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Dual quaternion theory over $\mathcal{HGC}$ numbers

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Abstract

Knowing the applications of quaternions in various fields, such as robotics, navigation, computer visualization and animation, in this study, we give the theory of dual quaternions considering Hyperbolic-Generalized Complex ($\mathcal{HGC}$) numbers as coefficients via generalized complex and hyperbolic numbers. We account for how $\mathcal{HGC}$ number theory can extend dual quaternions to $\mathcal{HGC}$ dual quaternions. Some related theoretical results with $\mathcal{HGC}$ Fibonacci/Lucas numbers are established, including their dual quaternions. Given $\mathcal{HGC}$ Fibonacci/Lucas numbers, their special matrix correspondences have been identified and these are carried out to $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions. Furthermore, we provide a more accurate way to quickly calculate $\mathcal{HGC}$ Fibonacci numbers and associate this with $\mathcal{HGC}$ generalized Fibonacci numbers. For implementation, we produce an algorithm in Maple. Lastly, we put the theory into practice.

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1. Introduction

The real (Hamiltonian) quaternions are introduced in the form $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ are components and $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ are versors with the conditions $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, [1]. The set of real quaternions, which extends the complex numbers, is an associative but non-commutative Clifford algebra. The Hamilton quaternions are basic examples of quaternion algebra, which have turned out to be a really important concept in many fields in applied mathematics. However, lack of commutativity sometimes makes working with quaternion algebras complicated. The investigations on quaternions gave rise to thought-provoking hypercomplex studies.

Bicomplex numbers [2] are presented by replacing the real coefficients of complex numbers with complex numbers. Compared with quaternions, bicomplex numbers have the commutative property. Considering the basics of the classical theory of hypercomplex numbers, bicomplex numbers are generalized in [3]. With the help of the idea that every hypercomplex number generates its own geometry, these new numbers (generalized commutative hypercomplex numbers/commutative quaternions) are discussed to study Minkowski space-time. They are isomorphic to Segre's bicomplex numbers as a special case. Additionally, the commutative hypercomplex numbers are introduced in different dimensions in [4].

Considering components of bidimensional number systems to construct new number systems is an attractive area for researchers. From this point of view, hyperbolic-complex numbers are examined in [5-8]. n-dimensional hyperbolic-complex and bicomplex numbers are investigated in [9] and [10, 11], respectively. Bicomplex functional analysis is also explained in detail in [10]. Dual-complex numbers are examined in [8, 12-14]. Hyperbolic-dual numbers are investigated in [8], and some properties of hyperbolic-dual numbers and hyperbolic-complex numbers are examined in [15]. Bihyperbolic numbers are examined in [3, 4, 16-18]. Motivated by the above papers and using the Cayley-Dickson doubling procedure for construction, dual-generalized complex, Hyperbolic-Generalized Complex ($\mathcal{HGC}$) and complex-generalized complex numbers are investigated in [19].

There is a huge backlog and complexity in the literature about 4-dimensional numbers generated by doubling patterns. Amongst others, hypercomplex commutative numbers can be classified into three isomorphic algebra families that accept the pure hyperbolic versor as their backbone: tessarines [5] *(or hyperbolic-complex numbers, or commutative* *quaternions)* with $\mathbf{i}^2 = \mathbf{k}^2 = -1, \mathbf{j}^2 = 1$, bicomplex numbers *(or reduced biquaternions* [20]) with $\mathbf{i}^2 = \mathbf{j}^2 = -1, \mathbf{k}^2 = 1$ and complex-hyperbolic numbers $\mathbf{i}^2 = 1, \mathbf{j}^2 = \mathbf{k}^2 = -1$ where $\mathbf{ij} = \mathbf{ji} = \mathbf{k}$. Similar studies are examined according to these different nomenclatures. It is worth mentioning that the general set of commutative quaternions is algebraically isomorphic to the fundamental algebra of $\mathcal{HGC}$ numbers as in the relations between the three algebra families above.

$\mathcal{HGC}$ numbers originated by doubling hyperbolic numbers over generalized complex numbers are defined as follows:

$$\mathbb{HC}_{\mathfrak{p}} := \{w = z_1 + z_2 j : j^2 = 1, z_1, z_2 \in \mathbb{C}_{\mathfrak{p}}, j \neq \pm 1, j \notin \mathbb{R}\},$$

where $\mathbb{HC}_{\mathfrak{p}} := \mathbb{H} \otimes \mathbb{C}_{\mathfrak{p}}$, [19]. Here, $\mathbb{C}_{\mathfrak{p}}$ is the set of generalized complex numbers (for details see [6, 21, 22]):

$$\mathbb{C}_{\mathfrak{p}} := \{z = a_1 + a_2 J : J^2 = \mathfrak{p}, a_1, a_2 \in \mathbb{R}, \mathfrak{p} \in \mathbb{R}, J \notin \mathbb{R}\},$$

which analogue to the complex numbers ($\mathbb{C}$) for $\mathfrak{p} = -1$ [23], the hyperbolic numbers ($\mathbb{H}$) for $\mathfrak{p} = 1$ [24, 25] and the dual numbers[1] ($\mathbb{D}$) for $\mathfrak{p} = 0$ [27, 28]. For any $w_1 = z_{11} + z_{12} j$, $w_2 = z_{21} + z_{22} j \in \mathbb{HC}_{\mathfrak{p}}$ and $\lambda \in \mathbb{R}$, the operations are given by [19]: $w_1 = w_2 \Leftrightarrow z_{11} = z_{21}, z_{12} = z_{22}$, $w_1 + w_2 = (z_{11} + z_{21}) + (z_{12} + z_{22}) j$, $\lambda w_1 = \lambda z_{11} + \lambda z_{12} j$, $w_1 w_2 = (z_{11} z_{21} + z_{12} z_{22}) + (z_{11} z_{22} + z_{12} z_{21}) j$. Here and elsewhere, J denotes the generalized complex unit, j represents the pure hyperbolic unit and Jj represents the generalized complex-hyperbolic unit where $Jj = jJ$.

The Fibonacci and Lucas sequences are the sequences that fascinated the researchers considerably for centuries and a tremendous amount of information is available related to them. They are regarded as two shining stars in a wide array of integer sequences. The largest study of Fibonacci and Lucas sequences is Koshy's book [29], which includes many sources. It is natural to study Fibonacci and Lucas versions of above-mentioned types of numbers and quaternions. One can find Fibonacci and Lucas quaternions in some earlier works: [30-32]. In other respects, bicomplex Fibonacci numbers are examined in [33]. In [34] bicomplex Pell/Pell-Lucas numbers and their several identities are discussed. Dual-complex Fibonacci and Lucas numbers and their properties are presented in [35]. In [36], the hyperbolic-dual versions of Fibonacci and Lucas numbers are investigated. In [37], bihyperbolic numbers of Fibonacci type are obtained.

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[1] Complex, hyperbolic and dual numbers are linked to Euclidean, Lorentzian, Galilean geometries, respectively. See detailed information in [23-28].

In [38], some new identities with respect to bihyperbolic Fibonacci and Lucas numbers are given. Also, one can find Lucas regular matrices in [39] considering sequence spaces as an another aspect.

By searching literature up to the present, one can see that the properties and applications of quaternions are remarkable areas for researchers. Just because of this reason, obtaining new results in this respect is not always an easy problem. This study aims to connect the above two concepts of $\mathcal{HGC}$ numbers and dual quaternions. Section 2 examines the basic properties of the $\mathcal{HGC}$ dual quaternions, composed of dual quaternions and $\mathcal{HGC}$ numbers. The remaining part of this paper is organized as follows: Section 3, as a further approach, presents firstly $\mathcal{HGC}$ Fibonacci/Lucas numbers obtaining several theoretical identities and secondly implements this theory to $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions. Section 3.2 presents a matrix application to calculate quickly $\mathcal{HGC}$ Fibonacci/Lucas numbers. Finally, Section 4 concludes the results and classifies the special cases.

2. $\mathcal{HGC}$ Dual Quaternions

In this section, utilizing $\mathcal{HGC}$ numbers and dual quaternions, the general expression of dual quaternions with $\mathcal{HGC}$ components (or $\mathcal{HGC}$ numbers with dual quaternions) is introduced, and their properties are obtained. Firstly, the generalized complex numbers with dual quaternions coefficients are established, and then the hyperbolic number is established over this.

The set of dual quaternions is denoted by Q_D and its elements are associative and commutative . In this set, versors satisfy the following conditions:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0,\ \mathbf{ij} = \mathbf{ji} = \mathbf{jk} = \mathbf{kj} = \mathbf{ki} = \mathbf{ik} = 0. \tag{1}$$

Definition 1: The set of dual quaternions with $\mathcal{HGC}$ components is denoted by $\hat{Q}_D$ and the mathematical formulation of $\mathcal{HGC}$ dual quaternions is of the form

$$\hat{q} = \mathbf{q} + \mathbf{q}_h j, \tag{2}$$

where $\mathbf{q} = q_0 + q_1 J$, $\mathbf{q}_h = q_2 + q_3 J$ are generalized complex dual quaternions, and q_0, q_1, q_2, q_3 are dual quaternions with the versors $\mathbf{i}, \mathbf{j}, \mathbf{k} \notin \mathbb{R}$ that satisfy the conditions in equation (1). This is similar to the Cayley–Dickson form.

Axiomatically, J and j commutes with the three quaternion versors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, that is

$$\mathbf{i}J = J\mathbf{i},\ \mathbf{j}J = J\mathbf{j},\ \mathbf{k}J = J\mathbf{k},\ \mathbf{i}j = j\mathbf{i},\ \mathbf{j}j = j\mathbf{j},\ \mathbf{k}j = j\mathbf{k}. \tag{3}$$

It is evident that for $\mathfrak{p} = 0$, usual dual operator distinct from quaternion versors.

We give a simple approach to define elementary arithmetic operations. Consider $\hat{q}, \hat{p} \in \hat{\mathcal{Q}}_{\mathcal{D}}$ and $c \in \mathbb{R}$. The equality of two extended quaternions is as follows: $\hat{q} = \hat{p} \Leftrightarrow \mathbf{q} = \mathbf{p}, \mathbf{q}_h = \mathbf{p}_h$. The addition (and hence subtraction) of $\hat{q}$ to another quaternion $\hat{p}$ acts in a componentwise way $\hat{q} + \hat{p} = (\mathbf{q} + \mathbf{p}) + (\mathbf{q}_h + \mathbf{p}_h)j$. The scalar multiplication of $\hat{q}$ with a scalar c gives another $\mathcal{HGC}$ dual quaternion $c\hat{q} = c\mathbf{q} + c\mathbf{q}_h j$. The multiplication of the two quaternions is performed as $\hat{q}\hat{p} = (\mathbf{q}\mathbf{p} + \mathbf{q}_h\mathbf{p}_h) + (\mathbf{q}\mathbf{p}_h + \mathbf{q}_h\mathbf{p})j$. Here, the multiplication follows the dual quaternion multiplication rule of the versors (see in equation (1).

Theorem 1: *$\mathcal{HGC}$ dual quaternion is algebraically isomorphic to*

$$\mathrm{A} := \left\{ \begin{bmatrix} \mathbf{q} & \mathbf{q}_h \\ \mathbf{q}_h & \mathbf{q} \end{bmatrix} : \mathbf{q}, \mathbf{q}_h \text{ are generalized complex dual quaternion} \right\}$$

through the bijective map $\mathcal{A} : \hat{\mathcal{Q}}_{\mathcal{D}} \to \mathrm{A}, \hat{q} = \mathbf{q} + \mathbf{q}_h j \mapsto \mathcal{A}_{\hat{q}} = \begin{bmatrix} \mathbf{q} & \mathbf{q}_h \\ \mathbf{q}_h & \mathbf{q} \end{bmatrix}.$

We will also write any $\hat{p} = \mathbf{p} + \mathbf{p}_h j \cong \begin{bmatrix} \mathbf{p} \\ \mathbf{p}_h \end{bmatrix}$. Upon this, the following multiplication can be written with the help of the ordinary matrix multiplication as:

$$\hat{p}\hat{q} = \hat{q}\hat{p} = \begin{bmatrix} \mathbf{q} & \mathbf{q}_h \\ \mathbf{q}_h & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{p}_h \end{bmatrix} = \mathcal{A}_{\hat{q}}\hat{p}, \tag{4}$$

where $\mathcal{A}_{\hat{q}}$ is called the fundamental matrix of $\hat{q}$. This matrix multiplication leads to the same quaternion multiplication result.

The generalized complex and hyperbolic conjugations play a significant role both for the algebraic and geometric properties of $\mathcal{HGC}$ dual quaternions. The conjugates of $\hat{q}$ are the dual quaternions

$$\hat{q}^{\dagger_1} = \overline{\mathbf{q}} + \overline{\mathbf{q}}_h j,\ \hat{q}^{\dagger_2} = \mathbf{q} - \mathbf{q}_h j,\ \hat{q}^{\dagger_3} = \overline{\mathbf{q}} - \overline{\mathbf{q}}_h j,$$

where the overline is the standard generalized complex conjugate. These different conjugates give rise to the related norm operations as $N_{\hat{q}}^{\dagger} = \hat{q}\hat{q}^{\dagger_s}$,

$s = 1, 2, 3$. Also, the inverse^{†_s} of $\hat{q}$ is defined by $\hat{q}^{-1}_{\dagger_s} = \dfrac{\hat{q}^{\dagger_s}}{N^{\dagger_s}_{\hat{q}}}$, where $N^{\dagger_s}_{\hat{q}}$ non-null[2] number.

Proposition 1: Let $\hat{q}, \hat{p} \in \hat{\mathcal{Q}}_{\mathcal{D}}$ and $c_1, c_2 \in \mathbb{R}$. The conjugates and norms satisfy the following properties for $s = 1, 2, 3$:

(1) $(\hat{q}^{\dagger_s})^{\dagger_s} = \hat{q}$

(2) $(c_1\hat{q} \pm c_2\hat{p})^{\dagger_s} = c_1\hat{q}^{\dagger_s} \pm c_2\hat{p}^{\dagger_s}$,

(3) $(\hat{q}\,\hat{p})^{\dagger_s} = \hat{q}^{\dagger_s}\,\hat{p}^{\dagger_s}$,

(4) $\hat{q} + \hat{q}^{\dagger_1} = 2(q_0 + q_2 j)$,

(5) $\hat{q} + \hat{q}^{\dagger_2} = 2(q_0 + q_1 J) = 2\mathbf{q}$,

(6) $\hat{q} + \hat{q}^{\dagger_3} = 2(q_0 + q_3 Jj)$,

(7) $N^{\dagger_s}_{c_1\hat{q}} = c_1^2 N^{\dagger_s}_{\hat{q}}$,

(8) $N^{\dagger_s}_{\hat{q}\hat{p}} = N^{\dagger_s}_{\hat{q}} N^{\dagger_s}_{\hat{p}}$.

Theorem 2: *For $\hat{q}, \hat{p} \in \hat{\mathcal{Q}}_{\mathcal{D}}$ and $\lambda \in \mathbb{R}$, the fundamental matrix satisfies the following universal identities:*

(1) $\hat{q} = \hat{p} \Leftrightarrow \mathcal{A}_{\hat{q}} = \mathcal{A}_{\hat{p}}$,

(2) $\mathcal{A}_{(\hat{q}+\hat{p})} = \mathcal{A}_{\hat{q}} + \mathcal{A}_{\hat{p}}$,

(3) $\mathcal{A}_{(\hat{q}\hat{p})} = \mathcal{A}_{(\mathcal{A}_{\hat{q}}\hat{p})} = \mathcal{A}_{\hat{q}}\mathcal{A}_{\hat{p}}$,

(4) $\mathcal{A}_{\lambda\hat{q}} = \lambda\mathcal{A}_{\hat{q}}$,

(5) $\mathcal{A}_1 = I_2$,

(6) $\hat{q}$ is invertible $\Leftrightarrow \mathcal{A}_{\hat{q}}$ is invertible, in which case, $\mathcal{A}_{\hat{q}^{-1}_{\dagger_2}} = \mathcal{A}^{-1}_{\hat{q}}$,

(7) $det\mathcal{A}_{\hat{q}} = \mathbf{q}^2 - \mathbf{q}_h^2 = \hat{q}\hat{q}^{\dagger_2}$, $tr\mathcal{A}_{\hat{q}} = \hat{q} + \hat{q}^{\dagger_2}$.

Proof: According to the fundamental matrix, the proof is quite easy to verify.

Example 1: Consider $\hat{q} = (1 + \mathbf{k}J) + (2 - \mathbf{i}J)j$. According to $\mathcal{A}_{\hat{q}} = \begin{bmatrix} 1+\mathbf{k}J & 2-\mathbf{i}J \\ 2-\mathbf{i}J & 1+\mathbf{k}J \end{bmatrix}$, and $det\mathcal{A}_{\hat{q}} = -3 + 2\left(2\mathbf{i} + \mathbf{k}\right)J$ (it is a non-null number), we can assert that $\mathcal{A}_{\hat{q}}$ is invertible and its inverse is

[2] Null numbers are characterized by having zero norm in $\mathcal{HGC}$.

$$\mathcal{A}_{\hat{q}}^{-1} = \frac{1}{-3+2(2\mathbf{i}+\mathbf{k})J}\begin{bmatrix} 1+\mathbf{k}J & -2+\mathbf{i}J \\ -2+\mathbf{i}J & 1+\mathbf{k}J \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -3-(4\mathbf{i}+5\mathbf{k})J & 6+(5\mathbf{i}+4\mathbf{k})J \\ 6+(5\mathbf{i}+4\mathbf{k})J & -3-(4\mathbf{i}+5\mathbf{k})J \end{bmatrix}.$$

Moreover, the inverse ^{†_2} of is $\hat{q}_{\dagger_2}^{-1} = \frac{1}{9}((-3-(4\mathbf{i}+5\mathbf{k})J)+(6+(5\mathbf{i}+4\mathbf{k})J)j).$ Based on the map $\mathcal{A}$, equality $\mathcal{A}_{\hat{q}_{\dagger_2}^{-1}} = \mathcal{A}_{\hat{q}}^{-1}$ is verified.

One can present that $\hat{q} = \mathbf{q} + \mathbf{q}_h j$ is rewritten as $\hat{q} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where a_0, a_1, a_2, a_3 are $\mathcal{HGC}$ numbers as an alternative statement. The $\mathcal{HGC}$ scalar part $S_{\hat{q}}$ is a_0 and the $\mathcal{HGC}$ vector part $V_{\hat{q}} = (a_1, a_2, a_3)$. The basic quaternion conjugate is defined as $\tilde{\hat{q}} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$; i.e., $\tilde{\hat{q}} = S_{\hat{q}} - V_{\hat{q}}$. Within the framework of the quaternion theory, the quaternionic inverse is $\hat{q}^{-1} = \frac{\tilde{\hat{q}}}{N_{\hat{q}}}$ for non-null $N_{\hat{q}}$. Here, $N_{\hat{q}} = \hat{q}\tilde{\hat{q}}$ is the norm of $\hat{q}$. The quaternionic conjugate and the norm satisfy the above fundamental properties.

Another significant thing to note is the similarity of $\mathcal{HGC}$ dual quaternions.

Definition 2: For $\hat{q}, \hat{p} \in \hat{\mathcal{Q}}_D$, if $\hat{q}$ and $\hat{p}$ are similar if there exists $\hat{r} \in \hat{\mathcal{Q}}_D$ such that $\hat{r}^{-1}\hat{q}\hat{r} = \hat{p}$.

Theorem 3: *If $\hat{q}$ and $\hat{p}$ are similar $\mathcal{HGC}$ dual quaternions, then $N_{\hat{q}} = N_{\hat{p}}$ or $S_{\hat{q}}^2 = S_{\hat{p}}^2$.*

Example 2: The quaternions $\hat{q} = (1-J+j)+(-3+Jj)\mathbf{i}+(-1+j-Jj)\mathbf{j}$ and $\hat{p} = (-1+J-j)+(J+j-Jj)\mathbf{i}+(1-Jj)\mathbf{k}$ are similar $\mathcal{HGC}$ dual quaternions.

Over and above, $\hat{\mathcal{Q}}_D$ is an abelian group and a commutative ring with unity and is a vector space over $\mathbb{R}$ with base $\{1, J, j, Jj, \mathbf{i}, J\mathbf{i}, j\mathbf{i}, Jj\mathbf{i}, \mathbf{j}, J\mathbf{j}, j\mathbf{j}, Jj\mathbf{j}, \mathbf{k}, J\mathbf{k}, j\mathbf{k}, Jj\mathbf{k}\}$. It also has module structures on different sets. It is an $8-$dimensional module over $\mathbb{C}_\mathrm{p}$ with base $\{1, j, \mathbf{i}, j\mathbf{i}, \mathbf{j}, j\mathbf{j}, \mathbf{k}, j\mathbf{k}\}$, a $4-$dimensional module over $\mathbb{HC}_\mathrm{p}$ with base $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, over Q_D with base $\{1, J, j, Jj\}$. Finally, it is a $2-$dimensional module over generalized complex dual quaternions with base $\{1, j\}$.

2.1 *Matrix Correspondences*

Theorem 4: $\hat{\mathcal{Q}}_{\mathcal{D}}$ is the subset of 4×4 matrices with $\mathcal{HGC}$ numbers (as dual quaternion).

Proof: Consider the linear transformation $\mathcal{B}_{\hat{q}}(\hat{p})=\hat{q}\hat{p}$ for every $\hat{q}=a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\in\hat{\mathcal{Q}}_{\mathcal{D}}$. The $\mathcal{HGC}$ matrix correspondence of $\hat{q}$ concerning the standard quaternion basis $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ can be obtained as follows: $\mathcal{B}_{\hat{q}}(1)=\hat{q}=a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathcal{B}_{\hat{q}}(\mathbf{i})=\hat{q}\mathbf{i}=a_0\mathbf{i}$, $\mathcal{B}_{\hat{q}}(\mathbf{j})=\hat{q}\mathbf{j}=a_0\mathbf{j}$, $\mathcal{B}_{\hat{q}}(\mathbf{k})=\hat{q}\mathbf{k}=a_0\mathbf{k}$. As a result of this, we have:

$$\mathcal{B}_{\hat{q}}=\begin{bmatrix} a_0 & 0 & 0 & 0\\ a_1 & a_0 & 0 & 0\\ a_2 & 0 & a_0 & 0\\ a_3 & 0 & 0 & a_0\end{bmatrix}.$$

Here, $\mathcal{B}$ is an isomorphism[3] between $\hat{\mathcal{Q}}_{\mathcal{D}}$ and the set consists of the elements $\mathcal{B}_{\hat{q}}$.

It is a simple matter to give the proofs of Theorem 5-Theorem 7 using $\mathcal{B}_{\hat{q}}$. Also Theorem 7 is an analog of Theorem 2.

Theorem 5: *For* $\hat{q}=a_0+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}\in\hat{\mathcal{Q}}_{\mathcal{D}}$, *then* $\mathcal{B}_{\tilde{\hat{q}}}=\sigma\mathcal{B}_{\hat{q}}\sigma$, *where* $\sigma=diag$ $(1,-1,-1,-1)$.

Theorem 6: *For* $\hat{q}\in\hat{\mathcal{Q}}_{\mathcal{D}}$, *then we have:* $\mathcal{B}_{\hat{q}^{-1}}=\frac{1}{\sqrt{det(\mathcal{B}_{\hat{q}})}}\mathcal{B}_{\tilde{\hat{q}}}$.

Theorem 7: *For any* $\hat{q},\hat{p}\in\hat{\mathcal{Q}}_{\mathcal{D}}$ *and* $\lambda\in\mathbb{R}$, *the following properties are satisfied:*

(1) $\hat{q}=\hat{p}\Leftrightarrow\mathcal{B}_{\hat{q}}=\mathcal{B}_{\hat{p}}$,

(2) $\mathcal{B}_{(\hat{q}+\hat{p})}=\mathcal{B}_{\hat{q}}+\mathcal{B}_{\hat{p}}$,

(3) $\mathcal{B}_{(\hat{q}\hat{p})}=\mathcal{B}_{(\mathcal{B}_{\hat{q}}\hat{p})}=\mathcal{B}_{\hat{q}}\mathcal{B}_{\hat{p}}$,

(4) $\mathcal{B}_{\lambda\hat{q}}=\lambda\mathcal{B}_{\hat{q}}$,

(5) $\mathcal{B}_1=I_4$,

(6) $\hat{q}$ is invertible $\Leftrightarrow\mathcal{B}_{\hat{q}}$ is invertible, in which case, $\mathcal{B}_{\hat{q}^{-1}}=\mathcal{B}_{\hat{q}}^{-1}$,

(7) $det\mathcal{B}_{\hat{q}}=a_0^4=S_{\hat{q}}^4$, $tr\mathcal{B}_{\hat{q}}=4S_{\hat{q}}$.

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[3] The matrix $\mathcal{B}_{\hat{q}}$ is called left matrix representations of $\hat{q}$. Also, the right matrix representation of $\hat{q}$ is equal to its left representation, as is easy to check.

Remark 1: Considering $\mathcal{B}_q$, any $\mathcal{HGC}$ dual quaternion $\hat{q}$ can be represented by a 16×16 real matrix.

Another 4×4 matrix representation of $\hat{q}$ is that:

Theorem 8: *$\hat{\mathcal{Q}}_D$ is the subset of 4×4 matrices with dual quaternions (as $\mathcal{HGC}$ number).*

Proof: For any $\hat{q}=q_0+q_1J+q_2j+q_3Jj\in\hat{\mathcal{Q}}_D$, suppose that $\mathcal{C}_{\hat{q}}(\hat{p})=\hat{q}\hat{p}$ is a linear transformation. The dual quaternion matrix correspondence of $\hat{q}$ for the standard $\mathcal{HGC}$ basis $\{1,J,j,Jj\}$ can be obtained as follows:

$$\begin{cases} \mathcal{C}_{\hat{q}}(1) &= \hat{q} &= q_0+q_1J+q_2j+q_3Jj \\ \mathcal{C}_{\hat{q}}(J) &= \hat{q}J &= \mathfrak{p}q_1+q_0J+\mathfrak{p}q_3j+q_2Jj \\ \mathcal{C}_{\hat{q}}(j) &= \hat{q}j &= q_2+q_3J+q_0j+q_1Jj \\ \mathcal{C}_{\hat{q}}(Jj) &= \hat{q}Jj &= \mathfrak{p}q_3+q_2J+\mathfrak{p}q_1j+q_0Jj. \end{cases}$$

Consequently, we get $\mathcal{C}$ is an isomorphism between $\hat{\mathcal{Q}}_D$ and the set

$$\mathrm{C}:=\left\{\mathcal{C}_{\hat{q}}=\begin{bmatrix} q_0 & \mathfrak{p}q_1 & q_2 & \mathfrak{p}q_3 \\ q_1 & q_0 & q_3 & q_2 \\ q_2 & \mathfrak{p}q_3 & q_0 & \mathfrak{p}q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} : q_0,q_1,q_2,q_3\in\mathcal{Q}_D\right\}.$$

Theorem 7 is also valid for the matrix $\mathcal{C}_{\hat{q}}$ and also the part 7) can be calculated easily.

Corollary 1: *The matrix forms satisfy the following statements:*

- $\mathcal{A}_{\hat{q}}$ is also of the form $\mathcal{A}_{\hat{q}}=\mathbf{q}I_2+\mathbf{q}_hj$, where $j=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. There is no doubt that these matrices are isomorphic to the hyperbolic basis.
- $\mathcal{B}_{\hat{q}}$ is also of the form

$$\mathcal{B}_{\hat{q}}=a_0I_4+a_1\mathcal{I}+a_2\mathcal{J}+a_3\mathcal{K},$$

where $\mathcal{I}=\begin{bmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$, $\mathcal{J}=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}$, $\mathcal{K}=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}$. It is evident that these matrices are isomorphic to the dual quaternion basis.

- $\mathcal{C}_{\hat{q}}$ is also of the form
$$\mathcal{C}_{\hat{q}} = q_0 I_4 + q_1 \mathrm{J} + q_2 \mathrm{j} + q_3 \mathrm{Jj},$$
where $\mathrm{J} = \begin{bmatrix} 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{p} \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\mathrm{j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $\mathrm{Jj} = \begin{bmatrix} 0 & 0 & 0 & \mathfrak{p} \\ 0 & 0 & 1 & 0 \\ 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. One can see that these matrices are obviously isomorphic to the hyperbolic-generalized complex basis.

3.1 *Examination of $\mathcal{HGC}$ Fibonacci/Lucas Numbers*

As in the theory of the previous section, the $\mathcal{HGC}$ Fibonacci/Lucas numbers are defined using the hyperbolic number theory over the generalized complex Fibonacci/Lucas numbers.

Definition 3: The *nth* $\mathcal{HGC}$ Fibonacci/Lucas numbers are defined by:
$$\hat{\mathcal{F}}_n = \hat{F}_n + \hat{F}_{n+1} j \text{ and } \hat{\mathcal{L}}_n = \hat{L}_n + \hat{L}_{n+1} j,$$
where $\hat{F}_n = F_n + F_{n+1} J$ is a generalized complex Fibonacci number[4] and $\hat{L}_n = L_n + L_{n+1} J$ is generalized complex Lucas number. We will denote the set of $\mathcal{HGC}$ Fibonacci and $\mathcal{HGC}$ Lucas numbers by the notations $\mathbb{HC}_{\mathfrak{p}}\mathbb{F}$ and $\mathbb{HC}_{\mathfrak{p}}\mathbb{L}$, respectively.

In the sequel, we always assume $\hat{\mathcal{F}}_n$, $\hat{\mathcal{F}}_m \in \mathbb{HC}_{\mathfrak{p}}\mathbb{F}$ and $\hat{\mathcal{L}}_n$, $\hat{\mathcal{L}}_m \in \mathbb{HC}_{\mathfrak{p}}\mathbb{L}$. The standard algebraic operations are defined as follows: $\hat{\mathcal{F}}_n = \hat{\mathcal{F}}_m \Leftrightarrow \hat{F}_n = \hat{F}_m, \hat{F}_{n+1} = \hat{F}_{m+1}$. Addition (and hence subtraction) acts component-wise, i.e., $\hat{\mathcal{F}}_n + \hat{\mathcal{F}}_m = (\hat{F}_n + \hat{F}_m) + (\hat{F}_{n+1} + \hat{F}_{m+1}) j$. The product of two $\mathcal{HGC}$ Fibonacci numbers: $\hat{\mathcal{F}}_n \hat{\mathcal{F}}_m = (\hat{F}_n \hat{F}_m + \hat{F}_{n+1} \hat{F}_{m+1}) + (\hat{F}_n \hat{F}_{m+1} + \hat{F}_{n+1} \hat{F}_m) j$. The conjugates[5] of $\hat{\mathcal{F}}_n$ are defined by:
$$\hat{\mathcal{F}}_n^{\dagger_1} = \overline{\hat{F}}_n + \overline{\hat{F}}_{n+1} j,\ \hat{\mathcal{F}}_n^{\dagger_2} = \hat{F}_n - \hat{F}_{n+1} j,\ \hat{\mathcal{F}}_n^{\dagger_3} = \overline{\hat{F}}_n - \overline{\hat{F}}_{n+1} j.$$

The norms of $\hat{\mathcal{F}}_n$ are given by: $N^{\dagger_s}_{\hat{\mathcal{F}}_n} = \hat{\mathcal{F}}_n \hat{\mathcal{F}}_n^{\dagger_s}$ for $s = 1, 2, 3$. The standard operations for $\mathcal{HGC}$ Lucas numbers are analog of the above operations.

Any two consecutive $\mathcal{HGC}$ Fibonacci numbers have no common integer factors, i.e., one of them is not an integer multiple of the other.

---

4 See definition and basic properties of generalized complex Fibonacci numbers $\hat{F}_n = F_n + F_{n+1} J$, $J^2 = \mathfrak{p}$ in [44].

5 The overline represents the standard generalized complex conjugate.

Proposition 2: The below conjugate properties can be given:

(1) $\hat{\mathcal{F}}_n + \hat{\mathcal{F}}_n^{\dagger_1} = 2(F_n + F_{n+1} j),$

(2) $\hat{\mathcal{F}}_n \hat{\mathcal{F}}_n^{\dagger_1} = F_{2n+1} - \mathfrak{p}F_{2n+3} + 2(F_n F_{n+1} - \mathfrak{p}F_{n+1}F_{n+2})j,$

(3) $\hat{\mathcal{F}}_n + \hat{\mathcal{F}}_n^{\dagger_2} = 2\hat{F}_n,$

(4) $\hat{\mathcal{F}}_n \hat{\mathcal{F}}_n^{\dagger_2} = -(F_{n+2}F_{n-1} + \mathfrak{p}F_{n+3}F_n) + 2(F_n F_{n+1} - F_{n+1}F_{n+2})J,$

(5) $\hat{\mathcal{F}}_n + \hat{\mathcal{F}}_n^{\dagger_3} = 2(F_n + F_{n+2} Jj),$

(6) $\hat{\mathcal{F}}_n \hat{\mathcal{F}}_n^{\dagger_3} = (-F_{n+2}F_{n-1} + \mathfrak{p}F_{n+3}F_n) + 2(-1)^{n+1} Jj.$

The following elementary properties will be useful in the sequel. They extend the familiar relations of Fibonacci/Lucas numbers to $\mathcal{HGC}$ versions. We omit the proofs of some of the theorems because they are quite easy to verify using Fibonacci/Lucas relations (see in [29]).

Theorem 9: *For $n, m, r \geq 0$, the following additional relations hold:*

(1) $\hat{\mathcal{F}}_n + \hat{\mathcal{F}}_{n+1} = \hat{\mathcal{F}}_{n+2},$

(2) $\hat{\mathcal{F}}_{n+r} + \hat{\mathcal{F}}_{n-r} = \begin{cases} L_r \hat{\mathcal{F}}_n, r = 2k \\ F_r \hat{\mathcal{L}}_n, r = 2k+1, \end{cases}$

(3) $\hat{\mathcal{F}}_{n+r} - \hat{\mathcal{F}}_{n-r} = \begin{cases} F_r \hat{\mathcal{L}}_n, r = 2k \\ L_r \hat{\mathcal{F}}_n, r = 2k+1, \end{cases}$

(4) $\hat{\mathcal{L}}_n + \hat{\mathcal{L}}_{n+1} = \hat{\mathcal{L}}_{n+2},$

(5) $\hat{\mathcal{L}}_{n+r} + \hat{\mathcal{L}}_{n-r} = \begin{cases} L_r \hat{\mathcal{L}}_n, r = 2k \\ 5F_r \hat{\mathcal{F}}_n, r = 2k+1, \end{cases}$

(6) $\hat{\mathcal{L}}_{n+r} - \hat{\mathcal{L}}_{n-r} = \begin{cases} 5F_r \hat{\mathcal{F}}_n, r = 2k \\ L_r \hat{\mathcal{L}}_n, r = 2k+1. \end{cases}$

Theorem 10: For $n, m, r \geq 0$, the following order 2 formulas hold:

(1) $\hat{\mathcal{F}}_n^2 + \hat{\mathcal{F}}_{n+1}^2 = L_{2n+2} + \mathfrak{p}L_{2n+4} + 2L_{2n+3}J + 2(F_{2n+2} + \mathfrak{p}F_{2n+4} + 2F_{2n+3}J)j,$

(2) $\hat{\mathcal{F}}_{n+1}^2 - \hat{\mathcal{F}}_{n-1}^2 = L_{2n+1} + \mathfrak{p}L_{2n+3} + 2L_{2n+2}J + 2(F_{2n+1} + \mathfrak{p}F_{2n+3} + 2F_{2n+2}J)j,$

(3) $\hat{\mathcal{F}}_n \hat{\mathcal{F}}_m + \hat{\mathcal{F}}_{n+1}\hat{\mathcal{F}}_{m+1} =$
$L_{m+n+2} + \mathfrak{p}L_{m+n+4} + 2L_{m+n+3}J + 2(F_{m+n+2} + \mathfrak{p}F_{m+n+4} + 2F_{m+n+3}J)j,$

(4) $\hat{\mathcal{L}}_n^2 + \hat{\mathcal{L}}_{n+1}^2 = 5(\hat{\mathcal{F}}_n^2 + \hat{\mathcal{F}}_{n+1}^2),$

(5) $\hat{\mathcal{L}}^2_{n+1} - \hat{\mathcal{L}}^2_{n-1} = 5(\hat{\mathcal{F}}^2_{n+1} - \hat{\mathcal{F}}^2_{n-1})$,
(6) $\hat{\mathcal{L}}_n \hat{\mathcal{L}}_m + \hat{\mathcal{L}}_{n+1} \hat{\mathcal{L}}_{m+1} = 5(\hat{\mathcal{F}}_n \hat{\mathcal{F}}_m + \hat{\mathcal{F}}_{n+1} \hat{\mathcal{F}}_{m+1})$.

Parts 3) and 6) in Theorem 10 are often referred to as Hornsberger's identities.

Theorem 11: *The generating functions are as follows:*

- $\mathrm{F}(x) = \sum_{n=0}^{\infty} \hat{\mathcal{F}}_n x^n = \frac{\hat{\mathcal{F}}_0 + (\hat{\mathcal{F}}_1 - \hat{\mathcal{F}}_0)x}{1-x-x^2}$,

- $\mathrm{L}(x) = \sum_{n=0}^{\infty} \hat{\mathcal{L}}_n x^n = \frac{\hat{\mathcal{L}}_0 + (\hat{\mathcal{L}}_1 - \hat{\mathcal{L}}_0)x}{1-x-x^2}$.

Proof: Assume that $\mathrm{F}(x) = \sum_{n=0}^{\infty} \hat{\mathcal{F}}_n x^n$ is a generating function of $\hat{\mathcal{F}}_n$. Then, we have:

$$\begin{aligned}(1-x-x^2)\mathrm{F}(x) = \; & \hat{\mathcal{F}}_0 + (\hat{\mathcal{F}}_1 - \hat{\mathcal{F}}_0)x + (\hat{\mathcal{F}}_2 - \hat{\mathcal{F}}_1 - \hat{\mathcal{F}}_0)x^2 + (\hat{\mathcal{F}}_3 - \hat{\mathcal{F}}_2 - \hat{\mathcal{F}}_1)x^3 \\ & + \ldots + (\hat{\mathcal{F}}_{n+3} - \hat{\mathcal{F}}_{n+2} - \hat{\mathcal{F}}_{n+1})x^{n+3} + \ldots\end{aligned}$$

Then using Theorem 9, the formula for $\mathrm{F}(x)$ is clear. The same proof works for $\mathrm{L}(x)$.

In the sequel, $\hat{\alpha} = \alpha^* + \alpha\alpha^* j$, $\hat{\beta} = \beta^* + \beta\beta^* j$, $\alpha^* = 1 + \alpha J$, and $\beta^* = 1 + \beta J$ with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Recall the Binet's formulas: $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, $L_n = \alpha^n + \beta^n$.

Theorem 12: For $n \geq 1$, the Golden-like ratio approximation is

$$\lim_{n\to\infty} \frac{\hat{\mathcal{F}}_n}{F_n} = \lim_{n\to\infty} \frac{\hat{\mathcal{L}}_n}{L_n} = \hat{\alpha}.$$

Proof: Using the Golden ratio (see in [29]) $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha$, the proof is clear.

Lemma 1: *For* $n \geq 1$, *the Golden-like ratio relations are*

- $\hat{\alpha}\alpha^{n} = \alpha\hat{\mathcal{F}}_{n} + \hat{\mathcal{F}}_{n-1}$,
- $\hat{\beta}\beta^{n} = \beta\hat{\mathcal{F}}_{n} + \hat{\mathcal{F}}_{n-1}$
- $\sqrt{5}\hat{\alpha}\alpha^{n} = \alpha\hat{\mathcal{L}}_{n} + \hat{\mathcal{L}}_{n-1}$,
- $\sqrt{5}\hat{\beta}\beta^{n} = -\beta\hat{\mathcal{L}}_{n} - \hat{\mathcal{L}}_{n-1}$.

Theorem 13: *For $n \geq 1$, the general Binet's formulas can be calculated as follows:*

- $\hat{\mathcal{F}}_{n} = \frac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}$,
- $\hat{\mathcal{L}}_{n} = \hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}$.

The study [45] motivates the exponential generating function of $\mathcal{HGC}$ Fibonacci/Lucas numbers.

Theorem 14: The *exponential generating functions* are as follows:

- $\sum_{n=0}^{\infty} \hat{\mathcal{F}}_{n} \frac{t^{n}}{n!} = \frac{\hat{\alpha}e^{\alpha t} - \hat{\beta}e^{\beta t}}{\alpha-\beta}$,
- $\sum_{n=0}^{\infty} \hat{\mathcal{L}}_{n} \frac{t^{n}}{n!} = \hat{\alpha}e^{\alpha t} + \hat{\beta}e^{\beta t}$.

Proof: Considering Theorem 13, we have:

$$\sum_{n=0}^{\infty} \hat{\mathcal{F}}_{n} \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta} \right) \frac{t^{n}}{n!} = \frac{\hat{\alpha}}{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(\alpha t)^{n}}{n!} - \frac{\hat{\beta}}{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(\beta t)^{n}}{n!} = \frac{\hat{\alpha}e^{\alpha t} - \hat{\beta}e^{\beta t}}{\alpha-\beta}.$$

Theorem 15: *For $n, m \geq 0$, the special case of Tagiuri identities can be given as follows:*

- $\hat{\mathcal{F}}_{m}\hat{\mathcal{F}}_{n} - \hat{\mathcal{F}}_{m+r}\hat{\mathcal{F}}_{n-r} = (-1)^{n-r} F_{m-n+r} F_{r} [(1-\mathfrak{p}) + J] j$,
- $\hat{\mathcal{L}}_{m}\hat{\mathcal{L}}_{n} - \hat{\mathcal{L}}_{m+r}\hat{\mathcal{L}}_{n-r} = 5(-1)^{n-r+1} F_{m-n+r} F_{r} [(1-\mathfrak{p}) + J] j$.

Proof: The proof is straightforward.

Theorem 16: *For $n, m \geq 0$, general D'Ocagne's identities can be given as follows:*

- $\hat{\mathcal{F}}_{m}\hat{\mathcal{F}}_{n+1} - \hat{\mathcal{F}}_{m+1}\hat{\mathcal{F}}_{n} = (-1)^{n} F_{m-n} [(1-\mathfrak{p}) + J] j$,
- $\hat{\mathcal{L}}_{m}\hat{\mathcal{L}}_{n+1} - \hat{\mathcal{L}}_{m+1}\hat{\mathcal{L}}_{n} = 5(-1)^{n+1} F_{m-n} [(1-\mathfrak{p}) + J] j$.

Proof: By writing $n \to n+1$ and $r = 1$ in Theorem 15, the proof is completed.

Theorem 17: The general *Catalan's identities* are as below:

- $\hat{\mathcal{F}}_n^2 - \hat{\mathcal{F}}_{n+r}\hat{\mathcal{F}}_{n-r} = (-1)^{n-r} F_r^2[(1-\mathfrak{p})+J]j,$
- $\hat{\mathcal{L}}_n^2 - \hat{\mathcal{L}}_{n+r}\hat{\mathcal{L}}_{n-r} = 5(-1)^{n-r+1} F_r^2[(1-\mathfrak{p})+J]j.$

Proof: By writing $m \to n$ in Theorem 15, the proof is obvious.

Theorem 18: The general *Cassini's identities* (sometimes called *Simson's identities*) are given as:

- $\hat{\mathcal{F}}_n^2 - \hat{\mathcal{F}}_{n+1}\hat{\mathcal{F}}_{n-1} = (-1)^{n-1}[(1-\mathfrak{p})+J]j,$
- $\hat{\mathcal{L}}_n^2 - \hat{\mathcal{L}}_{n+1}\hat{\mathcal{L}}_{n-1} = 5(-1)^{n}[(1-\mathfrak{p})+J]j.$

Proof: Substituting $r=1$ in the Catalan's identities, the proof is completed.

Theorem 19: $\forall k, n \in \mathbb{N}$, *the following summation properties are obtained:*

(1) $\sum_{n=1}^{k}\hat{\mathcal{F}}_n = \hat{\mathcal{F}}_{k+2} - \hat{\mathcal{F}}_2,$ (2) $\sum_{n=1}^{k}\hat{\mathcal{F}}_{2n} = \hat{\mathcal{F}}_{2k+1} - \hat{\mathcal{F}}_1,$

(3) $\sum_{n=1}^{k}\hat{\mathcal{F}}_{2n-1} = \hat{\mathcal{F}}_{2k} - \hat{\mathcal{F}}_0,$ (4) $\sum_{n=0}^{k}\hat{\mathcal{F}}_{2n+1} = \hat{\mathcal{F}}_{2k+2} - \hat{\mathcal{F}}_0,$

(5) $\sum_{n=0}^{k}\hat{\mathcal{F}}_{i+n} = \hat{\mathcal{F}}_{i+k+2} - \hat{\mathcal{F}}_{i+1},$ (6) $\sum_{n=1}^{k}\hat{\mathcal{L}}_n = \hat{\mathcal{L}}_{k+2} - \hat{\mathcal{L}}_2,$

(7) $\sum_{n=1}^{k}\hat{\mathcal{L}}_{2n} = \hat{\mathcal{L}}_{2k+1} - \hat{\mathcal{L}}_1,$ (8) $\sum_{n=1}^{k}\hat{\mathcal{L}}_{2n-1} = \hat{\mathcal{L}}_{2k} - \hat{\mathcal{L}}_0,$

(9) $\sum_{n=0}^{k}\hat{\mathcal{L}}_{2n+1} = \hat{\mathcal{L}}_{2k+2} - \hat{\mathcal{L}}_0,$ (10) $\sum_{n=0}^{k}\hat{\mathcal{L}}_{i+n} = \hat{\mathcal{L}}_{i+k+2} - \hat{\mathcal{L}}_{i+1}.$

Proof:

(1) Using the recurrence relation in Theorem 9-part 1), we have:

$$\hat{\mathcal{F}}_1 = \hat{\mathcal{F}}_2 - \hat{\mathcal{F}}_0, \hat{\mathcal{F}}_2 = \hat{\mathcal{F}}_3 - \hat{\mathcal{F}}_1, \ldots, \hat{\mathcal{F}}_{k-1} = \hat{\mathcal{F}}_k - \hat{\mathcal{F}}_{k-2}, \hat{\mathcal{F}}_k = \hat{\mathcal{F}}_{k+1} - \hat{\mathcal{F}}_{k-1}.$$

Then, we obtain $\sum_{n=1}^{k}\hat{\mathcal{F}}_n = \hat{\mathcal{F}}_k + \hat{\mathcal{F}}_{k+1} - (\hat{\mathcal{F}}_0 + \hat{\mathcal{F}}_1) = \hat{\mathcal{F}}_{k+2} - \hat{\mathcal{F}}_2.$

(2) Using even and odd sums of Fibonacci numbers (see in [29]), we obtain:

$$\begin{aligned}
\sum_{n=1}^{k}\hat{\mathcal{F}}_{2n} &= \sum_{n=1}^{k}F_{2n} + \sum_{n=1}^{k}F_{2n+1}J + \sum_{n=1}^{k}F_{2n+1}j + \sum_{n=1}^{k}\mathcal{F}_{2n+2}Jj,\\
&= (F_{2k+1} + F_{2k+2}J + F_{2k+2}j + F_{2k+3}Jj) - (1 + J + j + 2Jj)\\
&= \hat{\mathcal{F}}_{2k+1} - \hat{\mathcal{F}}_1
\end{aligned}$$

The other parts can be proved similarly using the study [29].

Theorem 20: $\forall k, n \in \mathbb{N}$, *the following binomial properties are given:*

1. $\sum_{n=0}^{k} \binom{k}{n} \hat{\mathcal{F}}_n = \hat{\mathcal{F}}_{2k}$,
2. $\sum_{n=0}^{k} \binom{k}{n} (-1)^n \hat{\mathcal{F}}_n = (-1)^{k-1} \hat{\mathcal{F}}_k$,
3. $\sum_{n=0}^{k} \binom{k}{n} \hat{\mathcal{L}}_n = \hat{\mathcal{L}}_{2k}$,
4. $\sum_{n=0}^{k} \binom{k}{n} (-1)^n \hat{\mathcal{L}}_n = (-1)^k \hat{\mathcal{L}}_k$.

Proof: Using the Binet formula for $\mathcal{HGC}$ Fibonacci numbers in Theorem 13, we have:

$$
\begin{aligned}
2) \sum_{n=0}^{k} \binom{k}{n} (-1)^n \hat{\mathcal{F}}_n &= \sum_{n=0}^{k} \binom{k}{n} (-1)^n \left(\frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \right) \\
&= \frac{\hat{\alpha}}{\alpha - \beta}(1-\alpha)^k - \frac{\hat{\beta}}{\alpha - \beta}(1-\beta)^k \\
&= (-1)^{k-1} \frac{\hat{\alpha}\alpha^k - \hat{\beta}\beta^k}{\alpha - \beta} \\
&= (-1)^{k-1} \hat{\mathcal{F}}_k .
\end{aligned}
$$

The other parts can be shown similarly.

3.2 *How to calculate quickly $\mathcal{HGC}$ Fibonacci /Lucas numbers*

We can present a close link between matrices and $\mathcal{HGC}$ Fibonacci numbers. As observed in Theorem 9-part 1), we can write the recurrence relation of $\mathcal{HGC}$ Fibonacci numbers in matrix notation as:

$$
\begin{bmatrix} \hat{\mathcal{F}}_{n+1} \\ \hat{\mathcal{F}}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathcal{F}}_n \\ \hat{\mathcal{F}}_{n-1} \end{bmatrix}.
$$

Each time we multiply by the matrix, we move from one term of the $\mathcal{HGC}$ Fibonacci number to the next. If we take $\hat{\mathrm{F}}_n = \begin{bmatrix} \hat{\mathcal{F}}_{n+1} \\ \hat{\mathcal{F}}_n \end{bmatrix}$ and $\mathrm{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then we have a dynamic linear system $\hat{\mathrm{F}}_n = \mathrm{Q}\hat{\mathrm{F}}_{n-1}$. This matrix, is called the $\mathrm{Q}-$matrix. Q is the transformation matrix that moves us from one solution to the next. We can see a pattern emerging. Generally, we have the following intriguing theorem.

Theorem 21: *The solution of the dynamic linear system* $\hat{\mathrm{F}}_n = \mathrm{Q}\hat{\mathrm{F}}_{n-1}$ *can also be given in terms of the powers of the* Q*-matrix. That is* $\hat{\mathrm{F}}_n = \mathrm{Q}^n\hat{\mathrm{F}}_0$*, where* $\hat{\mathrm{F}}_0 = \begin{bmatrix} 1+J+j+2Jj \\ J+j+Jj \end{bmatrix}$ *is the initial solution and* $\mathrm{Q}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ *(see details Q and* Q^n *in [29]).*

To implement the presented matrix method above, we produce an algorithm for $\hat{\mathrm{F}}_5$ in Maple.

Algorithm 1: How to calculate quickly $\mathcal{HGC}$ Fibonacci number

```
restart
with (linalg):
interface(rtablesize=infinity):
n:=5;
Q:=Matrix(fibonacci(2));
Qn:=Q^n:
Qn;
F0:=Matrix(2,1):
F0[1,1]:=(1+J+j+2*J*j):
F0[2,1]:=(J+j+J*j):
F0;
Fn:=Matrix(2,2):
Fn:=Qn.F0;
```

Also, Theorem 21 provides an alternate version of Cassini's formula, as the next corollary shows.

Corollary 2: (Cassini's Identity Revisited). For $n \geq 1$, $\hat{\mathcal{F}}_n^2 - \hat{\mathcal{F}}_{n+1}\hat{\mathcal{F}}_{n-1} = det\mathrm{Q}^n.det\mathbf{Q}$, where $\mathbf{Q} = \begin{bmatrix} \hat{\mathcal{F}}_2 & \hat{\mathcal{F}}_1 \\ \hat{\mathcal{F}}_1 & \hat{\mathcal{F}}_0 \end{bmatrix}$.

We can apply the $\mathrm{Q}-$matrix to derive $\mathcal{HGC}$ Fibonacci number identities. They are basically the similar. It is worth noting that this approach can be extended to $\mathcal{HGC}$ Lucas numbers. From Theorem 9, we can write the following:

$$\begin{bmatrix} \hat{\mathcal{L}}_{n+1} \\ \hat{\mathcal{L}}_{n} \end{bmatrix} = \mathrm{Q}_L \begin{bmatrix} \hat{\mathcal{F}}_{n} \\ \hat{\mathcal{F}}_{n-1} \end{bmatrix} \text{ and } 5\begin{bmatrix} \hat{\mathcal{F}}_{n+1} \\ \hat{\mathcal{F}}_{n} \end{bmatrix} = \mathrm{Q}_L \begin{bmatrix} \hat{\mathcal{L}}_{n} \\ \hat{\mathcal{L}}_{n-1} \end{bmatrix},$$

where $\mathrm{Q}_L = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ (see [46] for more details about Q_L.)

$\mathcal{HGC}$ Generalized Fibonacci Transform

We can study $\mathcal{HGC}$ generalized Fibonacci number $\hat{\mathcal{G}}_n$ by investigating a $\mathcal{HGC}$ number sequence that satisfies the $\mathcal{HGC}$ Fibonacci recurrence relation, but with arbitrary initial conditions.

Definition 4: For $n \geq 3$, the $\mathcal{HGC}$ generalized Fibonacci sequence is defined as $\hat{\mathcal{G}}_n = \hat{G}_n + \hat{G}_{n+1} j$, where $\hat{G}_n = G_n + G_{n+1} J$ and satisfy the relation $\hat{\mathcal{G}}_n = \hat{\mathcal{G}}_{n-1} + \hat{\mathcal{G}}_{n-2}$, where $G_1 = \mathrm{a}$ and $G_2 = \mathrm{b}$.

If you take a close look at the coefficients of a and b in the various terms of this sequence, you can realize the interesting pattern: the coefficients are $\mathcal{HGC}$ Fibonacci numbers. In fact, we can pinpoint these coefficients.

Theorem 22: *For $n \geq 3$, the $\mathcal{HGC}$ generalized Fibonacci recurrence is in the form:*

$$\hat{\mathcal{G}}_n = \mathrm{a}\hat{\mathcal{F}}_{n-2} + \mathrm{b}\hat{\mathcal{F}}_{n-1}.$$

When viewed from this aspect, we could extrapolate the above idea to $\mathcal{HGC}$ generalized Fibonacci numbers using the following generalized matrix form: $\hat{\mathcal{G}}_n = [\mathrm{a} \;\; \mathrm{b}] \begin{bmatrix} \hat{\mathcal{F}}_{n-2} \\ \hat{\mathcal{F}}_{n-1} \end{bmatrix}$.

3.3 $\mathcal{HGC}$ *Fibonacci/Lucas Dual Quaternions*

This section applies the above results to the construction of $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions.

Definition 5: The $\mathcal{HGC}$ numbers with Fibonacci dual quaternion coefficient is defined as follows:

$$\hat{\mathcal{Q}}_n = \mathbf{Q}_n + \mathbf{Q}_{n+1} j,$$

where $\mathbf{Q}_n = Q_n + Q_{n+1}J$, $Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+1}\mathbf{j} + F_{n+2}\mathbf{k}$ is the nth Fibonacci dual quaternion[6]. Similarly, the $\mathcal{HGC}$ numbers with Lucas dual quaternion coefficient is of the form:

$$\hat{\mathcal{K}}_n = \mathbf{K}_n + \mathbf{K}_{n+1}j,$$

where $\mathbf{K}_n = K_n + K_{n+1}J$, $K_n = L_n + L_{n+1}\mathbf{i} + L_{n+1}\mathbf{j} + L_{n+2}\mathbf{k}$ is the nth Lucas dual quaternion. Here, the versors $\mathbf{i},\mathbf{j},\mathbf{k} \notin \mathbb{R}$ satisfy the conditions in equation (1). It is worthy to note that equation (3) is valid. We write the notations $Q\mathbb{HC}_{\mathfrak{p}}$ and $K\mathbb{HC}_{\mathfrak{p}}$ for the set of $\mathcal{HGC}$ Fibonacci dual quaternions and $\mathcal{HGC}$ Lucas dual quaternions, respectively.

In the sequel, we assume that $\hat{\mathcal{Q}}_n, \hat{\mathcal{Q}}_m \in Q\mathbb{HC}_{\mathfrak{p}}$, $\hat{\mathcal{K}}_n, \hat{\mathcal{K}}_m \in K\mathbb{HC}_{\mathfrak{p}}$ and $c \in \mathbb{R}$. Let us give a simple way to define elementary arithmetic operations: $\hat{\mathcal{Q}}_n = \hat{\mathcal{Q}}_m \Leftrightarrow \mathbf{Q}_n = \mathbf{Q}_m$, $\mathbf{Q}_{n+1} = \mathbf{Q}_{m+1}$, $\hat{\mathcal{Q}}_n + \hat{\mathcal{Q}}_m = (\mathbf{Q}_n + \mathbf{Q}_m) + (\mathbf{Q}_{n+1} + \mathbf{Q}_{m+1})j$, $c\hat{\mathcal{Q}}_n = c\mathbf{Q}_n + c\mathbf{Q}_{n+1}$, $\hat{\mathcal{Q}}_n\hat{\mathcal{Q}}_m = (\mathbf{Q}_n\mathbf{Q}_m + \mathbf{Q}_{n+1}\mathbf{Q}_{m+1}) + (\mathbf{Q}_n\mathbf{Q}_{m+1} + \mathbf{Q}_{n+1}\mathbf{Q}_m)j$. Additionally, $\hat{\mathcal{Q}}_n$ can also be represented 2×2 matrix for the hyperbolic basis $\{1, j\}$ as $\mathcal{A}_{\hat{\mathcal{Q}}_n} = \begin{bmatrix} \mathbf{Q}_n & \mathbf{Q}_{n+1} \\ \mathbf{Q}_{n+1} & \mathbf{Q}_n \end{bmatrix}$. Using equation (4), another quaternion multiplication formulae also can be written for multiplication. Furthermore, the different conjugates of $\hat{\mathcal{Q}}_n$ are:

$$\hat{\mathcal{Q}}_n^{\dagger_1} = \overline{\mathbf{Q}_n} + \overline{\mathbf{Q}_{n+1}}j,\ \hat{\mathcal{Q}}_n^{\dagger_2} = \mathbf{Q}_n - \mathbf{Q}_{n+1}j,\ \hat{\mathcal{Q}}_n^{\dagger_3} = \overline{\mathbf{Q}_n} - \overline{\mathbf{Q}_{n+1}}j.$$

The norms of $\hat{\mathcal{Q}}_n$ are calculated by: $N^{\dagger_s}_{\hat{\mathcal{Q}}_n} = \hat{\mathcal{Q}}_n\hat{\mathcal{Q}}_n^{\dagger_s}$ for $s = 1,2,3$. The inverse ^{†_s} is $\hat{\mathcal{Q}}^{-1}_{n\dagger_s} = \dfrac{\hat{\mathcal{Q}}_n^{\dagger_s}}{N^{\dagger_s}_{\hat{\mathcal{Q}}_n}}$, for non-null $N^{\dagger_s}_{\hat{\mathcal{Q}}_n}$.

Remark 2: As an alternative representation, every $\hat{\mathcal{Q}}_n$ and $\hat{\mathcal{K}}_n$ can also be rewritten quaternion form as, respectively:

$$\hat{\mathcal{Q}}_n = \hat{\mathcal{F}}_n + \hat{\mathcal{F}}_{n+1}\mathbf{i} + \hat{\mathcal{F}}_{n+1}\mathbf{j} + \hat{\mathcal{F}}_{n+2}\mathbf{k},$$

$$\hat{\mathcal{K}}_n = \hat{\mathcal{L}}_n + \hat{\mathcal{L}}_{n+1}\mathbf{i} + \hat{\mathcal{L}}_{n+1}\mathbf{j} + \hat{\mathcal{L}}_{n+2}\mathbf{k},$$

where $\hat{\mathcal{F}}_n$ is nth $\mathcal{HGC}$ Fibonacci number and $\hat{\mathcal{L}}_n$ is nth $\mathcal{HGC}$ Lucas number. The $\mathcal{HGC}$ scalar part is $S_{\hat{\mathcal{Q}}_n} = \hat{\mathcal{F}}_n$ and the $\mathcal{HGC}$ vector part is $V_{\hat{\mathcal{Q}}_n} = (\hat{\mathcal{F}}_{n+1}, \hat{\mathcal{F}}_{n+1}, \hat{\mathcal{F}}_{n+2})$. The conjugate, the norm operator and the inverse are defined by the same concepts in Section 2.

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6 A different form of the Fibonacci dual quaternion can be seen in the study [47].

Due to their relations with $\mathcal{HGC}$ Fibonacci/Lucas numbers, it is natural to seek the recurrence relations of $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions.

Theorem 23: *The following quaternion conjugate relations hold for* $n \geq 0$:

(1) $\hat{\mathcal{Q}}_n + \tilde{\hat{\mathcal{Q}}}_n = 2\hat{\mathcal{F}}_n$,

(2) $\hat{\mathcal{Q}}_n \tilde{\hat{\mathcal{Q}}}_n = \hat{\mathcal{F}}_n^2$,

(3) $\hat{\mathcal{Q}}_n \tilde{\hat{\mathcal{Q}}}_n + \hat{\mathcal{Q}}_{n+1} \tilde{\hat{\mathcal{Q}}}_{n+1} = \hat{\mathcal{F}}_n^2 + \hat{\mathcal{F}}_{n+1}^2$,

(4) $\hat{\mathcal{Q}}_n^2 = 2\hat{\mathcal{F}}_n \hat{\mathcal{Q}}_n - \hat{\mathcal{F}}_n^2$,

(5) $\hat{\mathcal{K}}_n + \tilde{\hat{\mathcal{K}}}_n = 2\hat{\mathcal{L}}_n$,

(6) $\hat{\mathcal{K}}_n \tilde{\hat{\mathcal{K}}}_n = \hat{\mathcal{L}}_n^2$,

(7) $\hat{\mathcal{K}}_n \tilde{\hat{\mathcal{K}}}_n + \hat{\mathcal{K}}_{n+1} \tilde{\hat{\mathcal{K}}}_{n+1} = 5(\hat{\mathcal{F}}_n^2 + \hat{\mathcal{F}}_{n+1}^2)$,

(8) $\hat{\mathcal{K}}_n^2 = 2\hat{\mathcal{L}}_n \hat{\mathcal{K}}_n - \hat{\mathcal{L}}_n^2$.

Proof: The proof is clear using operations on $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions.

Theorem 24: *The following additional recurrence relations hold for* $n, r \geq 0$:

(1) $\widehat{\mathcal{Q}}_n + \widehat{\mathcal{Q}}_{n+1} = \widehat{\mathcal{Q}}_{n+2}$,

(2) $\widehat{\mathcal{Q}}_{n+r} + \widehat{\mathcal{Q}}_{n-r} = \begin{cases} L_r \widehat{\mathcal{Q}}_n, & r = 2k \\ F_r \widehat{\mathcal{K}}_n, & r = 2k+1, \end{cases}$

(3) $\widehat{\mathcal{Q}}_{n+r} - \widehat{\mathcal{Q}}_{n-r} = \begin{cases} F_r \widehat{\mathcal{K}}_n, & r = 2k \\ L_r \widehat{\mathcal{Q}}_n, & r = 2k+1, \end{cases}$

(4) $\widehat{\mathcal{Q}}_n - \widehat{\mathcal{Q}}_{n+1}\mathbf{i} - \widehat{\mathcal{Q}}_{n+1}\mathbf{j} - \widehat{\mathcal{Q}}_{n+2}\mathbf{k} = \widehat{\mathcal{F}}_n$,

(5) $\widehat{\mathcal{K}}_n + \widehat{\mathcal{K}}_{n+1} = \widehat{\mathcal{K}}_{n+2}\$$,

(6) $\widehat{\mathcal{K}}_{n+r} + \widehat{\mathcal{K}}_{n-r} = \begin{cases} L_r \widehat{\mathcal{K}}_n, & r = 2k \\ 5F_r \widehat{\mathcal{Q}}_n, & r = 2k+1, \end{cases}$

(7) $\widehat{\mathcal{K}}_{n+r} - \widehat{\mathcal{K}}_{n-r} = \begin{cases} 5F_r \widehat{\mathcal{Q}}_n, & r = 2k \\ L_r \widehat{\mathcal{K}}_n, & r = 2k+1, \end{cases}$

(8) $\widehat{\mathcal{K}}_n - \widehat{\mathcal{K}}_{n+1}\mathbf{i} - \widehat{\mathcal{K}}_{n+1}\mathbf{j} - \widehat{\mathcal{K}}_{n+2}\mathbf{k} = \widehat{\mathcal{L}}_n$

Proof: Considering Theorem 9 and definitions of $\widehat{\mathcal{Q}}_n$ and $\widehat{\mathcal{K}}_n$, the proof is clear.

Theorem 25: *The following order 2 relations hold for* $n, m, r \geq 0$:

(1) $$\hat{\mathcal{Q}}_n \hat{\mathcal{Q}}_m + \hat{\mathcal{Q}}_{n+1} \hat{\mathcal{Q}}_{m+1} = K_{m+n+2} + \mathfrak{p} K_{m+n+4} + 2K_{m+n+3} J + 2(Q_{m+n+2} + \mathfrak{p} Q_{m+n+4} + 2Q_{m+n+3} J) j + V_{K_{m+n+2}} + \mathfrak{p} V_{K_{m+n+4}} + 2V_{K_{m+n+3}} J + 2(V_{Q_{m+n+2}} + \mathfrak{p} V_{Q_{m+n+4}} + 2V_{Q_{m+n+3}}) j,$$

(2) $$\hat{\mathcal{Q}}_n^2 + \hat{\mathcal{Q}}_{n+1}^2 = K_{2n+2} + \mathfrak{p} K_{2n+4} + 2K_{2n+3} J + 2(Q_{2n+2} + \mathfrak{p} Q_{2n+4} + 2Q_{2n+3} J) j + V_{K_{2n+2}} + \mathfrak{p} V_{K_{2n+4}} + 2V_{K_{2n+3}} J + 2(V_{Q_{2n+2}} + \mathfrak{p} V_{Q_{2n+4}} + 2V_{Q_{2n+3}}) j,$$

(3) $$\hat{\mathcal{K}}_n \hat{\mathcal{K}}_m + \hat{\mathcal{K}}_{n+1} \hat{\mathcal{K}}_{m+1} = 5(\hat{\mathcal{Q}}_n \hat{\mathcal{Q}}_m + \hat{\mathcal{Q}}_{n+1} \hat{\mathcal{Q}}_{m+1}),$$

(4) $$\hat{\mathcal{K}}_n^2 + \hat{\mathcal{K}}_{n+1}^2 = 5(\hat{\mathcal{Q}}_n^2 + \hat{\mathcal{Q}}_{n+1}^2).$$

Proof: Part 1) can be proved by applying the definition of $\mathcal{HGC}$ Fibonacci dual quaternion and Theorem 10 into direct multiplication $\hat{\mathcal{Q}}_n \hat{\mathcal{Q}}_m + \hat{\mathcal{Q}}_{n+1} \hat{\mathcal{Q}}_{m+1}$. Also, part 2) can be proved by writing $m \to n$ in part 1). The other parts can be proved similarly.

Theorem 26: *The followings are given:*

$$\text{Generating functions: } \begin{cases} \hat{\mathrm{Q}}(x) = \sum_{n=0}^{\infty} \hat{\mathcal{Q}}_n x^n = \frac{\hat{\mathcal{Q}}_0 + (\hat{\mathcal{Q}}_1 - \hat{\mathcal{Q}}_0) x}{1 - x - x^2}, \\ \hat{\mathrm{K}}(x) = \sum_{n=0}^{\infty} \hat{\mathcal{K}}_n x^n = \frac{\hat{\mathcal{K}}_0 + (\hat{\mathcal{K}}_1 - \hat{\mathcal{K}}_0) x}{1 - x - x^2}, \end{cases}$$

$$\text{Binet's formulas: } \begin{cases} \hat{\mathcal{Q}}_n = \frac{\hat{\alpha}^* \alpha^n - \hat{\beta}^* \beta^n}{\alpha - \beta}, \\ \hat{\mathcal{K}}_n = \hat{\alpha}^* \alpha^n + \hat{\beta}^* \beta^n, \end{cases}$$

$$\text{Exponential generating functions: } \begin{cases} \sum_{n=0}^{\infty} \hat{\mathcal{Q}}_n \frac{t^n}{n!} = \frac{\hat{\alpha}^* e^{\alpha t} - \hat{\beta}^* e^{\beta t}}{\alpha - \beta}, \\ \sum_{n=0}^{\infty} \hat{\mathcal{K}}_n \frac{t^n}{n!} = \hat{\alpha}^* e^{\alpha t} + \hat{\beta}^* e^{\beta t}, \end{cases}$$

where $\hat{\alpha}^* = \hat{\alpha}(1 + \alpha \mathbf{i} + \alpha \mathbf{j} + \alpha^2 \mathbf{k})$ and $\hat{\beta}^* = \hat{\beta}(1 + \beta \mathbf{i} + \beta \mathbf{j} + \beta^2 \mathbf{k})$.

The proof of the following theorem can be conducted in a similar way to the proofs of Theorem 15-Theorem 18, so we omit it.

Theorem 27: *For* $n, m \geq 0$ *and* $\hat{\mathcal{R}} = [(1-\mathfrak{p})j + Jj]\,[1+\mathbf{i}+\mathbf{j}+3\mathbf{k}]$, *the followings are given:*

$$\text{Tagiuri identities: } \begin{cases} \hat{\mathcal{Q}}_m\hat{\mathcal{Q}}_n - \hat{\mathcal{Q}}_{m+r}\hat{\mathcal{Q}}_{n-r} = (-1)^{n-r}F_{m-n+r}F_r\hat{\mathcal{R}}, \\ \hat{\mathcal{K}}_m\hat{\mathcal{K}}_n - \hat{\mathcal{K}}_{m+r}\hat{\mathcal{K}}_{n-r} = 5(-1)^{n-r+1}F_{m-n+r}F_r\hat{\mathcal{R}}, \end{cases}$$

$$\text{D'Ocagne's identities: } \begin{cases} \hat{\mathcal{Q}}_m\hat{\mathcal{Q}}_{n+1} - \hat{\mathcal{Q}}_{m+1}\hat{\mathcal{Q}}_n = (-1)^{n}F_{m-n}\hat{\mathcal{R}}, \\ \hat{\mathcal{K}}_m\hat{\mathcal{K}}_{n+1} - \hat{\mathcal{K}}_{m+1}\hat{\mathcal{K}}_n = 5(-1)^{n+1}F_{m-n}\hat{\mathcal{R}}, \end{cases}$$

$$\text{Catalan's identities: } \begin{cases} \hat{\mathcal{Q}}_n^2 - \hat{\mathcal{Q}}_{n+r}\hat{\mathcal{Q}}_{n-r} = (-1)^{n-r}F_r^2\hat{\mathcal{R}}, \\ \hat{\mathcal{K}}_n^{\,2} - \hat{\mathcal{K}}_{n+r}\hat{\mathcal{K}}_{n-r} = 5(-1)^{n-r+1}F_r^2\hat{\mathcal{R}}, \end{cases}$$

$$\text{Cassini's identities: } \begin{cases} \hat{\mathcal{Q}}_n^2 - \hat{\mathcal{Q}}_{n+1}\hat{\mathcal{Q}}_{n-1} = (-1)^{n-1}\hat{\mathcal{R}}, \\ \hat{\mathcal{K}}_n^{\,2} - \hat{\mathcal{K}}_{n+1}\hat{\mathcal{K}}_{n-1} = 5(-1)^{n}\hat{\mathcal{R}}. \end{cases}$$

Example 3: Let find the special identities for $\hat{\mathcal{Q}}_n$ and $\hat{\mathcal{K}}_n$ as follows:

Tagiuri identities, $m=4, n=2, r=1,\ \mathfrak{p}=1$
$\hat{\mathcal{K}}_4\hat{\mathcal{K}}_2 - \hat{\mathcal{K}}_5\hat{\mathcal{K}}_1 = -5(\hat{\mathcal{Q}}_4\hat{\mathcal{Q}}_2 - \hat{\mathcal{Q}}_5\hat{\mathcal{Q}}_1) = 10Jj(1+\mathbf{i}+\mathbf{j}+3\mathbf{k})$
D'ocagne's identities, $m=3, n=1,\ \mathfrak{p}=-\frac{1}{2}$
$\hat{\mathcal{K}}_3\hat{\mathcal{K}}_2 - \hat{\mathcal{K}}_4\hat{\mathcal{K}}_1 = -5(\hat{\mathcal{Q}}_3\hat{\mathcal{Q}}_2 - \hat{\mathcal{Q}}_4\hat{\mathcal{Q}}_1) = 5(\frac{3}{2}j + Jj)(1+\mathbf{i}+\mathbf{j}+3\mathbf{k})$
Catalan's identities, $n=r=2,\ \mathfrak{p}=0$
$\hat{\mathcal{K}}_2^2 - \hat{\mathcal{K}}_4\hat{\mathcal{K}}_0 = -5(\hat{\mathcal{Q}}_2^2 - \hat{\mathcal{Q}}_4\hat{\mathcal{Q}}_0) = -5(j + Jj)(1+\mathbf{i}+\mathbf{j}+3\mathbf{k})$
Cassini's identities, $n=2,\ \mathfrak{p}=\frac{1}{3}$
$\hat{\mathcal{K}}_2^2 - \hat{\mathcal{K}}_3\hat{\mathcal{K}}_1 = -5(\hat{\mathcal{Q}}_2^2 - \hat{\mathcal{Q}}_3\hat{\mathcal{Q}}_1) = 5(\frac{2}{3}j + Jj)(1+\mathbf{i}+\mathbf{j}+3\mathbf{k})$

A natural question to ask is if Theorem 19, Theorem 20 and Section 2.1 can be generalized for $\mathcal{HGC}$ Fibonacci/Lucas numbers. One can see their analogs easily.

Example 4: Using Corollary 1, the following matrix forms of $\hat{\mathcal{Q}}_2$ are given for $\mathfrak{p}=2$:

- $\mathcal{A}_{\hat{\mathcal{Q}}_2} = ((1+2\mathbf{i}+2\mathbf{j}+3\mathbf{k}) + (2+3\mathbf{i}+3\mathbf{j}+5\mathbf{k})J)I_2,$
  $+((2+3\mathbf{i}+3\mathbf{j}+5\mathbf{k}) + (3+5\mathbf{i}+5\mathbf{j}+8\mathbf{k})J)j$
- $\mathcal{B}_{\hat{\mathcal{Q}}_2} = (1+2J+2j+3Jj)I_4 + (2+3J+3j+5Jj)\mathcal{I} + (2+3J+3j+5Jj)\mathcal{J},$
  $+(3+5J+5j+8Jj)\mathcal{K},$

- $$\mathcal{C}_{\hat{\mathcal{Q}}_2} = (1+2\mathbf{i}+2\mathbf{j}+3\mathbf{k})I_4 + (2+3\mathbf{i}+3\mathbf{j}+5\mathbf{k})\mathrm{J} + (2+3\mathbf{i}+3\mathbf{j}+5\mathbf{k})\mathrm{j}, \\ +(3+5\mathbf{i}+5\mathbf{j}+8\mathbf{k})\mathrm{Jj}.$$

Additionally, the Fibonacci matrix approximation method (see Section 3.2) can also be carried over to $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions.

4. Conclusion

Because of the wide range of applicability of quaternions, they have attracted interest among researchers for a long time. Motivated by this point of view, in this present paper, the Cayley-Dickson doubling process is used to generate $\mathcal{HGC}$ dual quaternions and $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions. Based on the idea that the usage of the Fibonacci/Lucas extension will be interesting, we emphasize the theory of $\mathcal{HGC}$ Fibonacci/Lucas numbers. To increase comprehension, we give a more accurate way to quickly calculate $\mathcal{HGC}$ Fibonacci/Lucas numbers. As compared to the standard ones, one can see extended versions of Fibonacci/Lucas numbers by taking this approach into consideration. To extend the framework, even more, we give the basics for $\mathcal{HGC}$ Fibonacci/Lucas dual quaternions.

The main advantage of introducing the $\mathcal{HGC}$ dual quaternions is that many dual quaternions have celebrated numbers, such as hyperbolic-complex for $\mathfrak{p}=-1$, hyperbolic-dual for $\mathfrak{p}=0$ and bihyperbolic for $\mathfrak{p}=1$, can be deduced as particular cases.

As part of further research, we intend to continue studying different types of non-commutative quaternions and their Horadam number extension expecting to introduce new and interesting results.

Supporting information

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