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Burning, edge burning & chromatic burning classification of some graph family

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Abstract

Graph 'G' is a Simple and undirected graph, which has a lowest number of color that is required to color the edge is called chromatic index. It is denoted by the symbol $\chi^1(G)$. In order to burn a graph, a minimum source of edges is required which is called edge burning index or edge burning number and it is denoted by the symbol $b^1(G)$. A minimum cardinality of color graph set and the source of the burning index set are called edge chromatic burning and let us assume that it is represented by the symbol $\chi b^1(G)$.

In this paper we are mainly concentrating on identifying the chromatic burning and edge chromatic burning graphs and their chromatic burning index using a specific graph family. They are Petersen graph, Helm graph and n- Sunlet graph.

Subject Classification: *05C15, 05C99.*

Keywords: *Burning, Edge burning, Chromatic burning number, Edge chromatic burning index.*

1. Introduction

In this paper, we will consider a simple and undirected graph denoted as G. A graph G consist of a set of vertices (V(G)) and a set of edges (E(G)). The edges represent connections or relationships between the vertices [8]. It is essential for any graph G to have at least one edge; otherwise, it is classified as a trivial graph. This means that the graphs has no edges

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Coloring Problem originated from the Four-color Conjecture almost in the year 1852. In 1840, August Möbius (1700-1868) presented a problem known as "The Problem of the Five Princes"[3]. This problem involved a king with five sons who specified in his will that upon his death, his kingdom should be divided into five regions. The challenge was to divide the kingdom in such a way that each region shared borders with all four others. Möbius posed the question: How is this possible? In other words, how can a map be colored with five colors so that no adjacent regions share the same color?

Despite the king's wishes, it seemed impossible to find a configuration where the five regions were all neighboring on any map. This prompted further exploration into Coloring Graphs and their chromatic number - the minimum number of colors needed to properly color a given graph. Through further observations and analysis, significant insights can be gained about this problem and its solutions.

To properly color a graph, at least two colors are necessary for any given edge. In other words, the chromatic number (χ) of a graph G is equal to 2. Proper coloring [7] means assigning distinct colors to each vertex in such a way that adjacent vertices have different colors[9]. The smallest possible number of colors required for this type of coloring is known as the Chromatic Number [3] [4].

Anthony Bonato introduced the concept of vertex burning [1]. Here I'm introducing the concept of edge burning and Chromatic burning.

A minimum cardinality of color classification graph set and the source of the burning number set is called chromatic burning and it is represented by the symbol $\chi b(G)$.

2. Definitions

2.1 *Burning number*

Burning is a discrete step by step process, to burn a graph minimum cardinality of source vertex set is required and it is denoted by $b(G)$[5].

2.2 *Edge Burning number*

Burning is a discrete step by step process, to burn a graph minimum cardinality of source edge set is required and it is denoted by $b^1(G)$.

2.3 *Chromatic burning*

Chromatic burning to the equivalence between the minimum cardinality of color classification vertex set and the minimum cardinality of a source vertex burning set, denoted as $\chi b(G)$. This term encompasses situation in which both sets have equal sizes.

2.4 *Chromatic edge burning*

The minimum cardinality of color classification edge set [2] is equivalent to the minimum cardinality of source of edge burning set, it is referred to as Chromatic edge burning and it is denoted by $\chi b^1(G)$.

3. Burning number Conjecture

There is a conjecture for any connected graph with n vertices, folks are attempted to prove, the burning number at most ceiling function $\sqrt{n}$. For any path graph, the burning is ceiling function of $\sqrt{n}$ [6].

The following some of the observations.

- For non trivial graph chromatic burning number at least 2.
- For a trivial graph, the chromatic burning number is 1.
- If G is either a cycle C_n or path P_n, then to burn G, we required ceiling of $\sqrt{n}$. It has been proven that the chromatic number of odd cycles is always 3, and for an even cycle is 2.
- For a cycle graph, chromatic burning attempts only when n = 4, 5, 7 and 9. Otherwise it is $b(Cn) \geq \chi(Cn)$.

Some times $\chi(G) \geq b(G)$ or $b(G) \geq \chi(G)$.

4. Chromatic burning number for some graphs

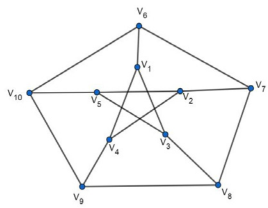

Figure 1

Petersen graph.

The Petersen graph Figure 1, is an undirected and regular graph consisting of 10 vertices and 15 edges. Each vertex has a degree 3.

4.1.1. Theorem

Petersen graph chromatic burning number is 3.

Proof: Let G the Petersen graph. Here Assign a color classes to vertex of Petersen graph. i.e., C1={V_1,V_4,V_7,V_8}, C2 ={V_2,V_5,V_6} and C3={V_3,V_9,V_{10}} and the cardinality of Color graph is 3.

Let us consider V_1 vertex as source vertex to burn, before it get burn take V_9 vertex as source of second vertex then, the first vertex V_1 adjacent vertices get burnt, choose V_{10} vertex as a source of the third vertex. after selecting the third vertex, the second adjacent vertices would get burnt. The source of the burning set is {V_1,$V_3$$V_9$}, the cardinality of the burning number of Petersen graph is 3.

Therefore the Petersen graph chromatic cardinality and burning cardinality are the same, so it is a chromatic burning with the cardinality of a three. Petersen graph is a chromatic burning graph.

Result:

1. χ(G)=b(G)=3.
2. χb(G) =3.

4.2 *General Edge burning procedure*

Step 1: Choose any edge in the Graph G, called the source edge, and name it as b^1_1(G). (Ignite a fire on it)

Step 2: If possible choose any non adjacent edge of Step:1, otherwise any edge apart from step 1, call the source edge name it as b^1_2(G). (Adjacent to step 1 the edges would get burnt).

Step 3: Continue repeating the process until all edges have been burned

4.2.1. Edge Burn -guess- Petersen graph

The edge burning of the Petersen graph is based on our approximation edge burning guess.

Note: Before moving to the procedure we should first identify the adjacency of the edges in the Petersen graph.

Table 01

e_1

Table 02

<table>
<tr><td rowspan="4">e_8</td><td>e_2</td></tr>
<tr><td>e_5</td></tr>
<tr><td>e_{11}</td></tr>
<tr><td>e_{13}</td></tr>
</table>

Table 03

<table>
<tr><td rowspan="5">e_4</td><td>e_6</td></tr>
<tr><td>e_7</td></tr>
<tr><td>e_{10}</td></tr>
<tr><td>e_{14}</td></tr>
<tr><td>e_{15}</td></tr>
</table>

Table 04

<table>
<tr><td rowspan="2">e_9</td><td>e_3</td></tr>
<tr><td>e_5</td></tr>
</table>

Step 1. Choose an edge e_1 as shown in the table 01, then ignite a fire on it.

Step 2. Choose an edge e_8 as shown in the table 02, then Step 1 adjacent would get burnt.

Step 3. Choose an edge e_4 as shown in the table 03, then e_8 & Step 2 adjacent would get burn

Step 4. Choose an edge e_9 as shown in the table 04, then e_4 & Step 3 adjacent would get burnt.

4.1.2. Theorem

Petersen graph edge chromatic burning number is 4.

Proof: Let G be the Petersen graph. Here we Assign a color classes to edges of Petersen graph. i.e., C1 = $\{e_3,e_5,e_{10},e_{13}\}$, C2 =$\{e_1,e_4,e_6,e_8,e_{15}\}$, C3=$\{e_2,e_{11},e_{12},e_{14}\}$ and C4=$\{e_7,e_9\}$and the cardinality of Color graph is 4.

Let us consider e_1 edge as source edge to burn, before it gets burnt, take $e_8$edgeas second source of edge then, the first edge $e_1$adjacent edges get burn, choosee$_4$ edge as a third source of edge, after selecting the third source of edge, the second source of adjacent edges would get burned. The source of the burning edge set is $\{e_1, e_4, e_8, e_9\}$, the cardinality of edge burning number of Petersen graph is 4.

Therefore the Petersen graph edge chromatic cardinality and burning cardinality are same, so it is a chromatic edge burning with the cardinality of a four. Petersen graph is an edge chromatic burning graph.

Result:

1. $\chi^1(G) = b^1(G) = 4.$
2. $\chi^1 b^1(G) = 4.$

4.3 *Helm graph*

Helm graph H_n is a graph which is a modified structure of wheel graph, a pendent edge is added at each vertex of the cycle.

4.3.1. Theorem

For any Helm graph H_n, the chromatic burning $\chi b(H_n)$ cardinality is 3, only where n is an even.

Proof: Let H_n be the Helm graph with V_0 and $V(G) = (V_1,V_2,V_3,V_4,V_5,\ldots\ldots V_n)$. $\{u_1,u_2,u_3\ldots\ldots u_n)$ are the another pendent vertices attached to the cycle.

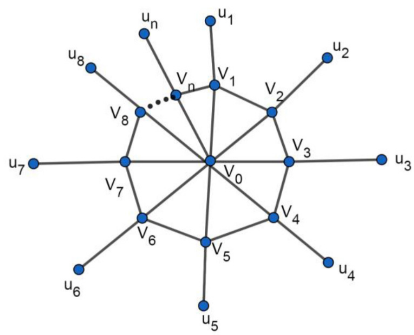

Figure 2

Helm graph (H_n)

The following procedure gives the Figure 2, coloring classification of H_n. Where n is an even graph. $C1 = \{V_0, u_1,u_2,u_3\ldots\ldots u_n)$, $C2 = \{V_1,V_3,V_5,V_7,\ldots\ldots\}$ and $C3 = \{V_2,V_4,V_4,\ldots\ldots\}$,the cardinality of H_n Color graph is 3.

Choose V_0 is the source of the initial vertex and let us consider any u_n vertex as the second source of vertex then only V_0 adjacent vertices would get burnt. Now take any unburnt vertex as the third source of vertex, the remaining unburnt vertices it would get burnt. The source of the burning set is $\{V_0, u_1, u_5\}$, the cardinality of the burning number of Helm $b(H_n)$ graph is 3.

Helm H_n graph color cardinality is three when n is an even otherwise it is 4 for an odd.

Color classification cardinality and burning number cardinality are same only when n is an even.

Therefore, the Helm graph $\chi b(H_n)$ chromatic burning with the cardinality of a three when n is an even.

Result:

1. $\chi(H_n)=b(H_n)=3$, where 'n' is an even integer.
2. $\chi b(H_n) = 3$, where 'n' is an even integer.

Observation:

1. For any Helm graph H_n burning number is 3. Therefore $b(H_n) = 3$.
2. For any Helm graph H_n chromatic number, it varies from arbitrary of a graph. Purely it depends upon the integer of 'n'.

$$\chi(H_n) = \begin{cases} 4, & n \text{ is an odd.} \\ 3, & n \text{ is an even.} \end{cases}$$

4.3.2 Theorem

For any Helm graph H_n, the edge burning $b^1(H_n)$ number is 3.

Proof: Let H_n be the Helm graph with V_0 and $V(G) = (V_1,V_2,V_3,V_4,V_5,\ldots\ldots V_n)$. $\{u_1,u_2,u_3\ldots\ldots u_n)$ are the another pendent vertices attached to the cycle.

Choose any edge in the wheel as a first source edge, set a fire to it, then again choose a second edge let it be any pendent edge as a second source, now first source adjacent edges would get burnt. Continue repeating the process until all edges have been burned. For Helm in graph H_n, the edge source of burning set is 3.

4.3.3. Theorem

For any Helm graph H_n, the edge chromatic $\chi^1(H_n)$ number is n.

Proof: Let H_n be the Helm graph with V_0 and $V(G) = (V_1,V_2,V_3,V_4,V_5,\ldots\ldots V_n)$. $\{u_1,u_2,u_3\ldots\ldots u_n)$ are the other pendent vertices attached to the cycle. Of course we have e_n edges as a spokes structure in the wheel, to assign colors for these edges we must required n colors. Here is the cardinality of the color graph is n.

Result:

1. $\chi^1(H_n)>b^1(H_n)$, therefore Helm graph H_n is not a edge chromatic burning graph.

4.4 *n-Sunlet graph*

A pendent vertices attached to a cycle is called sunlet graph.

4.4.1 Theorem

For any n- sunlet graph is chomatic burning only when n =5.

Proof: Let G be a n- Sunlet graph. For n-sunlet graph color classification of cardinality varies from an odd to an even vertices, same thing it follows to burning cardinality too, there is no correlation between color and burning cardinality of n-sunlet graph, except when n is 5.

When n =5, Color cardinality of n-Sunlet graph is 3 and the burning source set cardinality also 3. Therefore the cardinality of chromatic burning is 3.

Observation:

1. For any n- sunlet graph, chomatic number is

$$\chi(G) = \begin{cases} 2, & n \text{ is an even.} \\ 3, & n \text{ is an odd.} \end{cases}$$

2. For any n- sunlet graph, burning number is

Result:

1. $\chi(G) \neq b(G)$
2. Only when n is 5, n- sunlet graph is chromatic burning χb(G).

4.4.2 Theorem

For any n- sunlet graph is edge chromatic number3

Proof: Sunlet graph is in the structure of Cycle, so n-Sunlet graph has C_n cycle edges with n-pendent edges attached to it, to assign the color for the edges, it would require only 3 colors for the entire n-Sunlet graph. So, the cardinality of the color graph is 3.

Observation:

1. b(Cn) = ceiling of $\sqrt{n}$, Almost it is same for edge burning too.
2. There is no correlation between the edge color index and edge burning index for n- Sunlet graph. So, Here $\chi^1(G) \neq b^1(G)$.
3. $b^1(G) > \chi^1(G)$ therefore n- sunlet graph is not edge chromatic burning.

5. Conclusion

We have discussed burning b(G), edge burningb^1(G), chromatic χ(G) and edge chromatic χ^1(G) number of Petersen graph, Helm graph H_n and n-Sunlet graph. And we have successfully examined the chromatic burning χb(G) graphs and edge chromatic burning $\chi^1 b^1$(G)graphs. Chromatic burning & Edge chromatic burning can be extended to other graphs.

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