \in M_3$ and $n \geq 2$, we have*

$$D = \det(A^n)^{n-2} (-4 \operatorname{Tr}(A)^3 \det(A) + S_2^2 \operatorname{Tr}(A)^2 + 18 \operatorname{Tr}(A) \det(A) S_2 - 4S_2^3 - 27 \det(A)^2).$$

3. Special Cases

As a consequence of the Theorem 2.1, we record the following corollaries depending upon the combinations of $\operatorname{Tr}(A) = 0$, $\det(A) = 0$ and $S_2(A) = 0$.

Corollary 3.1: *The following are equivalent.*

1. $\operatorname{Tr}(A) = 0 = S_2(A) = \det(A)$.
2. $\operatorname{Tr}(A^n) = 0$ for every $n \in \mathbb{N}$.

Corollary 3.2: *If A is a singular matrix and $S_2(A) = 0$, then trace of power of a matrix is equal to the power of trace of the matrix, that is, $\operatorname{Tr}(A^n) = [\operatorname{Tr}(A)]^n$.*

Corollary 3.3: *For $\operatorname{Tr}(A) = 0$, and $n \in \mathbb{N}$,*

$$\operatorname{Tr}(A^n) = \sum_{i \geq j} 2i+j=n (-1)^{i-j} \frac{n}{n-i-j} \binom{n-i-j}{i} S_2^{i-j} \det(A)^j.$$

Corollary 3.4: *If $\operatorname{Tr}(A) = 0 = S_2(A)$, then for $n \in \mathbb{N}$,*

$$\operatorname{Tr}(A^n) = \begin{cases} 3 \det(A)^{\frac{n}{3}}, & \text{if } 3|n; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.5: *For $\det(A) = 0$ with $\operatorname{Tr}(A) = 0$,*

$$\operatorname{Tr}(A^n) = \begin{cases} 2(-S_2(A))^{\frac{n}{2}}, & \text{if } 2|n; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.6: *If A is a singular matrix, then for $n \in \mathbb{N}$,*

$$\operatorname{Tr}(A^n) = \sum_{2i \leq n} (-1)^i \frac{n}{n-i} \binom{n-i}{i} \operatorname{Tr}(A)^{n-2i} S_2^i.$$

Corollary 3.7: *If $S_2(A) = 0$, then for $n \in \mathbb{N}$,*

$$\operatorname{Tr}(A^n) = \sum_{3i \leq n} \frac{n}{n-2i} \binom{n-2i}{i} \operatorname{Tr}(A)^{n-3i} \det(A)^i.$$

4. Trace of Negative Powers of $A \in GL_3(\mathbb{K})$

Throughout this section, $A = [a_{ij}] \in GL_3$, the set of all nonsingular 3×3 matrices. We fix the eigenvalues of A to be λ_1, λ_2 and λ_3 . In this case, $\text{Tr}(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \frac{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}{\lambda_1\lambda_2\lambda_3}$. Thus,

$$\text{Tr}(A^{-1}) = \frac{S_2(A)}{\det(A)} \quad (8)$$

Also,

$$S_2(A^{-1}) = \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1} = \frac{\text{Tr}(A)}{\det(A)} \quad (9)$$

and

$$\det(A^{-1}) = \frac{1}{\det(A)}. \quad (10)$$

The following theorem states the formula for the trace of negative powers of A . The proof follows by replacing A by A^{-1} in Theorem 2.1 and using the equations (8), (9) and (10).

Theorem 4.1: For $A \in GL_3$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\text{Tr}(A^{-n}) \\ &= \frac{1}{\det(A)^n} \sum_{\substack{2i+j \leq n \\ i \geq j}} (-1)^{i-j} \frac{n}{n-i-j} \binom{n-i-j}{i} \binom{i}{j} \text{Tr}(A)^{i-j} S_2^{n-2i-j} \det(A)^{i+j}. \end{aligned} \quad (11)$$

Corollary 4.2: If $A \in GL_3$ and $\text{Tr}(A) = 0$, then for $n \in \mathbb{N}$,

$$\text{Tr}(A^{-n}) = \frac{1}{\det(A)^n} \sum_{3i \leq n} \frac{n}{n-2i} \binom{n-2i}{i} S_2^{n-2i} \det(A)^{2i}.$$

Analogous to Corollary 3.4, we have the following corollary.

Corollary 4.3: If $A \in GL_3$ and $\text{Tr}(A) = 0 = S_2(A)$, then for $n \in \mathbb{N}$,

$$\text{Tr}(A^{-n}) = \begin{cases} 3 \det(A)^{-\frac{n}{3}}, & \text{if } 3|n; \\ 0, & \text{otherwise.} \end{cases}$$

From Corollaries 3.4 and 4.3, we have the following.

Corollary 4.4: If $A \in GL_3$ and $\text{Tr}(A) = 0 = S_2(A)$, then for $n \in \mathbb{N}$,

$$\text{Tr}(A^n) \text{Tr}(A^{-n}) = \begin{cases} 9, & \text{if } 3|n; \\ 0, & \text{otherwise.} \end{cases}$$

5. Numerical Examples

In this section, we give examples to exhibit how our corollaries simplify and (tremendously) reduce the computation, leaving no room for errors due to truncation while computing the powers of a matrix followed by the trace using computer. The following Table 1 facilitates the overview of the examples at a glance.

Table 1
Trace 3×3

No.	Tr(A)	S_2	det(A)	$n; \text{Tr}(A^n)$	Reference
Positive Powers					
A_1	6	11	6	7; 2316	Th 2.1
A_2	0	0	0	$n; 0$	Cor 3.1
A_3	2	0	0	2022; 2^{2022}	Cor 3.2
A_4	0	0	14	2022; $3(14)^{674}$ & 2023; 0	Cor 3.4
A_5	0	-26	0	2345; $0 \& 2346; 2(26)^{1173}$	Cor 3.5
A_6	0	-32	85	9; 26909895	Cor 3.3
A_7	-2	-126	0	16; 2.790495×10^{17}	Cor 3.6
A_8	7	0	26	6; 173185	Cor 3.7
Negative Powers					
A_9	6	11	6	-5; 1.035365226	Th 4.1
A_{10}	0	0	14	-2020; 0	Cor 4.3

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} & A_2 &= \begin{bmatrix} 7 & -34 & 7 \\ 8 & -16 & 8 \\ 9 & 2 & 9 \end{bmatrix} & A_3 &= \begin{bmatrix} 2 & -6 & -2 \\ -1 & 3 & 1 \\ 3 & -9 & -3 \end{bmatrix} \\
 A_4 &= \begin{bmatrix} 3 & 11 & 6 \\ 1 & 1 & 0 \\ -4 & -7 & -4 \end{bmatrix} & A_5 &= \begin{bmatrix} 4 & 1 & 1 \\ 3 & -3 & 1 \\ -1 & 11 & -1 \end{bmatrix} & A_6 &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 5 \\ 6 & 1 & 0 \end{bmatrix} \\
 A_7 &= \begin{bmatrix} -1 & 6 & 7 \\ 8 & 4 & 2 \\ 9 & -2 & -5 \end{bmatrix} & A_8 &= \begin{bmatrix} 2 & 1 & 9 \\ 1 & 3 & 1 \\ 1 & 6 & 2 \end{bmatrix} & A_9 &= \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \\
 A_{10} &= \begin{bmatrix} 3 & 11 & 6 \\ 1 & 1 & 0 \\ -4 & -7 & -4 \end{bmatrix}
 \end{aligned}$$

6. Applications

In this section, we give some applications of Theorem 2.1 to problems in Combinatorics, Algebra and Graph Theory.

Now, we fix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$. Then $\text{Tr}(A) = x + y + z$, $S_2 = xy + yz + zx$ and $\det(A) = xyz$. So, by Theorem 2.1, we get

Theorem 6.1. For real numbers x, y, z ,

$$\begin{aligned}
 x^n + y^n + z^n &= \sum_{\substack{2i+j \leq n \\ i \geq j}} (-1)^{i-j} \frac{n}{n-i-j} \binom{n-i-j}{i} \binom{i}{j} (x+y+z)^{n-2i-j} \times \\
 &\quad (xy + yz + zx)^{i-j} (xyz)^j.
 \end{aligned}$$

Corollary 6.2: For $x, y, z \in \mathbb{R}$, $xy + yz + zx = 0$, then

$$x^n + y^n + z^n = \sum_{3i \leq n} \frac{n}{n-2i} \binom{n-2i}{i} (x+y+z)^{(n-3i)} (xyz)^i.$$

Corollary 6.3: *If for $x, y, z \in \mathbb{R}, x + y + z = 0$, then*

$$x^n + y^n + z^n = \sum_{i \geq j} 2i+j=n (-1)^{i-j} \frac{n}{n-i-j} \binom{n-i-j}{i} \binom{i}{j} (xy + yz + zx)^{i-j} (xyz)^j.$$

Taking $z = 0$ in Theorem 6.1, we recapture the following well-known combinatorics identity. This, and similar identities have also been obtained by Gaould [9] and by Carlitz in [3] and [4]. This amounts to the following result known as Waring's formula.

Theorem 6.3: *Let $x, y \in \mathbb{R}$. Then*

$$x^n + y^n = \sum_{2i \leq n} (-1)^i \frac{n}{n-i} \binom{n-i}{i} (x+y)^{n-2i} (xy)^i.$$

Corollary 6.4: *If $x + y = 1$, then $x^n + y^n = \sum_{2i \leq n} (-1)^i \frac{n}{n-i} \binom{n-i}{i} (xy)^i$.*

Next, we turn to an application of our formula to an algebraic problem. In fact, using our formulae, a number of algebraic problems could be solved. We exhibit one of them as a demonstration.

Example 6.5: Let $x, y, z \in \mathbb{C}$ be such that

$$x + y + z = 1;$$

$$x^2 + y^2 + z^2 = 2;$$

$$x^3 + y^3 + z^3 = 3.$$

Then,

$$x^5 + y^5 + z^5 = 6.$$

The conventional solution to the above involves a lengthy computation. However, our method makes it as simple as possible. Indeed, we need the following simple computation. Using Theorem 2.1 for A , we get,

$$2 = \text{Tr}(A^2) = x^2 + y^2 + z^2 = \text{Tr}(A)^2 - 2S_2 \Rightarrow S_2 = -\frac{1}{2}$$

and

$$3 = \text{Tr}(A^3) = x^3 + y^3 + z^3 = \text{Tr}(A)^3 - 3 \text{Tr}(A) S_2 + 3 \det(A) \Rightarrow \det(A) = \frac{1}{6}.$$

Thus,

$$x^5 + y^5 + z^5 = \text{Tr}(A)^5 - 5 \text{Tr}(A)^3 S_2 + 5 \text{Tr}(A) S_2^2 + (5 \text{Tr}(A)^2 - 5S_2) \det(A) = 6.$$

For application to Graph Theory, we recall the following theorem.

Theorem 6.6: *Total closed walks of length k in a simple undirected graph G from the vertices to themselves is given by the formula $\text{Tr}(A(G)^k)$, where $A(G)$ is the adjacency matrix of the graph G .*

For $A \in M_4$, we define the following.

1. $S_1 = \text{Tr}(A)$.
2. $S_2 =$ the sum of principal minors of A of order 2.
3. $S_3 =$ the sum of principal minors of A of order 3.
4. $S_4 = \det(A)$.

Using Cayley Hamilton Theorem for $A \in M_4$, we have the following equation analogous to the (4).

$$\text{Tr}(A^{n+4}) = S_1 \text{Tr}(A^{n+3}) - S_2 \text{Tr}(A^{n+2}) + S_3 \text{Tr}(A^{n+1}) - \det(A) \text{Tr}(A^n). \quad (12)$$

Example 6.7: Consider the graph shown in Figure 1. Then the adjacency matrix

of G is given by $A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

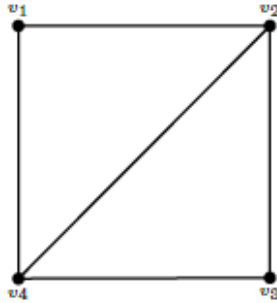


Figure 1
Undirected Graph

Here $\text{Tr}(A) = 0, S_2 = -5, S_3 = 4, \det(A) = 0$. Using (12),

$$\text{Tr}(A(G)^3) = \text{Tr}(A)^3 - 3 \text{Tr}(A) S_2 + 3 S_3 = 12.$$

Thus, the trace of $A(G)^3 = 12$. Consequently, there are all in all 12 closed walks of length 3 in the graph G . Also,

$$\begin{aligned} \text{Tr}(A(G)^5) &= \text{Tr}(A)^5 - 5 \text{Tr}(A)^3 S_2 + 5 \text{Tr}(A) S_2^2 + 5 \text{Tr}(A)^2 S_3 \\ &\quad - 5 \text{Tr}(A) \det(A) \\ &= 100. \end{aligned}$$

Thus, there are 100 closed walks of length 5 in the graph G .

At the first sight only on the graph, one is inclined to believe that there will be very few such closed walks of length 5. However, after algebraically obtaining the number 100 from the formula, one can verify that there are 12 closed walks of length 5 starting and ending at v_1 and there are 38 closed walks of length 5 starting and ending at v_2 . The symmetry of the graph gives yet another 50 closed walks at v_3 and v_4 yielding to a total of 100 closed walks of length 5 in the graph.

Just a curiosity leads us to perturb the same graph making it a directed graph and see what happens to the number of closed walks.

Example 6.8: Consider the graph shown in Figure 2. Then the adjacency matrix

of G is given by $A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

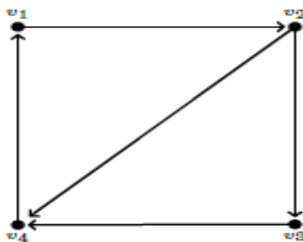


Figure 2
Directed Graph

Here $\text{Tr}(A) = 0 = S_2, S_3 = 1, \det(A) = -1$. Thus,

$$\text{Tr}(A(G)^4) = \text{Tr}(A)^4 - 4 \text{Tr}(A)^2 S_2 + 2S_2^2 + 4 \text{Tr}(A) S_3 - 4\det(A) = 4$$

and

$$\begin{aligned} \text{Tr}(A(G)^5) &= \text{Tr}(A)^4 - 4 \text{Tr}(A)^3 S_2 + 5 \text{Tr}(A) S_2^2 + 5 \text{Tr}(A)^2 S_3 - 5S_2 S_3 \\ &\quad - 5 \text{Tr}(A) \det(A) = 0. \end{aligned}$$

Thus, there are 4 closed walks of length 4 and there is no closed walk of length 5.

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