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True-False ideals of BCK/BCI-algebras

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Abstract

The notions of a (limited) T & F-ideal and a T & F- $\circ$ -subalgebra are introduced, and various properties are investigated. The relationship between a T & F-subalgebra and a T & F-ideal is established. Conditions for a T & F-structure (resp., T & F-subalgebra) to be a T & F-ideal are discussed. An example is provided to confirm that in a BCI-algebra, a T & F-ideal cannot be a T & F-subalgebra in general, and then conditions for a T & F-ideal to be a T & F-subalgebra in a BCI-algebra are considered. The relationship between a T & F-ideal and a T & F- $\circ$ -subalgebra in a BCK-algebra with condition (S) is discussed. Conditions for a T & F-structure over a BCK-algebra with condition (S) to be a T & F-ideal are provided. Using

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the notion of T & F-level sets, a characterization of a T & F-ideal is established. A novel T & F-ideal is constructed with an existing T & F-ideal using a characterization of a T & F-ideal.

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1. Introduction

In the fuzzy set introduced by Zadeh [23], the degree of membership is represented only by one function, called a membership function. An extension of the fuzzy set is made by Zadeh by using an interval-valued fuzzy set (see [24]). An intuitionistic fuzzy set is introduced by Atanassove [2], and it is a generalization of the fuzzy set. Jun et al. [11] introduced the notion of cubic sets by combining the interval-valued fuzzy sets and the fuzzy sets. In 1966, Iseki introduced BCK/BCI-algebras which are two essential classes of logical algebras (see [7], [8], and [14]). Since then, many results have been produced for the BCK/BCI-algebra theory by several researchers. Fuzzy mathematics is considered as the study of fuzzy subsets and its application to mathematical contexts. As an essential branch of fuzzy mathematics, we can consider fuzzy algebra. Rosenfeld [18] introduced the concept of fuzzy subgroups in 1971. Since Xi [21] applied the concept of fuzzy sets to BCK-algebra in 1991 and studied its properties, many researchers have been extensively studying fuzzy theory in BCI/BCK-algebra. Biswas [3] defined interval-valued fuzzy subgroups of Rosenfeld's nature, and investigated some elementary properties, and the notion of Interval-valued fuzzy sets, neutosophic sets and MBJ-sets has been studied in BCK/BCI-algebras, UP-algebras and groups (see [1], [4], [5], [9], [10], [12], [13], [17], [19], [20], [22], [25]). Using the interval-valued fuzzy sets and the fuzzy set, Mohseni Takallo et al. [16] constructed a new kind of structures, called True-False structures, and considered its application in BCK/BCI-algebras and groups. They introduced the concepts of (limited) T & F-groups and (limited) T & F-subalgebras, and investigated several properties. They considered characterizations of (limited) T & F-groups and (limited) T & F-subalgebras, and showed that the intersection of two T & F-groups is also a T & F-group, and the intersection of two T & F-subalgebras is also a T & F-subalgebra. They provided examples to show that the union of two T & F-groups (resp., T & F-subalgebras) may not be a T & F-group (resp., T & F-subalgebra).

In this article, we introduce the concepts of a (limited) T & F-ideal and a T & F-$\circ$-subalgebra, and investigate various related properties. We discuss relationship between a T & F-subalgebra and a T & F -ideal. We provide conditions for a T & F-structure (resp., T & F-subalgebra) to be a T & F-ideal. We use examples to confirm that in a BCI-algebra, a T & F-ideal cannot be a T & F-subalgebra in general, and then we consider conditions for a T & F-ideal to be a T & F-subalgebra in a BCI-algebra. We discuss relationship between a T & F-ideal and a T & F-$\circ$-subalgebra in a BCK-algebra with condition (S). We provide conditions for a T & F-structure over a BCK-algebra with condition (S) to be a T & F-ideal. Using the notion of T & F-level sets, we establish a characterization of a T & F-ideal. We set up a new T & F-ideal from the existing T & F-ideal using a characterization of a T & F-ideal.

2. Preliminaries

A set $\mathcal{K}$ together with a special element "0" and a binary operation "$*$" is said to be a *BCI-algebra* if the following assertions are satisfied.

(I) $(\forall q,z,r \in \mathcal{K})\ (((q*z)*(q*r))*(r*z)=0)$,

(II) $(\forall q,z \in \mathcal{K})\ ((q*(q*z))*z=0)$,

(III) $(\forall q \in \mathcal{K})\ (q*q=0)$,

(IV) $(\forall q,z \in \mathcal{K})\ (q*z=,\ z*q=0 \Rightarrow q=z)$.

If $\mathcal{K}$ is a BCI-algebra that satisfies the following identity:

(V) $(\forall q \in \mathcal{K})\ (0*q=0)$,

then X is called a *BCK-algebra.* An order relation "$\leq$" on a BCK/BCI-algebra $\mathcal{K}$ is defined by $(\forall q,z \in \mathcal{K})(q \leq z \Leftrightarrow *z=0)$. Every BCK/BCI-algebra $\mathcal{K}$ satisfies:

$$(\forall \xi \in \mathcal{K})(\xi*0=\xi), \tag{2.1}$$

$$(\forall \xi,\varsigma,\gamma \in \mathcal{K})(\xi \leq \varsigma \Rightarrow \xi*\gamma \leq \varsigma*\gamma,\ \gamma*\varsigma \leq \gamma*\xi), \tag{2.2}$$

$$(\forall \xi,\varsigma,\gamma \in \mathcal{K})((\xi*\varsigma)*\gamma=(\xi*\gamma)*\varsigma). \tag{2.3}$$

Every BCI-algebra $\mathcal{K}$ satisfies $(\forall \xi,\varsigma \in \mathcal{K})$ $(\xi*(\xi*(\xi*\varsigma)) = \xi*\varsigma)$ and $(\forall \xi,\varsigma \in \mathcal{K})$ $(0*(\xi*\varsigma)=(0*\xi)*(0*\varsigma))$. A subset M of a BCK/BCI-algebra $\mathcal{K}$ is called a *subalgebra* of $\mathcal{K}$ if $\xi*\varsigma \in M$ for all $\xi,\varsigma \in M$. A subset M of a BCK/BCI-algebra $\mathcal{K}$ is called an *ideal* of $\mathcal{K}$ if $0 \in M$ and $(\forall \xi,\varsigma \in \mathcal{K})$ $(\xi*\varsigma \in M,\ \varsigma \in M \Rightarrow \xi \in M)$. A BCK-algebra $\mathcal{K}$ is said to have *condition (S)* if given $\xi,\varsigma \in \mathcal{K}$ the set $\mathcal{K}(\xi,\varsigma):=\{q \in \mathcal{K} \mid q*\xi \leq \varsigma\}$ has a greatest element, say $\xi \circ b$. A subset M of a BCK $\mathcal{K}$ with condition (S) is called a $\circ$*-subalgebra* of $\mathcal{K}$ if $\xi \circ \varsigma \in M$ for all $\xi, \varsigma \in M$.

For more information on BCK/BCI-Alzebra, please refer to the books [6] and [14].

Let U be a universal set. A *True-False structure* (briefly, *T & F-structure*) over U (see [16]) is defined to be a pair $(U, \mathcal{A})$ with $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$ and it is the following function:

$$\begin{aligned} &\mathcal{A} : U \to [0,1] \times \operatorname{int}[0,1] \times [0,1] \times \operatorname{int}[0,1], \\ &\xi \mapsto (\psi_A(\xi), \tilde{\psi}_A(\xi), \upsilon_A(\xi), \tilde{\upsilon}_A(\xi)). \end{aligned} \tag{2.4}$$

A T & F-structure $(U, \mathcal{A})$ over U with $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$ is said to be *limited* (see [16]) if $\psi_A(\xi) + \upsilon_A(\xi) \le 1$ and $\sup \tilde{\psi}_A(\xi) + \sup \tilde{\upsilon}_A(\xi) \le 1$ for all $\xi \in U$.

Given a (limited) T & F-structure $(U, \mathcal{A})$ over U with $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$, consider the sets

$$\begin{gathered} U(\psi_A; \alpha) := \{\xi \in U \mid \psi_A(\xi) \ge \alpha\},\ U(\tilde{\psi}_A; \tilde{u}) := \{\xi \in U \mid \tilde{\psi}_A(\xi) \succcurlyeq \tilde{u}\}, \\ L(\upsilon_A; \beta) := \{\xi \in U \mid \upsilon_A(\xi) \le \beta\},\ L(\tilde{\upsilon}_A; \tilde{w}) := \{\xi \in U \mid \tilde{\upsilon}_A(\xi) \preccurlyeq \tilde{w}\}, \\ \mathcal{L}_A(\alpha, \tilde{u}, \beta, \tilde{w}) := U(\psi_A; \alpha) \cap U(\tilde{\psi}_A; \tilde{u}) \cap L(\upsilon_A; \beta) \cap L(\tilde{\upsilon}_A; \tilde{w}) \end{gathered}$$

where $\alpha, \beta \in [0,1]$ and $\tilde{u} = [t^-, t^+], \tilde{w} = [s^-, s^+] \in \operatorname{int}([0,1])$. We say that $\mathcal{L}_A(\alpha, \tilde{u}, \beta, \tilde{w})$ is a T & *F-level set* of $\mathcal{A}$ over U.

3. *T* & *F*-ideals

Given a T & F-structure $(\mathcal{K}, \mathcal{A})$ over a set $\mathcal{K}$ with $\mathcal{A} := (\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$, we consider the following sets:

$$\Omega_T^{\psi} := \{(q,z) \in \mathcal{K} \times \mathcal{K} : \psi_A(q) \ge \psi_A(z), \tilde{\psi}_A(q) \succcurlyeq \tilde{\psi}_A(z)\},$$

$$\Omega_F^{\upsilon} := \{(q,z) \in \mathcal{K} \times \mathcal{K} : \upsilon_A(q) \le \upsilon_A(z), \tilde{\upsilon}_A(q) \preccurlyeq \tilde{\upsilon}_A(z)\},$$

$$\Omega^{\psi}_{T(\min, r\min)} := \left\{ \tfrac{q}{(z,r)} \in \tfrac{\mathcal{K}}{\mathcal{K} \times \mathcal{K}} : \begin{matrix} \psi_A(q) \ge \min\{\psi_A(z), \psi_A(r)\} \\ \tilde{\psi}_A(q) \succcurlyeq r\min\{\tilde{\psi}_A(z), \tilde{\psi}_A(r)\} \end{matrix} \right\}$$

and

$$\Omega^{\upsilon}_{F(\max, r\max)} := \left\{ \tfrac{q}{(z,r)} \in \tfrac{\mathcal{K}}{\mathcal{K} \times \mathcal{K}} : \begin{matrix} \upsilon_A(q) \le \max\{\upsilon_A(z), \upsilon_A(r)\} \\ \tilde{\upsilon}_A(q) \preccurlyeq r\max\{\tilde{\upsilon}_A(z), \tilde{\upsilon}_A(r)\} \end{matrix} \right\}.$$

It is clear that

$$(\forall q,z \in \mathcal{K})\Big((q,z) \in \Omega_T^{\psi} \Leftrightarrow \tfrac{q}{(z,z)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)}\Big), \tag{3.1}$$

$$(\forall q,z,r \in \mathcal{K})\Big((q,z) \in \Omega_T^{\psi}, (z,r) \in \Omega_T^{\psi} \Rightarrow (q,r) \in \Omega_T^{\psi}\Big), \tag{3.2}$$

$$(\forall q,z,r \in \mathcal{K})\Big((q,z) \in \Omega_F^{\upsilon}, (z,r) \in \Omega_F^{\upsilon} \Rightarrow (q,r) \in \Omega_F^{\upsilon}\Big), \tag{3.3}$$

$$(\forall q,z \in \mathcal{K})\Big((q,z) \in \Omega_F^{\upsilon} \Leftrightarrow \tfrac{q}{(z,z)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}\Big), \tag{3.4}$$

$$(\forall q,z,r \in \mathcal{K})\Big(\tfrac{q}{(z,r)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)} \Leftrightarrow \tfrac{q}{(r,z)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)}\Big), \tag{3.5}$$

$$(\forall q,z,r \in \mathcal{K})\Big(\tfrac{q}{(z,r)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)} \Leftrightarrow \tfrac{q}{(r,z)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}\Big). \tag{3.6}$$

Proposition 3.1 : *Let* $(\mathcal{K},\mathcal{A})$ *be a T & F-structure over a set* $\mathcal{K}$ *with* $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$. *For any* $\xi, q, z, r \in \mathcal{K}$, *we have*

$$\begin{aligned} &\tfrac{\xi}{(q,z)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)}, (z,r) \in \Omega_T^{\psi} \Rightarrow \tfrac{\xi}{(q,r)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)}, \\ &\tfrac{\xi}{(q,z)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}, (z,r) \in \Omega_F^{\upsilon} \Rightarrow \tfrac{\xi}{(q,r)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}. \end{aligned} \tag{3.7}$$

***Proof* :** Let $\xi, q, z, r \in \mathcal{K}$ be such that $\frac{\xi}{(q,z)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)}$ and $(z,r) \in \Omega_T^{\psi}$. Then

$$\begin{aligned} &\psi_A(\xi) \geq \min\{\psi_A(q), \psi_A(z)\} \geq \min\{\psi_A(q), \psi_A(r)\}, \\ &\tilde{\psi}_A(\xi) \succcurlyeq \mathrm{r}min\{\tilde{\psi}_A(q), \tilde{\psi}_A(z)\} \succcurlyeq \mathrm{r}min\{\tilde{\psi}_A(q)\tilde{\psi}_A(r)\} \end{aligned}$$

and thus $\frac{\xi}{(q,r)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)}$. If $\frac{\xi}{(q,z)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}$ and $(z,r) \in \Omega_F^{\upsilon}$, then

$$\begin{aligned} &\upsilon_A(\xi) \leq \max\{\upsilon_A(q), \upsilon_A(z)\} \leq \max\{\upsilon_A(q), \upsilon_A(r)\}, \\ &\tilde{\upsilon}_A(\xi) \preccurlyeq \mathrm{r}max\{\tilde{\upsilon}_A(q), \tilde{\upsilon}_A(z)\} \preccurlyeq \mathrm{r}max\{\tilde{\upsilon}_A(q), \tilde{\upsilon}_A(r)\} \end{aligned}$$

Hence $\frac{\xi}{(q,r)} \in \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}$. □

In what follows, let $\mathcal{K}$ be a BCK/BCI-algebra unless otherwise stated.

Definition 3.2 : ([16]) A T & F-structure $(\mathcal{K},\mathcal{A})$ over $\mathcal{K}$ with $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$ is called a T & F-subalgebra of $\mathcal{K}$ if the following assertion is valid.

$$(\forall q,z \in \mathcal{K})\Big(\tfrac{*z}{(q,z)} \in \Omega^{\psi}_{T(\min,\mathrm{r}min)} \cap \Omega^{\upsilon}_{F(\max,\mathrm{r}max)}\Big). \tag{3.8}$$

If a T & F-subalgebra is limited, then we say that it is a *limited* T & *F-subalgebra*.

Definition 3.3 : A T & F-structure $(\mathcal{K},\mathcal{A})$ over $\mathcal{K}$ with $\mathcal{A}:=(\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ is called a T & F-ideal of $\mathcal{K}$ if the following assertions are valid.

$$(\forall q\in\mathcal{K})\Big((,q)\in\Omega_T^{\psi}\cap\Omega_F^{\upsilon}\Big). \tag{3.9}$$

$$(\forall q,z\in\mathcal{K})\Big(\tfrac{q}{(^*z,z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}\Big). \tag{3.10}$$

If a T & F-ideal is limited, then we say that it is a *limited T & F-ideal*.

Example 3.4 : Let $(\mathcal{J},*,0)$ be a BCI-algebra and $(\mathbb{Z},-,0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z},+,0)$ of integers. Then the direct product $\mathcal{K}:=\mathcal{J}\times\mathbb{Z}$ is a BCI-algebra in which the zero element is $(0,0)$ and the binary operation " $\rightsquigarrow$ " is given as follows:

$$(\forall(q,\xi),(z,\varsigma)\in\mathcal{K})((q,\xi)\rightsquigarrow(z,\varsigma)=(q*z,\xi-\varsigma)). \tag{3.11}$$

Let $(\mathcal{K},\mathcal{A})$ be a T & F-structure over $\mathcal{K}$ with $\mathcal{A}:=(\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ which is given as follows:

$$\mathcal{A}:\ \mathcal{K}\to[0,1]\times int[0,1]\times[0,1]\times int[0,1],$$
$$(q,\xi)\mapsto\begin{cases}(0.7,[0.4,0.8],0.3,[0.2,0.5]) & \text{if } (q,\xi)\in\mathcal{J}\times\mathbb{N}_0,\\ (0.5,[0.2,0.5],0.6,[0.6,0.8]) & \text{otherwise}\end{cases}$$

where $\mathbb{N}_0$ is the set of all nonnegative integers. It is routine to confirm that $(\mathcal{K},\mathcal{A})$ is a T & F-ideal of $\mathcal{K}$.

Example 3.5 : Consider a BCK-algebra $\mathcal{K}=\{0,b_1,b_2,b_3,b_4\}$ with the binary operation $*$ given by Table 1. Let $(\mathcal{K},\mathcal{A})$ be a T & F-structure over $\mathcal{K}$ with $\mathcal{A}:=(\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ given by Table 2. It is routine to verify that $(\mathcal{K},\mathcal{A})$ is a T & F -ideal of $\mathcal{K}$.

Table 1

Cayley table for the binary operation " * "

$*$	0	b_1	b_2	b_3	b_4
0	0	0	0	0	0
b_1	b_1	0	0	b_1	b_1
b_2	b_2	b_2	0	b_2	b_2
b_3	b_3	b_3	b_3	0	b_3
b_4	b_4	b_4	b_4	b_4	0

Table 2

Tabular representation of $(\mathcal{K}, \mathcal{A})$

$\mathcal{K}$	$\psi_A(q)$	$\tilde{\psi}_A(q)$	$\upsilon_A(q)$	$\tilde{\upsilon}_A(q)$
0	0.88	[0.6,0.9]	0.13	[0.11,0.25]
b_1	0.66	[0.4,0.7]	0.43	[0.47,0.56]
b_2	0.44	[0.2,0.5]	0.53	[0.53,0.77]
b_3	0.77	[0.3,0.6]	0.33	[0.23,0.35]
b_4	0.55	[0.1,0.2]	0.23	[0.28,0.49]

Lemma 3.6 : *If a* $T\&F$*-structure* $(\mathcal{K}, \mathcal{A})$ *over* $\mathcal{K}$ *with* $\mathcal{A} := (\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ *satisfies (3.9), then*

$$(\forall q, z \in \mathcal{K})\left(\tfrac{q}{(,z)} \in \Omega^{\psi}_{T(\min, r\mathit{min})} \cap \Omega^{\upsilon}_{F(\max, r\mathit{max})} \Rightarrow (q, z) \in \Omega^{\psi}_{T} \cap \Omega^{\upsilon}_{F}\right). \tag{3.12}$$

Proof: Assume that (3.9) is valid and $\frac{q}{(,z)} \in \Omega^{\psi}_{T(\min, r\mathit{min})} \cap \Omega^{\upsilon}_{F(\max, r\mathit{max})}$ for all $q, z \in \mathcal{K}$. Then $\psi_A(q) \geq \min\{\psi_A(0), \psi_A(z)\} = \psi_A(z), \tilde{\psi}_A(q) \succcurlyeq \mathrm{r}\mathit{min}\{\tilde{\psi}_A(0), \tilde{\psi}_A(z)\} = \tilde{\psi}_A(z)$, $\upsilon_A(q) \leq \max\{\upsilon_A(0), \ \upsilon_A(z)\} = \upsilon_A(z)$, $\tilde{\upsilon}_A(q) \preccurlyeq \mathrm{r}\mathit{max}\{\tilde{\upsilon}_A(0), \tilde{\upsilon}_A(z)\} = \tilde{\upsilon}_A(z)$ and so $(q, z) \in \Omega^{\psi}_{T} \cap \Omega^{\upsilon}_{F}$. □

Lemma 3.7 : *If a* $T\&F$*-ideal* $(\mathcal{K}, \mathcal{A})$ *of* $\mathcal{K}$ *satisfies:*

$$(\forall q, z, r \in \mathcal{K})\left(\tfrac{q*z}{(,r)} \in \Omega^{\psi}_{T(\min, r\mathit{min})} \cap \Omega^{\upsilon}_{F(\max, r\mathit{max})}\right), \tag{3.13}$$

then $\frac{q}{(z,r)} \in \Omega^{\psi}_{T(\min, r\mathit{min})} \cap \Omega^{\upsilon}_{F(\max, r\mathit{max})}$ for all $q, z, r \in \mathcal{K}$.

Proof: Assume that a $T\&F$-ideal $(\mathcal{K}, \mathcal{A})$ of $\mathcal{K}$ satisfies (3.13). Then

$$\psi_A(q*z) \geq \min\{\psi_A(0), \psi_A(r)\}, \tilde{\psi}_A(q*z) \propto r\min\{\tilde{\psi}_A(0), \tilde{\psi}_A(r)\},$$
$$\upsilon_A(q*z) \leq \max\{\upsilon_A(0), \upsilon_A(r)\}, \tilde{\upsilon}_A(q*z) \times r\max\{\tilde{\upsilon}_A(0), \tilde{\upsilon}_A(r)\}$$

for all $q, z, r \in \mathcal{K}$. This in combination with (3.9) and (3.10) leads to:

$$\psi_A(q) \geq \min\{\psi_A(z), \psi_A(r)\}, \tilde{\psi}_A(q) \propto r\min\{\tilde{\psi}_A(z), \tilde{\psi}_A(r)\}$$
$$\upsilon_A(q) \leq \max\{\upsilon_A(z), \upsilon_A(r)\}, \tilde{\upsilon}_A(q) \times r\max\{\tilde{\upsilon}_A(z), \tilde{\upsilon}_A(r)\},$$

that is, $\frac{q}{(z,r)} \in \Omega^{\psi}_{T(\min, r\mathit{min})} \cap \Omega^{\upsilon}_{F(\max, r\mathit{max})}$ for all $q, z, r \in \mathcal{K}$. □

Proposition 3.8 : Every $T\,\&\,F$-ideal $(\mathcal{K},\mathcal{A})$ of $\mathcal{K}$ satisfies:

(i) $(\forall q,z\in\mathcal{K})\left(q\le z\Rightarrow (q,z)\in\Omega_T^{\psi}\cap\Omega_F^{\upsilon}\right).$

(ii) $(\forall q,z,r\in\mathcal{K})\left(q*z\le r\Rightarrow \frac{q}{(z,r)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}\right).$

(iii) $(\forall q,z,r\in\mathcal{K})\left(\frac{*z}{(*r,*z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}\right).$

(iv) $(\forall q,z\in\mathcal{K})\Big(\frac{q}{(,z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)},\ (,z)\in\Omega_T^{\psi}\cap\Omega_F^{\upsilon}$
$\Rightarrow (q,z)\in\Omega_T^{\psi}\cap\Omega_F^{\upsilon}\Big).$

(v) $(\forall q,z\in\mathcal{K})\left((q*z,)\in\mathbb{A}\Rightarrow (q,z)\in\Omega_T^{\psi}\cap\Omega_F^{\upsilon}\right)$

where $\mathbb{A}=\{(q,z)\in\mathcal{K}\times\mathcal{K}\mid \psi_A(q)=\psi_A(z),\quad \tilde{\psi}_A(q)=\tilde{\psi}_A(z),\quad \upsilon_A(q)=\upsilon_A(z),$ $\tilde{\upsilon}_A(q)=\tilde{\upsilon}_A(z)\}.$

Proof :

(i) Let $q,z\in\mathcal{K}$ be such that $q\le z.$ Then $q*z=$ and so $\frac{q}{(,z)}=\frac{q}{(q*z,z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}$ by (3.10). It follows from Lemma 3.6 that $(q,z)\in\Omega_T^{\psi}\cap\Omega_F^{\upsilon}.$

(ii) Let $q,z,r\in\mathcal{K}$ be such that $q*z\le r.$ Then $(q*z)*r=,$ and thus

$$\frac{q*z}{(0,r)}=\frac{q*z}{((q*z)*r,r)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}$$

by (3.10). It follows from Lemma 3.7 that $\frac{q}{(z,r)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}.$

(iii) Since $(q*z)*(q*r)\le r*z$ for all $q,z,r\in\mathcal{K},$ it is induced by (ii).

(iv) Straightforward.

(v) is induced by using (3.9) and (3.10). □

Corollary 3.9 : *If $(\mathcal{K},\mathcal{A})$ is a $T\,\&\,F$-ideal of $\mathcal{K},$ then*

(i) $(\forall q,z\in\mathcal{K})\left(\frac{q}{(*z,z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}\right),$

(ii) $(\forall \xi,q,z,r\in\mathcal{K})\left(\frac{*(*r)}{(*(*\xi),*\xi)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}\right),$

(iii) *If $\mathcal{K}$ is a BCK-algebra, then* $\frac{q}{(z,q)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}$
for all $q,z\in\mathcal{K}.$

Theorem 3.10 : *If a $T\&F$-structure $(\mathcal{K},\mathcal{A})$ over $\mathcal{K}$ with $\mathcal{A}:=(\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ satisfies (3.9) and the condition (ii) in Proposition 3.8, then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$.*

Proof : Since $q*(q*z)\le z$ for all $q,z\in\mathcal{K}$, we have $\frac{q}{(*z,z)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}$ by the condition (ii) in Proposition (3.8). Hence $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$.

Corollary 3.11 : *If a $T\&F$-subalgebra $(\mathcal{K},\mathcal{A})$ over $\mathcal{K}$ with $\mathcal{A}:=(\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ satisfies the condition (ii) in Proposition 3.8, then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$.*

Proposition 3.12 : *Given a $T\&F$-ideal $(\mathcal{K},\mathcal{A})$ of $\mathcal{K}$, the following are equivalent.*

$$(\forall q,z\in\mathcal{K})\Big((q*z,(q*z)*z)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}\Big). \tag{3.14}$$

$$(\forall q,z,r\in\mathcal{K})\Big(((q*r)*(z*r),(q*z)*r)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}\Big). \tag{3.15}$$

Proof : Let $q,z,r\in\mathcal{K}$. If (3.14) is valid, then

$$\begin{aligned}&\psi_A(q*z)\ge\psi_A((q*z)*z),\tilde{\psi}_A(q*z)\succcurlyeq\tilde{\psi}_A((q*z)*z),\\&\upsilon_A(q*z)\le\upsilon_A((q*z)*z),\tilde{\upsilon}_A(q*z)\preccurlyeq\tilde{\upsilon}_A((q*z)*z).\end{aligned} \tag{3.16}$$

Using (I), (2.2) and (2.3), we have

$$((q*(z*r))*r)*r=((q*r)*(z*r))*r\le(q*z)*r.$$

It follows from , Proposition (i) and that

$$\begin{aligned}\psi_A((q*r)*(z*r))&=\psi_A((q*(z*r))*r)\ge\psi_A(((q*(z*r))*r)*r)\ge\psi_A((q*z)*r),\\\tilde{\psi}_A((q*r)*(z*r))&=\tilde{\psi}_A((q*(z*r))*r)\succcurlyeq\tilde{\psi}_A(((q*(z*r))*r)*r)\succcurlyeq\tilde{\psi}_A((q*z)*r),\\\upsilon_A((q*r)*(z*r))&=\upsilon_A((q*(z*r))*r)\le\upsilon_A(((q*(z*r))^*r)*r)\le\upsilon_A((q*z)*r),\\\tilde{\upsilon}_A((q*r)*(z*r))&=\tilde{\upsilon}_A((q*(z*r))*r)\preccurlyeq\tilde{\upsilon}_A(((q*(z*r))*r)*r)\preccurlyeq\tilde{\upsilon}_A((q*z)*r),\end{aligned}$$

that is, $((q*r)*(z*r),(q*z)*r)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}$.

Conversely, suppose that (3.15) is valid. If we take $z=r$ in (3.15) and use (III) and (2.1), then (3.14) is induced. □

Theorem 3.13 : *Every $T\&F$-ideal is a $T\&F$-subalgebra in a BCK-algebra.*

Proof : Let $(\mathcal{K},\mathcal{A})$ be a $T\&F$-ideal of a BCK-algebra $\mathcal{K}$. For every $q,z,r\in\mathcal{K}$, we have

$$\frac{q*z}{(0,q)} = \frac{q*z}{(0*z,q)} = \frac{q*z}{((q*q)*z,q)} = \frac{q*z}{((q*z)*q,q)} \in \Omega^{\psi}_{T(\min,r\min)} \cap \Omega^{\upsilon}_{F(\max,r\max)}$$

by (III), (V), (2.3) and (3.10), and so $\frac{q*z}{(q,0)} \in \Omega^{\psi}_{T(\min,r\min)} \cap \Omega^{\upsilon}_{F(\max,r\max)}$ by (3.5) and (3.6). Since $(0,z) \in \Omega^{\psi}_{T} \cap \Omega^{\upsilon}_{F}$ for all $z \in \mathcal{K}$, it follows from (3.7) that $\frac{q*z}{(q,z)} \in \Omega^{\psi}_{T(\min,r\min)} \cap \Omega^{\upsilon}_{F(\max,r\max)}$. Therefore $(\mathcal{K},\mathcal{A})$ is a $T\,\&\,F$-subalgebra of $\mathcal{K}$. □

The converse of Theorem 3.13 is not true in general as seen in the following example.

Example 3.14 : Consider a BCK-algebra $\mathcal{K} = \{0, b_1, b_2, b_3, b_4\}$ with the binary operation $*$ given by Table 3.

Table 3

Cayley table of the binary operation "$*$"

$*$	0	b_1	b_2	b_3	b_4
0	0	0	0	0	0
b_1	b_1	0	0	0	0
b_2	b_2	b_1	0	b_1	b_1
b_3	b_3	b_1	b_1	0	b_1
b_4	b_4	b_4	b_4	b_4	0

Let $(\mathcal{K},\mathcal{A})$ be a $T\,\&\,F$-structure over $\mathcal{K}$ with $\mathcal{A} := (\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$ which is given by Table 4.

Table 4

Tabular representation of $\mathcal{A} := (\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$

$\mathcal{K}$	$\psi(q)$	$\tilde{\psi}(q)$	$\upsilon(q)$	$\tilde{\upsilon}(q)$
0	0.7	[0.5, 0.7]	0.3	[0.2, 0.25]
b_1	0.5	[0.3, 0.5]	0.5	[0.4, 0.56]
b_2	0.6	[0.3, 0.4]	0.4	[0.4, 0.56]
b_3	0.5	[0.2, 0.3]	0.5	[0.2, 0.25]
b_4	0.3	[0.1, 0.2]	0.7	[0.6, 0.89]

It is routine to verify that $(\mathcal{K},\mathcal{A})$ is a $T\&F$-subalgebra of $\mathcal{K}$. But it is not a $T\&F$-ideal of $\mathcal{K}$ because (3.10) does not hold since $\frac{b_2}{(b_1*b_2,b_2)} \notin \Omega^{\psi}_{T(\min,rmin)} \cap \Omega^{\upsilon}_{F(\max,rmax)}$.

We consider conditions for a $T\&F$-subalgebra to be a $T\&F$-ideal in BCK-algebras.

Theorem 3.15 : *If a $T\&F$-subalgebra $(\mathcal{K},\mathcal{A})$ of a BCK-algebra $\mathcal{K}$ satisfies the condition (ii) in Proposition 3.8, then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$.*

Proof : If we take $q=z$ in (3.8) and use (III), then $(0,q)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}$ for all $q\in\mathcal{K}$. Since $q*(q*z)\le z$ for all $q,z\in\mathcal{K}$, it follows from Proposition 3.8(ii) that $\frac{q}{(q*z,z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}$ for all $q,z\in\mathcal{K}$. Therefore $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$. □

The following example shows that a $T\&F$-ideal may not be a $T\&F$-subalgebra in a BCI-algebra.

Example 3.16 : In Example 3.4, if we take $(0,0),(0,2)\in\mathcal{K}:=\mathcal{J}\times\mathbb{Z}$, then

$$\frac{(0,0)\rightsquigarrow(0,2)}{((0,0),(0,2))}=\frac{(0,-2)}{((0,0),(0,2))}\notin\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}.$$

Hence the $T\&F$-ideal $(\mathcal{K},\mathcal{A})$ is not a $T\&F$-subalgebra of $\mathcal{K}$.

We provide conditions for a $T\&F$-ideal to be a $T\&F$-subalgebra in BCI-algebra.

Theorem 3.17 : If a $T\&F$-ideal $(\mathcal{K},\mathcal{A})$ of a BCI-algebra $\mathcal{K}$ satisfies:

$$(\forall z\in\mathcal{K})\Big((0*z,z)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}\Big), \tag{3.17}$$

then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-subalgebra of $\mathcal{K}$.

Proof : Let $(\mathcal{K},\mathcal{A})$ be a $T\&F$-ideal of a BCI-algebra $\mathcal{K}$ that satisfies (3.17). For any $q,z\in\mathcal{K}$, we have

$$\frac{q*z}{(0*z,q)}=\frac{q*z}{((q*q)*z,q)}=\frac{q*z}{((q*z)*q,q)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)},$$

and so $\frac{q*z}{(q,0*z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}$ by (3.5) and (3.6). It follows from (3.7) and (3.17) that $\frac{q*z}{(q,z)}\in\Omega^{\psi}_{T(\min,rmin)}\cap\Omega^{\upsilon}_{F(\max,rmax)}$. Therefore $(\mathcal{K},\mathcal{A})$ is a $T\&F$-subalgebra of $\mathcal{K}$. □

Let $(\mathcal{K},*,0)$ be a BCI-algebra and $\mathcal{K}_+ := \{q \in \mathcal{K} \mid 0 \leq q\}$. For any $q \in \mathcal{K}$ and $n \in \mathbb{N}$, we define q^n by $q^1 = q, q^{n+1} = q*(0*q^n)$. The element q of $\mathcal{K}$ is said to be of *finite periodic* (see [15]) if there exists $n \in \mathbb{N}$ such that $q^n \in \mathcal{K}_+$. The period of q is denoted by $|q|$ and it is given by $|q| = \min\{n \in \mathbb{N} \mid q^n \in \mathcal{K}_+\}$. We provide conditions for a $T\&F$-ideal to be a $T\&F$-subalgebra in BCI-algebras.

Theorem 3.18 : *Let $\mathcal{K}$ be a BCI-algebra in which every element is of finite period. Then every $T\&F$-ideal is a $T\&F$-subalgebra.*

Proof : Let $(\mathcal{K},\mathcal{A})$ be a $T\&F$-ideal of $\mathcal{K}$. Let z be an element of $\mathcal{K}$ with $|z| = n$. Then $z^n \in \mathcal{K}_+$. Note that

$$(0*z^{n-1})*z = (0*(0*(0*z^{n-1})))*z = (0*z)*(0*(0*z^{n-1})) = 0*(z*(0*z^{n-1})) = 0*z^n = 0.$$

Hence $((0*z^{n-1})*z,z) = (0,z) \in \Omega_T^{\psi} \cap \Omega_F^{\upsilon}$ by (3.9). Since $\frac{0*z^{n-1}}{((0*z^{n-1})*z,z)} \in \Omega^{\psi}_{T(\min,rmin)} \cap \Omega^{\upsilon}_{F(\max,rmax)}$ by (3.10), it follows from (3.7) that $(0*z^{n-1},z) \in \Omega_T^{\psi} \cap \Omega_F^{\upsilon}$. Also, note that

$$(0*z^{n-2})*z = (0*(0*(0*z^{n-2})))*z = (0*z)*(0*(0*z^{n-2})) = 0*(z*(0*z^{n-2})) = 0*z^{n-1}$$

and so $((0*z^{n-2})*z,z) = (0*z^{n-1},z) \in \Omega_T^{\psi} \cap \Omega_F^{\upsilon}$. Since $\frac{0*z^{n-2}}{((0*z^{n-2})*z,z)} \in \Omega^{\psi}_{T(\min,rmin)} \cap \Omega^{\upsilon}_{F(\max,rmax)}$ by (3.10), it follows from (3.1), (3.4) and (3.7) that $(0*z^{n-2},z) \in \Omega_T^{\psi} \cap \Omega_F^{\upsilon}$. Continuing this process, we get $(0*z,z) \in \Omega_T^{\psi} \cap \Omega_F^{\upsilon}$. Using Theorem 3.17, we know that $(\mathcal{K},\mathcal{A})$ is a $T\&F$-subalgebra of $\mathcal{K}$. ☐

Definition 3.19 : A $T\&F$-structure $(\mathcal{K},\mathcal{A})$ over a BCK-algebra $\mathcal{K}$ with $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$ is called a $T\&F$-$\circ$-subalgebra of $\mathcal{K}$ if the following assertion is valid.

$$(\forall q,z \in \mathcal{K})\left(\tfrac{q\circ z}{(q,z)} \in \Omega^{\psi}_{T(\min,r\min)} \cap \Omega^{\upsilon}_{F(\max,r\max)}\right). \tag{3.18}$$

Theorem 3.20 : *In a BCK-algebra with condition (S), every $T\&F$-ideal is a $T\&F$-$\circ$-subalgebra.*

Proof : Let $(\mathcal{K},\mathcal{A})$ be a $T\&F$-ideal of a BCK-algebra $\mathcal{K}$ with condition (S). Since $(q\circ z)*q \leq z$ for all $q,z \in \mathcal{K}$, it follows from Proposition 3.8(ii) that $\frac{q\circ z}{(q,z)} \in \Omega^{\psi}_{T(\min,rmin)} \cap \Omega^{\upsilon}_{F(\max,rmax)}$. Therefore $(\mathcal{K},\mathcal{A})$ is a $T\&F$-$\circ$-subalgebra of $\mathcal{K}$.

Proposition 3.21 : If $\mathcal{K}$ is a BCK-algebra with condition (S), then every $T\&F$-ideal $(\mathcal{K},\mathcal{A})$ of $\mathcal{K}$ satisfies:

$$(\forall q,z,r\in\mathcal{K})\left(q\leq z\circ r\Rightarrow \tfrac{q}{(z,r)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}\right). \tag{3.19}$$

Proof : Assume that $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of a BCK-algebra $\mathcal{K}$ with condition (S). Then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-$\circ$-subalgebra of $\mathcal{K}$, and thus $\frac{z\circ r}{(z,r)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}$ for all $z,r\in\mathcal{K}$. Let $q,z,r\in\mathcal{K}$ be such that $q\leq z\circ r$. Then $q*(z\circ r)=0$, and so $\frac{q}{(0,z\circ r)}=\frac{q}{(q*(z\circ r),z\circ r)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}$ by (3.10). It follows from Lemma 3.6 that $(q,z\circ r)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}$. Hence $\frac{q}{(z,r)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}$.

We provide conditions for a $T\&F$-structure $(\mathcal{K},\mathcal{A})$ over a BCK-algebra $\mathcal{K}$ with condition (S) to be a $T\&F$-ideal of $\mathcal{K}$.

Theorem 3.22 : *Let $(\mathcal{K},\mathcal{A})$ be a $T\&F$-structure over a BCK-algebra $\mathcal{K}$ with condition (S). Then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$ if and only if it satisfies the condition (3.19).*

Proof : The necessity is by Proposition 3.21. Assume that $(\mathcal{K},\mathcal{A})$ satisfies the condition (3.19). Since $0\leq q\circ q$ for all $q\in\mathcal{K}$, we have $\frac{0}{(q,z)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}$ by (3.19) which implies that $(0,q)\in\Omega^{\psi}_{T}\cap\Omega^{\upsilon}_{F}$ for all $q\in\mathcal{K}$. Since $q\leq(q*z)\circ z$ for all $q,z\in\mathcal{K}$, it follows from (3.19) that $\frac{q}{(q*z,z)}\in\Omega^{\psi}_{T(\min,r\min)}\cap\Omega^{\upsilon}_{F(\max,r\max)}$. Therefore $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$. □

Using the notion of $T\&F$-level sets, we provide a characteization of a $T\&F$-ideal.

Theorem 3.23 *Let $(\mathcal{K},\mathcal{A})$ be a $T\&F$-structure over $\mathcal{K}$ with $\mathcal{A}:=(\psi_A,\ \tilde{\psi}_A,\ \upsilon_A,\ \tilde{\upsilon}_A)$. Then $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$ if and only if the nonempty $T\&F$-level sets $U(\psi_A;\alpha)$, $U(\tilde{\psi}_A;\tilde{u})$, $L(\upsilon_A;\beta)$ and $L(\tilde{\upsilon}_A;\tilde{w})$ of $\mathcal{A}$ over $\mathcal{K}$ are ideals of $\mathcal{K}$ for all $\alpha,\beta\in[0,1]$ and $\tilde{u},\tilde{w}\in\mathrm{int}([0,1])$.*

Proof. Assume that $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$ and let $\alpha,\beta\in[0,1]$ and $\tilde{u},\tilde{w}\in\mathrm{int}([0,1])$ be such that $U(\psi_A;\alpha)$, $U(\tilde{\psi}_A;\tilde{u})$, $L(\upsilon_A;\beta)$ and $L(\tilde{\upsilon}_A;\tilde{w})$ are nonempty. Then there exist $q,z,\xi,\varsigma\in\mathcal{K}$ such that $q\in U(\psi_A;\alpha)$, $z\in U(\tilde{\psi}_A;\tilde{u})$, $\xi\in L(\upsilon_A;\beta)$ and $\varsigma\in L(\tilde{\upsilon}_A;\tilde{w})$. It follows from (3.9) that $\psi_A(0)\geq\psi_A(q)\geq\alpha$, $\tilde{\psi}_A(0)\succcurlyeq\tilde{\psi}_A(z)\succcurlyeq\tilde{u}$, $\upsilon_A(0)\leq\upsilon_A(\xi)\leq\beta$ and $\tilde{\upsilon}_A(0)\preccurlyeq\tilde{\upsilon}_A(\varsigma)\preccurlyeq\tilde{w}$, that is, $\in U(\psi_A;\alpha)\cap U(\tilde{\psi}_A;\tilde{u})\cap L(\upsilon_A;\beta)\cap L(\upsilon_A;\tilde{w})$. Let $q,z\in\mathcal{K}$ be such that $q*z\in U(\psi_A;\alpha)\cap U(\tilde{\psi}_A;\tilde{u})\cap L(\upsilon_A;\beta)\cap L(\tilde{\upsilon}_A;\tilde{w})$

and $z \in U(\psi_A;\alpha) \cap U(\tilde{\psi}_A;\tilde{u}) \cap L(\upsilon_A;\beta) \cap L(\tilde{\upsilon}_A;\tilde{w})$. Then $\psi_A(q*z) \geq \alpha$, $\psi_A(z) \geq \alpha$, $\tilde{\psi}_A(q*z) \succcurlyeq \tilde{u}$, $\tilde{\psi}_A(z) \succcurlyeq \tilde{u}$, $\upsilon_A(q*z) \leq \beta$, $\upsilon_A(z) \leq \beta$, $\tilde{\upsilon}_A(q*z) \preccurlyeq \tilde{w}$, and $\tilde{\upsilon}_A(z) \preccurlyeq \tilde{w}$. Using (3.10), we have

$$\psi_A(q) \geq \min\{\psi_A(q*z), \psi_A(z)\} \geq \alpha,\ \tilde{\psi}_A(q) \succcurlyeq r\min\{\tilde{\psi}_A(q*z), \tilde{\psi}_A(z)\} \succcurlyeq \tilde{u},$$
$$\upsilon_A(q) \leq \min\{\upsilon_A(q*z), \upsilon_A(z)\} \leq \beta,\ \tilde{\upsilon}_A(q) \preccurlyeq r\min\{\tilde{\upsilon}_A(q*z), \upsilon_A(z)\} \preccurlyeq \tilde{w},$$

and so $q \in U(\psi_A;\alpha) \cap U(\tilde{\psi}_A;\tilde{u}) \cap L(\upsilon_A;\beta) \cap L(\tilde{\upsilon}_A;\tilde{w})p$. Therefore $U(\psi_A;\alpha)$, $U(\tilde{\psi}_A;\tilde{u})$, $L(\upsilon_A;\beta)$ and $L(\tilde{\upsilon}_A;\tilde{w})$ are ideals of $\mathcal{K}$ for all $\alpha,\beta \in [0,1]$ and $\tilde{u},\tilde{w} \in \mathrm{int}([0,1])$.

Conversely, suppose that the nonempty $T\&F$-level sets $U(\psi_A;\alpha)$, $U(\tilde{\psi}_A;\tilde{u})$, $L(\upsilon_A;\beta)$ and $L(\tilde{\upsilon}_A;\tilde{w})$ of $\mathcal{A}$ over $\mathcal{K}$ are ideals of $\mathcal{K}$ for all $\alpha,\beta \in [0,1]$ and $\tilde{u},\tilde{w} \in \mathrm{int}([0,1])$. If there exist $\xi,\varsigma,\gamma,\delta \in \mathcal{K}$ such that $\psi_A(0) < \psi_A(\xi)$, $\tilde{\psi}_A(0) \prec \tilde{\psi}_A(\varsigma)$, $\upsilon_A(0) > \psi_A(\gamma)$, and $\tilde{\upsilon}_A(0) \succ \tilde{\upsilon}_A(\delta)$. Then $\psi_A(0) < \alpha < \psi_A(\xi)$, $\tilde{\psi}_A(0) < \tilde{u} < \tilde{\psi}_A(\xi)$, $\upsilon_A(0) > \beta > \psi_A(\gamma)$ and $\tilde{\upsilon}_A(0) > \tilde{w} > \tilde{\psi}_A(\gamma)$ where $\alpha := \frac{1}{2}(\psi_A(0)+\psi_A(\xi))$, $\tilde{u} := \frac{1}{2}(\tilde{\psi}_A(0)+\tilde{\psi}_A(\varsigma))$, $\beta := \frac{1}{2}(\upsilon_A(0)+\upsilon_A(\gamma))$ and $\tilde{w} := \frac{1}{2}(\tilde{\upsilon}_A(0)+\tilde{\upsilon}_A(\delta))$. Hence $0 \notin U(\psi_A;\alpha) \cap U(\tilde{\psi}_A;\tilde{u}) \cap L(\upsilon_A;\beta) \cap L(\tilde{\upsilon}_A;\tilde{w})$ which is a contradiction. Thus $(0,q) \in \Omega_T^{\psi} \cap \Omega_F^{\upsilon}$ for all $q \in \mathcal{K}$. If (3.10) is false, then there exist $q_i, z_i \in \mathcal{K}$, $i=1,2,3,4$, such that $\psi_A(q_1) < \min\{\psi_A(q_1*z_1),\psi_A(z_1)\}$, $\tilde{\psi}_A(q_2) \prec \mathrm{r}min\{\tilde{\psi}_A(q_2*z_2),\tilde{\psi}_A(z_2)\}$, $\upsilon_A(q_3) > \max\{\upsilon_A(q_3*z_3),\upsilon_A(z_3)\}$, and $\tilde{\upsilon}_A(q_4) \succ \mathrm{r}max\{\tilde{\upsilon}_A(q_4*z_4),\tilde{\upsilon}_A(z_4)\}$. It follows that $q_1*z_1 \in U(\psi_A;\alpha)$ and $z_1 \in U(\psi_A;\alpha)$, but $q_1 \notin U(\psi_A;\alpha)$ for $\alpha := \min\{\psi_A(q_1*z_1),\psi_A(z_1)\}$; $q_2*z_2 \in U(\tilde{\psi}_A;\tilde{u})$ and $z_2 \in U(\tilde{\psi}_A;\tilde{u})$, but $q_2 \notin U(\tilde{\psi}_A;\tilde{u})$ for $\tilde{u} := \mathrm{r}min\{\tilde{\psi}_A(q_2*z_2),\tilde{\psi}_A(z_2)\}$; $q_3*z_3 \in L(\upsilon_A;\beta)$ and $z_3 \in L(\upsilon_A;\beta)$, but $q_3 \notin L(\upsilon_A;\beta)$ for $\beta := \max\{\upsilon_A(q_3*z_3),\upsilon_A(z_3)\}$; $q_4*z_4 \in L(\tilde{\upsilon}_A;\tilde{w})$ and $z_4 \in L(\tilde{\upsilon}_A;\tilde{w})$, but $q_4 \notin L(\tilde{\upsilon}_A;\tilde{w})$ for $\tilde{w} := r\max\{\tilde{\upsilon}_A(q_4*z_4),\tilde{\upsilon}_A(z_4)\}$. This is a contradiction, and so (3.10) is valid. Consequently, $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$. □

Corollary 3.24 : *If $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$, then $\mathcal{L}_A(\alpha,\tilde{u},\beta,\tilde{w})$ is an ideal of $\mathcal{K}$ for all $\alpha,\beta \in [0,1]$ and $\tilde{u},\tilde{w} \in int([0,1])$.*

We want to set up a new $T\&F$-ideal from the existing $T\&F$-ideal using Theorem 3.23.

Given a $T\&F$-structure $(\mathcal{K},\mathcal{A})$ over $\mathcal{K}$ with $\mathcal{A} := (\psi_A, \tilde{\psi}_A, \upsilon_A, \tilde{\upsilon}_A)$, let $(\mathcal{K},\mathcal{A}^*)$ be a $T\&F$-structure over $\mathcal{K}$ with $\mathcal{A}^* := (\psi_A^*, \tilde{\psi}_A^*, \upsilon_A^*, \tilde{\upsilon}_A^*)$ given as follows:

$$\psi_A^* : \mathcal{K} \to [0,1], q \mapsto \begin{cases} \psi_A(q) & \text{if } q \in U(\psi_A;\alpha), \\ m & \text{otherwise,} \end{cases}$$

$$\tilde{\psi}_A^* : \mathcal{K} \to int([0,1]), q \mapsto \begin{cases} \tilde{\psi}_A(q) & \text{if } q \in U(\tilde{\psi}_A;\tilde{u}), \\ \tilde{p} & \text{otherwise,} \end{cases}$$

$$\upsilon_A^* : \mathcal{K} \to [0,1], q \mapsto \begin{cases} \upsilon_A(q) & \text{if } q \in L(\upsilon_A;\beta), \\ n & \text{otherwise,} \end{cases}$$

$$\tilde{\upsilon}_A^* : \mathcal{K} \to int([0,1]), q \mapsto \begin{cases} \tilde{\upsilon}_A(q) & \text{if } q \in L(\tilde{\upsilon}_A;\tilde{w}), \\ \tilde{q} & \text{otherwise} \end{cases}$$

where $\tilde{u},\tilde{w},\tilde{p},\tilde{q} \in int([0,1])$ and $\alpha,\beta,m,n \in [0,1]$ with $\psi_A(q) > m$, $\tilde{\psi}_A(q) \succ \tilde{p}$, $\upsilon_A(q) < n$ and $\tilde{\upsilon}_A(q) \prec \tilde{q}$.

Theorem 3.25 : *If* $(\mathcal{K},\mathcal{A})$ *is a* $T\&F$*-ideal of* $\mathcal{K}$*, then so is* $(\mathcal{K},\mathcal{A}^*)$.

Proof : Assume that $(\mathcal{K},\mathcal{A})$ is a $T\&F$-ideal of $\mathcal{K}$. Then the nonempty $T\&F$-level sets $U(\psi_A;\alpha)$, $U(\tilde{\psi}_A;\tilde{u})$, $L(\upsilon_A;\beta)$ and $L(\tilde{\upsilon}_A;\tilde{w})$ of $\mathcal{A}$ over $\mathcal{K}$ are ideals of $\mathcal{K}$ for all $\alpha,\beta \in [0,1]$ and $\tilde{u},\tilde{w} \in int([0,1])$ by Theorem 3.23. Obviously, $(0,q) \in \Omega_T^{\psi^*} \cap \Omega_F^{\upsilon^*}$ for all $q \in \mathcal{K}$. Let $q,z \in \mathcal{K}$. If $q*z \in U(\psi_A;\alpha)$ and $z \in U(\psi_A;\alpha)$, then $q \in U(\psi_A;\alpha)$. Hence $\psi_A^*(q) = \psi_A(q) \geq \min\{\psi_A(q*z),\psi_A(z)\} = \min\{\psi_A^*(q*z),\psi_A^*(z)\}$. If $q*z \notin U(\psi_A;\alpha)$ or $z \notin U(\psi_A;\alpha)$, then $\psi_A(q*z) = m$ or $\psi_A(z) = m$. Thus $\psi_A^*(q) \geq m = \min\{\psi_A^*(q*z),\psi_A^*(z)\}$. If $q*z, z \in U(\tilde{\psi}_A;\tilde{u})$, then $q \in U(\tilde{\psi}_A;\tilde{u})$, and thus

$$\tilde{\psi}_A^*(q) = \tilde{\psi}_A(q) \succcurlyeq \mathrm{r}min\{\tilde{\psi}_A(q*z),\upsilon_A(z)\} = \mathrm{r}min\{\tilde{\psi}_A^*(q*z),\tilde{\psi}_A^*(z)\}.$$

Assume that $q*z \notin U(\tilde{\psi}_A;\tilde{u})$ or $z \notin U(\tilde{\psi}_A;\tilde{u})$. Then $\tilde{\psi}_A(q*z) = \tilde{p}$ or $\tilde{\psi}_A(z) = \tilde{p}$. Hence

$$\tilde{\psi}_A^*(q) \succcurlyeq \tilde{p} = \mathrm{r}min\{\tilde{\psi}_A^*(q*z),\tilde{\psi}_A^*(z)\}.$$

If $q*z \in L(\upsilon_A;\beta)$ and $z \in L(\upsilon_A;\beta)$, then $q \in L(\upsilon_A;\beta)$. Hence

$$\upsilon_A^*(q) = \upsilon_A(q) \leq \max\{\upsilon_A(q*z),\upsilon_A(z)\} = \max\{\upsilon_A^*(q*z),\upsilon_A^*(z)\}.$$

If $q*z \notin L(\upsilon_A;\beta)$ or $z \notin L(\upsilon_A;\beta)$, then $\upsilon_A(q*z) = n$ or $\upsilon_A(z) = n$. Thus $\upsilon_A^*(q) \leq n = \max\{\upsilon_A^*(q*z),\upsilon_A^*(z)\}$. If $q*z, z \in L(\tilde{\upsilon}_A;\tilde{w})$, then $q \in L(\tilde{\upsilon}_A;\tilde{w})$, and thus

$$\tilde{\upsilon}_A^*(q) = \tilde{\upsilon}_A(q) \preccurlyeq \mathrm{r}max\{\tilde{\upsilon}_A(q*z),\tilde{\upsilon}_A(z)\} = \mathrm{r}max\{\tilde{\upsilon}_A^*(q*z),\tilde{\upsilon}_A^*(z)\}.$$

If $q*z \notin L(\tilde{\upsilon}_A;\tilde{w})$ or $z \in L(\tilde{\upsilon}_A;\tilde{w})$, then $\tilde{\upsilon}^*_A(q*z)=\tilde{q}$ or $\tilde{\upsilon}^*_A(z)=\tilde{q}$. Hence

$$\tilde{\upsilon}^*_A(q) \preccurlyeq \tilde{q} = \mathrm{r}max\{\tilde{\upsilon}^*_A(q*z), \tilde{\upsilon}^*_A(z)\}.$$

Therefore $\frac{q}{(q*z,z)} \in \Omega^{\psi^*}_{T(\min,rmin)} \cap \Omega^{\upsilon^*}_{F(\max,rmax)}$, and consequently, $(\mathcal{K},\mathcal{A}^*)$ is a $T\&F$-ideal of $\mathcal{K}$. □

4. Conclusion

Mohseni Takallo et al. have constructed a new kind of structures, called True-False structures (briefly, $T\&F$-structures), by using the notion of (interval-valued) fuzzy sets, and have applied it to groups and BCK/BCI-algebras. In this paper, we have introduced the notions of a (limited) $T\&F$-ideal and a $T\&F$-$\circ$-subalgebra, and have investigated several properties. We have discussed relationship between a $T\&F$-subalgebra and a $T\&F$-ideal. We have provided conditions for a $T\&F$-structure (resp., $T\&F$-subalgebra) to be a $T\&F$-ideal. We have given examples to confirm that in a BCI-algebra, a $T\&F$-ideal cannot be a $T\&F$-subalgebra in general, and then we have considered conditions for a $T\&F$-ideal to be a $T\&F$-subalgebra in a BCI-algebra. We have discussed relationship between a $T\&F$-ideal and a $T\&F$-$\circ$-subalgebra in a BCK-algebra with condition (S). We have provided conditions for a $T\&F$-structure over a BCK-algebra with condition (S) to be a $T\&F$-ideal. Using the notion of $T\&F$-level sets, we have established a characterization of a $T\&F$-ideal. We have set up a new $T\&F$-ideal from the existing $T\&F$-ideal using a characterization of a $T\&F$-ideal.

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