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Bidimensional extensions of balancing and Lucas-balancing numbers

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Abstract

In this article bidimensional extensions of balancing and Lucas-balancing numbers are introduced, as well as some properties of these new bidimensional sequences.

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1. Introduction

Numerical sequences have been the subject of several studies by many researchers in recent decades. Great emphasis has been given to Fibonacci and Lucas sequences, but many other numerical sequences have contributed to the increase of researchers interested in this area and in its several applications. The literature related to the studies of sequences

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of the most varied types, such as sequences of numbers, polynomials, quaternions, octonions, sedenions, etc., is vast and to exemplify some of these studies, the reader can consult the works of Catarino [3] where the sequence of *k*–Pell hybrid numbers was introduced, of Kizilates [12] with the study of another sequence of hybrid numbers (Fibonacci and Lucas), of Koshy [13] with the study of some applications of the Fibonacci and Lucas sequences, of Catarino and Borges [5] with the introduction of the numerical sequence constituted by Leonardo's numbers and in [6] of the incomplete version of these same numbers, of Catarino and Vasco [8] with the study of the *k*–Pell generalized numbers of order m and the work of Catarino [4] with the introduction of the sequence of bicomplex quaternions whose terms are the *k*–Pell numbers, among many other works. Also noteworthy are the works of Ayse Karakas and Murat Karakas [9] with the introduction of Lucas sequence space, where some properties are investigated, of Karataş and Halici [10] with the study of the bicomplex version of the Pell and Pell-Lucas numbers and the work of *Qi* [18] with the study of determinantal expressions and recurrence relations for generalized Fubini and Eulerian polynomials. Many identities and properties of sequences come from using the so-called Binet formula for that sequence. This formula allows the determination of the general term of a sequence without having to resort to other terms of the sequence and was deduced by Levesque in 1985, [14].

In this work it is our purpose to introduce and study two new bidimensional sequences and present some of their properties. One of the sequences that will serve as the basis for the study that we will present in this article is the sequence of balancing numbers that was introduced and studied by Behera and Panda in [1]. This sequence of balancing numbers is denoted by $\{B_n\}_{n=0}^{\infty}$ and the authors have studied some of its interesting properties which we will recall in the next section. Also the sequence of the Lucas-balancing numbers $\{C_n\}_{n=0}^{\infty}$ will serve to study its bidimensional version and obtain a new sequence. As one of the consequences of Binet's formula for balancing numbers and Lucas-balancing numbers, we will present some identities related to these bidimensional numerical sequences.

Another sequence to be used is the sequence $\{ST_n\}_{n=0}^{\infty}$ of the square triangular numbers.

There are already many works related to the study of the bidimensional, tridimensional and n–dimensional versions of several sequences. For example, in [19] we have a brief approach to Leonardo's sequence, as well as a discussion related to the recurrent bidimensional relations of this type of number from its one-dimensional model. Also in [2], the authors work on the recurrent bidimensional and tridimensional relationships defined from the one-dimensional recursive model of the Narayana sequence. In [16] we find a study on bidimensional and tridimensional identities for Fibonacci numbers in complex form and in [15] Gaussian Fibonacci numbers and recurrent bidimensional relations are introduced and studied. So, it was these and other works that served as motivation to the study of new bidimensional versions of two more numerical sequences, based on their one-dimensional model.

The structure of the article is as follows: in the next section we will incorporate the background of the balancing and Lucas-balancing numbers that will be necessary for our study. The bidimensional versions of these two sequences will be defined in Section 3 for the balancing numbers and in Section 5 for the Lucas-balancing numbers. Sections 4 and 6 will be dedicated to the study of some identities that are satisfied by these two sequences, respectively.

2. Balancing and Lucas-balancing numbers

According to [1], a positive integer n is a balancing number with balancer r if it is a solution of the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$.

The recurrence relation of the balancing numbers $\{B_n\}_{n=0}^{\infty}$ is given by

$$B_{n+2}=6B_{n+1}-B_n,$$

with the initial conditions $B_0 = 0$ and $B_1 = 1$.

Regarding Lucas-balancing numbers $\{C_n\}_{n=0}^{\infty}$, the recurrence relation is the same as that of the balancing numbers, but is different in the initial conditions that, in this case, are $C_0 = 1$ and $C_1 = 3$. Panda showed in [17] that an existing relationship between the n order terms of these two sequences is $C_n=\sqrt{8B_n^2+1}$.

The sequence of square triangular numbers $\{ST_n\}_{n=0}^{\infty}$ is a sequence that satisfies the following recurrence relation

$$ST_{n+1}=34ST_n-ST_{n-1}+2, n\geq 1,$$

with initial conditions $ST_0 = 0$ and $ST_1 = 1$.

The recurrence relations of the first two sequences are associated with the characteristic equation $r^2 - 6r + 1 = 0$, whose roots are $r_1=3+2\sqrt{2}$, and

$r_2 = 3 - 2\sqrt{2}$. Note that $r_1 + r_2 = 6$, $r_1 - r_2 = 4\sqrt{2}$, $r_1 r_2 = 1$ and $\frac{1}{r_1} = r_2, \frac{1}{r_2} = r_1$. These roots will be very useful in the Binet formula of each sequence, including also in the case of the last sequence, as we will see later.

Many properties of these two numerical sequences have been studied. We will recall below some of these properties that will be used in the study of the bidimensional version of each one. However, for example, in [7] and in [11] we can find other properties. Thus, according to Catarino, Campos and Vasco [7], Binet's formula for the sequence of balancing numbers is given by:

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

Through this formula we can define the terms of the sequence with negative index, obtaining the following result:

Proposition 2.1: If B_n represents the balancing number of order $n, n \geq 1$, then

$$B_{-n} = -B_n.$$

***Proof*:** We have:

$$\begin{aligned} B_{-n} &= \frac{r_1^{-n} - r_2^{-n}}{r_1 - r_2} \\ &= \frac{\left(\frac{1}{r_1}\right)^n - \left(\frac{1}{r_2}\right)^n}{r_1 - r_2} \\ &= \frac{r_2^n - r_1^n}{r_1 - r_2} \\ &= -\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right), \end{aligned}$$

hence, $B_{-n} = -B_n$. □

With regard to the recurrence relation of the Lucas-balancing numbers, Binet's formula is given by

$$C_n = \frac{r_1^n + r_2^n}{2}.$$

It is also possible to define the terms of this sequence with negative index

$$\begin{aligned}
C_{-n} &= \frac{r_1^{-n}+r_2^{-n}}{2} \\
&= \frac{\left(\frac{1}{r_1}\right)^n+\left(\frac{1}{r_2}\right)^n}{2} \\
&= \frac{r_2^n+r_1^n}{2} \\
&= \frac{r_1^n+r_2^n}{2},
\end{aligned}$$

following, therefore, that

Proposition 2.2: If C_n represents the Lucas-balancing number of order $n, n \geq 1,$ then

$$C_{-n} = C_n.$$

With the sequence $\{ST_n\}_{n=0}^{\infty}$ and using the known fact that $B_n^2 = ST_n$, we easily find its Binet formula from the Binet formula of balancing numbers. So we have

$$\begin{aligned}
ST_n &= B_n^2 \\
&= \left(\frac{r_1^n-r_2^n}{r_1-r_2}\right)^2 \\
&= \frac{r_1^{2n}+r_2^{2n}-2(r_1r_2)^n}{(r_1-r_2)^2} \\
&= \frac{r_1^{2n}+r_2^{2n}-2}{32} \\
&= \frac{r_1^{2n}+r_2^{2n}}{32}-\frac{1}{16},
\end{aligned}$$

therefore,

$$ST_n = \frac{1}{16}(C_{2n}-1).$$

Next, we present a sequence of results related to these two types of balancing numbers and also to the square triangular numbers that we will use later in proofs of other results related to the bidimensional versions of the balancing and Lucas-balancing numbers.

So we have the following properties:

Proposition 2.3: If B_n, C_n and ST_n represent, respectively, the balancing, the Lucas-balancing and the square triangular numbers of order $n, n \geq 1$, then the following properties are true:

1. $B_{2n} = B_n(B_{n+1} - B_{n-1})$;
2. $\sum_{l=1}^{n} B_{2l} = B_n B_{n+1}$;
3. $\sum_{l=1}^{n} B_{2l-1} = 3(B_{n-1}B_n) + \frac{1}{2}(B_{2n-1} + 1)$;
4. $\sum_{l=1}^{n} C_{2l} = \frac{-1-ST_n+ST_{n+1}}{2}$;
5. $\sum_{l=1}^{n} C_{2l-1} = \frac{4-C_n+C_{n+1}+2ST_n-2ST_{n+1}}{4}$.

***Proof*:** The proof of item 1 is done using Binet's formula of balancing numbers. Thus we have:

$$\begin{aligned}
B_n(B_{n+1} - B_{n-1}) &= \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} - \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}\right) \\
&= \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1} - r_1^{n-1} + r_2^{n-1}}{r_1 - r_2}\right) \\
&= \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1^n(r_1 - r_1^{-1}) - r_2^n(r_2 - r_2^{-1})}{r_1 - r_2}\right) \\
&= \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1^n(r_1 - r_2) - r_2^n(r_2 - r_1)}{r_1 - r_2}\right) \\
&= \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)(r_1^n + r_2^n) \\
&= \left(\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2}\right) \\
&= B_{2n}.
\end{aligned}$$

For the proof of item 2, using item 1, we have:

$$\begin{aligned}
\sum_{l=1}^{n} B_{2l} &= B_2 + B_4 + \cdots + B_{2n-2} + B_{2n} \\
&= (B_1(B_2 - B_0)) + (B_2(B_3 - B_1)) + \cdots + (B_n(B_{n+1} - B_{n-1})) \\
&= -B_1B_0 + B_nB_{n+1}.
\end{aligned}$$

For item 3, we will use the recurrence relation of the balancing numbers and the result found in 2 and the result follows:

$$\begin{aligned}
\sum_{l=1}^{n} B_{2l-1} &= B_1 + B_3 + B_5 + \cdots + B_{2n-1} \\
&= B_1 + (6B_2 - B_1) + (6B_4 - B_3) + \cdots + (6B_{2n-2} - B_{2n-3}) \\
&= 6\sum_{l=1}^{n-1} B_{2l} - \sum_{l=1}^{n} B_{2l-1} + B_{2n-1} + B_1.
\end{aligned}$$

Hence

$$\begin{aligned}2\sum_{l=1}^{n} B_{2l-1} &= 6\sum_{l=1}^{n-1} B_{2l} + B_{2n-1} + B_1\\ &= 6B_{n-1}B_n + B_{2n-1} + 1.\end{aligned}$$

Therefore, $\sum_{l=1}^{n} B_{2l-1} = 3B_{n-1}B_n + \frac{1}{2}(B_{2n-1} + 1)$.

For the proof of item 4, using items 2, 4 and 8 of Proposition 2.6, in [7], we have:

$$\begin{aligned}\sum_{l=1}^{n} C_{2l} &= \sum_{l=1}^{n}(16B_n^2 + 1)\\ &= 16\sum_{l=1}^{n} B_n^2 + \sum_{l=1}^{n} 1\\ &= 16\sum_{l=1}^{n} ST_n + n\\ &= 16\left(\frac{-1-ST_n+ST_{n+1}-2n}{32}\right) + n\\ &= \frac{-1-ST_n+ST_{n+1}-2n+2n}{2}\\ &= \frac{-1-ST_n+ST_{n+1}}{2}.\end{aligned}$$

Finally, for the proof of item 5, let's use item 7 of Proposition 2.6 of [7] and also the result found in 4 and, therefore, we have

$$\begin{aligned}\sum_{l=1}^{n} C_{2l-1} &= \sum_{l=1}^{n} C_l - \sum_{l=1}^{n} C_{2l}\\ &= \frac{2-C_n+C_{n+1}}{4} - \frac{-1-ST_n+ST_{n+1}}{2}\\ &= \frac{2-C_n+C_{n+1}}{4} + \frac{1+ST_n-ST_{n+1}}{2}\\ &= \frac{2-C_n+C_{n+1}+2+2ST_n-2ST_{n+1}}{4}\\ &= \frac{4-C_n+C_{n+1}+2ST_n-2ST_{n+1}}{4}.\end{aligned}$$

□

3. Recurrence relations of bidimensional balancing numbers

The bidimensional balancing numbers and the bidimensional Lucas-balancing numbers denoted by $B_{(n,m)}$ and $C_{(n,m)}$, respectively, will be defined on the set of Gaussian integers $(n, m) = n + mi$, where n and m are non-negative integers.

These numbers constitute an extension to complex dimension and the recurrent relation of the bidimensional case of these sequences will be explained throughout this paper.

Definition 3.1: Numbers of the form $B(n, m)$ will represent the bidimensional balancing numbers and must satisfy the following bidimensional recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} B_{(n+1,m)} &= 6B_{(n,m)} - B_{(n-1,m)}, \\ B_{(n,m+1)} &= 6B_{(n,m)} - B_{(n,m-1)}, \end{cases}$$

with the following initial conditions $B_{(0,0)} = 0$, $B_{(1,0)} = 1$, $B_{(0,1)} = i$, $B_{(1,1)} = 1 + i$, where $i^2 = -1$.

Below we present some properties of the bidimensional balancing numbers.

Lemma 3.2: *The following properties are true for every non-negative integers n and m:*

1. $B_{(n,0)} = B_n$;
2. $B_{(0,m)} = B_m i$;
3. $B_{(n,1)} = B_n + (B_n - B_{n-1})\, i$;
4. $B_{(1,m)} = (B_m - B_{m-1}) + B_m i$.

***Proof*:** For the property 1, the proof is done by induction on n. For $n = 0$ and $n = 1$ we have $B_{(0,0)} = 0 = B_0$ and $B_{(1,0)} = 1 = B_1$ and so the property is true.

Suppose the property is true for values less than or equal to n. We will show that it remains true for $n + 1$. Then we have:

$$\begin{aligned} B_{(n+1,0)} &= 6B_{(n,0)} - B_{(n-1,0)} \\ &= 6B_n - B_{n-1} \end{aligned}$$

and by the recurrence relation of balancing numbers,

$$B_{(n+1,0)} = B_{n+1}.$$

Therefore, the property 1 is true.

The proof of 2 will be done by induction on m. For $m = 0$ and $m = 1$ we have $B_{(0,0)} = 0i = B_0 i$ and $B_{(0,1)} = i = B_1 i$ and thus the property is true.

Suppose the identity is valid for values less than or equal to m. We have:

$$\begin{aligned} B_{(0,m+1)} &= 6B_{(0,m)} - B_{(0,m-1)} \\ &= 6B_m i - B_{m-1} i \end{aligned}$$

and by the recurrence relation of balancing numbers,

$$\begin{aligned} B_{(0,m+1)} &= (6B_m - B_{m-1})i \\ &= B_{m+1} i. \end{aligned}$$

Thus, it follows 2.

The proof of 3 will also be done by induction on n. For $n = 0$ we have that $B_{(0,1)} = i$ and

$$\begin{aligned} B_0 + (B_0 - B_{-1})i &= 0 + (0 - (-B_1))i \\ &= B_1 i = i. \end{aligned}$$

For $n = 1$ we have that $B_{(1,1)} = 1 + i$ and

$$\begin{aligned} B_1 + (B_1 - B_0)i &= 1 + (1 - 0)i \\ &= 1 + i. \end{aligned}$$

Suppose that the identity is valid for all $k \leq n$. Then

$$\begin{aligned} B_{(n+1,1)} &= 6B_{(n,1)} - B_{(n-1,1)} \\ &= 6(B_n + (B_n - B_{n-1})i) - (B_{n-1} + (B_{n-1} - B_{n-2})i) \\ &= (6B_n - B_{n-1}) + 6(B_n - B_{n-1})i - (B_{n-1} - B_{n-2})i \\ &= B_{n+1} + 6B_n i - 6B_{n-1} i - B_{n-1} i + B_{n-2} i \\ &= B_{n+1} + (6B_n - B_{n-1})i - (6B_{n-1} - B_{n-2})i \\ &= B_{n+1} + B_{n+1} i - B_n i \\ &= B_{n+1} + (B_{n+1} - B_n)i. \end{aligned}$$

Thus, 3 is true.

The proof of 4 is also done by induction on m. For $m = 0$ we have that $B_{(1,0)} = 1$ and

$$\begin{aligned} (B_0 - B_{-1}) + B_0 i &= (0 - (-B_1)) + 0i \\ &= B_1 = 1. \end{aligned}$$

For $m = 1$ we have that $B_{(1,1)} = 1 + i$ and

$$(B_1 - B_0) + B_1 i = (1 - 0) + i = 1 + i.$$

and the property is true.

Suppose that the identity is valid for all values $k \leq m$. So we have:

$$\begin{aligned} B_{(1,m+1)} &= 6B_{(1,m)} - B_{(1,m-1)} \\ &= 6((B_m - B_{m-1}) + B_m i) - ((B_{m-1} - B_{m-2}) + B_{m-1} i) \end{aligned}$$

and by the recurrence relation of balancing numbers,

$$\begin{aligned}B_{(1,m+1)} &= (6B_m - B_{m-1}) - 6B_{m-1} + 6B_m i + B_{m-2} - B_{m-1} i\\ &= B_{m+1} - (6B_{m-1} - B_{m-2}) + (6B_m - B_{m-1})i\\ &= (B_{m+1} - B_m) + B_{m+1} i.\end{aligned}$$

Therefore, the property 4 follows. □

The following result is a relationship that the bidimensional balancing numbers must satisfy.

Theorem 3.3: *For non-negative integers n, m, the bidimensional balancing numbers are described as follows:*

$$B_{(n,m)} = B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i.$$

***Proof*:** We start by using induction on m, fixing n. For $m = 0$ and taking into account the initial conditions of the sequence $\{B_n\}_{n\geq 0}$ and the Proposition 2.1, we have that

$$\begin{aligned}B_{(n,0)} &= B_n(B_0 - B_{-1}) + (B_n - B_{n-1})B_0 i\\ &= B_n(0-(-B_1)) + (B_n - B_{n-1})0i\\ &= B_n(0+B_1)\\ &= B_n(0+1) = B_n,\end{aligned}$$

which is true by Lemma 3.2, item 1. For $m = 1$ and once again taking into account the initial conditions of the sequence $\{B_n\}_{n\geq 0}$, we have

$$\begin{aligned}B_{(n,1)} &= B_n(B_1 - B_0) + (B_n - B_{n-1})B_1 i\\ &= B_n(1-0) + (B_n - B_{n-1})(1)i\\ &= B_n + (B_n - B_{n-1})i,\end{aligned}$$

what turns out to be true by Lemma 3.2, item 3.

Suppose the theorem is true for $k \leq m$. Let us prove that:

$$B_{(n,m+1)} = B_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{m+1} i.$$

Now, by the recurrence relation of $\{B_{n,m}\}_{n,m\geq 0}$

$$B_{(n,m+1)} = 6B_{(n,m)} - B_{(n,m-1)}.$$

By the induction hypothesis, we have

$$\begin{aligned}B_{(n,m+1)} = {} & 6(B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i)\\ & -(B_n(B_{m-1} - B_{m-2}) + (B_n - B_{n-1})B_{m-1} i),\end{aligned}$$

which is equivalent to

$$B_{(n,m+1)} = 6B_n(B_m - B_{m-1}) + 6(B_n - B_{n-1})B_m i - B_n(B_{m-1} - B_{m-2}) \\ -(B_n - B_{n-1})B_{m-1}i,$$

which is, once more, equivalent to

$$\begin{aligned} B_{(n,m+1)} &= 6B_nB_m - 6B_nB_{m-1} + 6(B_n + B_{n-1})B_m i - B_nB_{m-1} + B_nB_{m-2} \\ &\quad -(B_n + B_{n-1})B_{m-1}i \\ &= 6B_nB_m - B_nB_{m-1} - 6B_nB_{m-1} + B_nB_{m-2} + 6(B_n + B_{n-1})B_m i \\ &\quad -(B_n + B_{n-1})B_{m-1}i \\ &= (6B_m - B_{m-1})B_n - (6B_{m-1} - B_{m-2})B_n \\ &\quad +(6B_m i - B_{m-1}i)(B_n - B_{n-1}) \\ &= B_{m+1}B_n - B_mB_n + B_{m+1}i(B_n - B_{n-1}) \\ &= B_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{m+1}i, \end{aligned}$$

as we wanted to show.

Let's now fix m and perform the induction on n. For $n = 0$ and given the initial conditions of $\{B_n\}_{n\geq 0}$, we have

$$\begin{aligned} B_{(0,m)} &= B_0(B_m - B_{m-1}) + (B_0 - B_{-1})B_m i \\ &= (0 + B_1)B_m i \\ &= (0+1)B_m i = B_m i, \end{aligned}$$

which is true by Lemma 3.2, item 2. For $n = 1$ and again from the initial conditions of $\{B_n\}_{n\geq 0}$, we have

$$\begin{aligned} B_{(1,m)} &= B_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i \\ &= (1)(B_m - B_{m-1}) + (1-0)B_m i \\ &= (B_m - B_{m-1}) + B_m i, \end{aligned}$$

which is true in view of Lemma 3.2, item 4.

Suppose the theorem is true for $k \leq n$. We will prove that:

$$B_{(n+1,m)} = B_{n+1}(B_m - B_{m-1}) + (B_{n+1} - B_n)B_{m+1}i.$$

We have, by the recurrence relation of $\{B_{n,m}\}_{n,m\geq 0}$

$$B_{(n+1,m)} = 6B_{(n,m)} - B_{(n-1,m)}.$$

Using the induction hypothesis

$$
\begin{aligned}
B_{(n+1,m)} &= 6(B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i) \\
&\quad -(B_{n-1}(B_m - B_{m-1}) + (B_{n-1} - B_{n-2})B_m i) \\
&= 6B_n(B_m - B_{m-1}) + 6(B_n + B_{n-1})B_m i - B_{n-1}(B_m - B_{m-1}) \\
&\quad -(B_{n-1} - B_{n-2})B_m i \\
&= 6B_n(B_m - B_{m-1}) + 6B_nB_m i - 6B_{n-1}B_m i - B_{n-1}(B_m - B_{m-1}) \\
&\quad -B_{n-1}B_m i + B_{n-2}B_m i \\
&= 6B_n(B_m - B_{m-1}) - B_{n-1}(B_m - B_{m-1}) + 6B_nB_m i - B_{n-1}B_m i \\
&\quad -6B_{n-1}B_m i + B_{n-2}B_m i \\
&= (6B_n - B_{n-1})(B_m - B_{m-1}) + (6B_n - B_{n-1})B_m i \\
&\quad -(6B_{n-1} - B_{n-2})B_m i \\
&= B_{n+1}(B_m - B_{m-1}) + B_{n+1}B_m i - B_nB_m i \\
&= B_{n+1}(B_m - B_{m-1}) + (B_{n+1} - B_n)B_m i.
\end{aligned}
$$

□

4. Some identities of bidimensional balancing numbers in complex form

From now on, we will explore some bidimensional identities of this sequence.

Proposition 4.1: The sum of the numbers $B(p, m)$ of odd index p can be described by:

$$\sum_{l=1}^{n} B_{(2l-1,m)} = \left(3(B_{n-1}B_n) + \frac{1}{2}(B_{2n-1}+1)\right)(B_m - B_{m-1} + B_m i) - B_{n-1}B_nB_m i.$$

***Proof*:** By theorem 3.3, we have

$$\sum_{l=1}^{n} B_{(2l-1,m)} = \sum_{l=1}^{n}(B_{2l-1}(B_m - B_{m-1}) + (B_{2l-1} - B_{2l-2})B_m i).$$

Hence

$$
\begin{aligned}
\sum_{l=1}^{n} B_{(2l-1,m)} &= (B_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i) + \cdots \\
&\quad + (B_{2n-1}(B_m - B_{m-1}) + (B_{2n-1} - B_{2n-2})B_m i) \\
&= (B_1 + B_3 + \cdots + B_{2n-1})(B_m - B_{m-1}) \\
&\quad + ((B_1 - B_0) + (B_3 - B_2) + \cdots + (B_{2n-1} - B_{2n-2}))B_m i \\
&= \sum_{l=1}^{n} B_{2l-1}(B_m - B_{m-1}) + \sum_{l=1}^{n} B_{2l-1}B_m i - \sum_{l=1}^{n-1} B_{2l}B_m i,
\end{aligned}
$$

and taking into account the items 2 and 3 of Proposition 2.3, the desired result follows immediately. □

Proposition 4.2: The sum of the numbers $B(p, m)$ of even and not null index p can be described by:

$$\sum_{l=1}^{n} B_{(2l,m)} = B_n B_{n+1}(B_m - B_{m-1} + B_m i) - \left(3(B_{n-1}B_n) + \tfrac{1}{2}(B_{2n-1}+1)\right)B_m i.$$

***Proof*:** Using Theorem 3.3, we have

$$\sum_{l=1}^{n} B_{(2l,m)} = \sum_{l=1}^{n}(B_{2l}(B_m - B_{m-1}) + (B_{2l} - B_{2l-1})B_m i).$$

Thus the result follows similarly to the previous proposition

$$\sum_{l=1}^{n} B_{(2l,m)} = \sum_{l=1}^{n} B_{2l}(B_m - B_{m-1}) + \sum_{l=1}^{n} B_{2l}B_m i - \sum_{l=1}^{n} B_{2l-1}B_m i$$

and taking into account items 2 and 3 of Proposition 2.3, the result follows. □

Proposition 4.3: The sum of the first p numbers $B(p, m)$, can be described as follows:

$$\sum_{l=1}^{n} B_{(l,m)} = \left(\frac{-1-B_n+B_{n+1}}{4}\right)(B_m - B_{m-1}) + B_n B_m i.$$

***Proof*:** Similarly to the proof of the previous identities, by Theorem 3.3, we get:

$$\sum_{l=1}^{n} B_{(l,m)} = \sum_{l=1}^{n}(B_l(B_m - B_{m-1}) + (B_l - B_{l-1})B_m i).$$

Thus, we have:

$$\begin{aligned}\sum_{l=1}^{n} B_{(l,m)} &= (B_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i) + \cdots \\ &\quad + (B_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i) \\ &= (B_1 + B_2 + \cdots + B_n)(B_m - B_{m-1}) \\ &\quad + ((B_1 - B_0) + (B_2 - B_1) + \cdots + (B_n - B_{n-1}))B_m i \\ &= \sum_{l=1}^{n} B_l(B_m - B_{m-1}) + B_n B_m i\end{aligned}$$

and finally, by Proposition 2.6, item 6 in [7], the result follows. □

5. Recurrence relations of bidimensional Lucas-balancing numbers

In this section we introduce the bidimensional version of Lucas-balancing sequence with the following definition:

Definition 5.1: Numbers of the form $C(n, m)$ will represent the bidimensional Lucas-balancing numbers and must satisfy the following bidimensional recurrence conditions, where n and m are non-negative integers:

$$\begin{cases} C_{(n+1,m)} &= 6C_{(n,m)} - C_{(n-1,m)}, \\ C_{(n,m+1)} &= 6C_{(n,m)} - C_{(n,m-1)}, \end{cases}$$

with the following initial conditions $C_{(0,0)} = 1$, $C_{(1,0)} = 3$, $C_{(0,1)} = 1 + i$, $C_{(1,1)} = 3 + i$, where $i^2 = -1$.

The next result states some interesting properties involving these sequences.

Lemma 5.2: *The following properties are true for every non-negative integers n and m:*

1. $C_{(n,0)} = C_n$;
2. $C_{(0,m)} = (B_m - B_{m-1}) + B_m i$;
3. $C_{(n,1)} = C_n + (B_n - B_{n-1})\, i$;
4. $C_{(1,m)} = 3(B_m - B_{m-1}) + B_m i$.

Proof: For the proof of property 1, we will use induction on n. For $n = 0$ and $n = 1$ we have $C_{(0,0)} = 1 = C_0$ and $C_{(1,0)} = 3 = C_1$ and so the property is true.

Suppose the property is true for values less than or equal to n, and let us show that it remains true for $n + 1$:

$$\begin{aligned} C_{(n+1,0)} &= 6C_{(n,0)} - C_{(n-1,0)} \\ &= 6C_n - C_{n-1} \\ &= C_{n+1}. \end{aligned}$$

Therefore, 1 follows.

The proof of 2 is also done by induction on m. For $m = 0$ we have $C_{(0,0)} = 1$ and $(B_0 - B_{-1}) + B_0 i = (0 + 1) + 0i = 1$. For $m = 1$ we have that $C_{(0,1)} = 1 + i$ and $(B_1 - B_0) + B_1 i = (1 - 0) + i = 1+ i$.

Suppose the property holds for every $k \le m$. Then:

$$\begin{aligned} C_{(0,m+1)} &= 6C_{(0,m)} - C_{(0,m-1)} \\ &= 6((B_m - B_{m-1}) + B_m i) - ((B_{m-1} - B_{m-2}) + B_{m-1} i) \\ &= 6(B_m - B_{m-1}) + 6B_m i - (B_{m-1} - B_{m-2}) - B_{m-1} i \end{aligned}$$

$$\begin{aligned}
&= 6B_m - 6B_{m-1} + 6B_m i - B_{m-1} + B_{m-2} - B_{m-1} i \\
&= (6B_m - B_{m-1}) - (6B_{m-1} - B_{m-2}) + (6B_m i - B_{m-1} i) \\
&= B_{m+1} - B_m + B_{m+1} i \\
&= (B_{m+1} - B_m) + B_{m+1} i.
\end{aligned}$$

Thus, 2 follows.

The proof of 3 is also done by using induction on n. For $n = 0$ we have $C_{(0,1)} = 1 + i$ and $C_0 + (B_0 - B_{-1})\, i = 1 + (0 + B_1)\, i = 1 + i$. For $n = 1$ we have $C_{(1,1)} = 3 + i$ and $C_1 + (B_1 - B_0)\, i = 3 + (1 - 0)\, i = 3 + i$.

Suppose the property holds for all $k \leq n$. Then:

$$\begin{aligned}
C_{(n+1,1)} &= 6C_{(n,1)} - C_{(n-1,1)} \\
&= 6(C_n + (B_n - B_{n-1})i) - (C_{n-1} + (B_{n-1} - B_{n-2})i) \\
&= 6C_n + 6(B_n - B_{n-1})i - C_{n-1} - (B_{n-1} - B_{n-2})i \\
&= 6C_n + 6B_n i - 6B_{n-1} i - C_{n-1} - B_{n-1} i + B_{n-2} i \\
&= (6C_n - C_{n-1}) + (6B_n i - B_{n-1} i) - (6B_{n-1} i - B_{n-2} i) \\
&= C_{n+1} + B_{n+1} i - B_n i \\
&= C_{n+1} + (B_{n+1} - B_n) i.
\end{aligned}$$

Hence, 3 is true.

For property 4 we also use induction on m. For $m = 0$ we have that $C_{(1,0)} = 3$ and $3(B_0 - B_{-1}) + B_0 i = 3\,(0 + 1) + 0i = 3$. For $m = 1$ we have that $C_{(1,1)} = 3 + i$ and $3(B_1 - B_0) + B_1 i = 3\,(1 - 0) + i = 3 + i$.

Suppose the property is true for every $k \leq m$. Then:

$$\begin{aligned}
C_{(1,m+1)} &= 6C_{(1,m)} - C_{(1,m-1)} \\
&= 6(3(B_m - B_{m-1}) + B_m i) - (3(B_{m-1} - B_{m-2}) + B_{m-1} i) \\
&= 18(B_m - B_{m-1}) + 6B_m i - 3(B_{m-1} - B_{m-2}) - B_m i \\
&= 18B_m - 18B_{m-1} + 6B_m i - 3B_{m-1} + 3B_{m-2} - B_m i \\
&= 18B_m - 3B_{m-1} - 18B_{m-1} + 3B_{m-2} + 6B_m i - B_m i \\
&= 3(6B_m - B_{m-1}) - 3(6B_{m-1} - B_{m-2}) + (6B_m i - B_m i) \\
&= 3B_{m+1} - 3B_m + B_{m+1} i \\
&= 3(B_{m+1} - B_m) + B_{m+1} i.
\end{aligned}$$

Thus, the property 4 holds. □

The following result gives us a relationship that the bidimensional Lucas-balancing numbers must satisfy.

Theorem 5.3: *For non-negative integers n, m, the bidimensional Lucas-balancing numbers are described as follows:*

$$C_{(n,m)} = C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i.$$

***Proof*:** We start the proof applying induction on m, fixing n. For $m = 0$ and given the initial conditions of the sequence of balancing numbers and the Proposition 2.1, we have

$$\begin{aligned} C_{(n,0)} &= C_n(B_0 - B_{-1}) + (B_n - B_{n-1})B_0 i \\ &= C_n(1) = C_n, \end{aligned}$$

which is true by Lemma 5.2, item 1. For $m = 1$ and once again considering the initial conditions of the balancing numbers, we have

$$\begin{aligned} C_{(n,1)} &= C_n(B_1 - B_0) + (B_n - B_{n-1})B_1 i \\ &= C_n(1-0) + (B_n - B_{n-1})(1)i \\ &= C_n + (B_n - B_{n-1})i, \end{aligned}$$

which is true by Lemma 5.2, item 3.

Suppose the Theorem is true for $k \le m$. We then have to show that

$$C_{(n,m+1)} = C_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{m+1} i.$$

By the recurrence relation of $\{C_{n,m}\}_{n,m\ge 0}$,

$$C_{(n,m+1)} = 6C_{(n,m)} - C_{(n,m-1)}.$$

By the induction hypothesis

$$\begin{aligned} C_{(n,m+1)} &= 6(C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i) \\ &\quad -(C_n(B_{m-1} - B_{m-2}) + (B_n - B_{n-1})B_{m-1} i) \\ &= 6C_n(B_m - B_{m-1}) + 6(B_n - B_{n-1})B_m i - C_n(B_{m-1} - B_{m-2}) \\ &\quad -(B_n - B_{n-1})B_{m-1} i \\ &= 6C_nB_m - 6C_nB_{m-1} - C_nB_{m-1} + C_nB_{m-2} \\ &\quad +(B_n - B_{n-1})(6B_m i - B_{m-1} i) \\ &= 6C_nB_m - C_nB_{m-1} - 6C_nB_{m-1} + C_nB_{m-2} \\ &\quad +(B_n - B_{n-1})(6B_m - B_{m-1})i \\ &= C_n(6B_m - B_{m-1}) - C_n(6B_{m-1} - B_{m-2}) + (B_n - B_{n-1})B_{m+1} i \\ &= C_nB_{m+1} - C_nB_m + (B_n - B_{n-1})B_{m+1} i \\ &= C_n(B_{m+1} - B_m) + (B_n - B_{n-1})B_{m+1} i, \end{aligned}$$

as we intended to show.

We now fix m and perform the induction on n. For $n=0$ the proposition is true since given the initial conditions of the sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ and taking into account the Proposition 2.1, we have

$$\begin{aligned} C_{(0,m)} &= C_0(B_m - B_{m-1}) + (B_0 - B_{-1})B_m i \\ &= (B_m - B_{m-1} 0 + B_m i, \end{aligned}$$

which is true by Lemma 5.2, item 2. For $n = 1$ the proposition is also true since by considering the initial conditions of the sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$, we have

$$\begin{aligned} C_{(1,m)} &= C_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i \\ &= 3(B_m - B_{m-1}) + B_m i, \end{aligned}$$

which is true by Lemma 5.2, item 4.

Suppose the Theorem is true for $k \leq n$. We then have to show that

$$C_{(n+1,m)} = C_{n+1}(B_m - B_{m-1}) + (B_{n+1} - B_n)B_m i.$$

Now, by the recurrence relation of $\{C_{n,m}\}_{n,m\geq 0}$,

$$C_{(n+1,m)} = 6C_{(n,m)} - C_{(n-1,m)}.$$

By the induction hypothesis and taking into account the recurrence relations of $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$

$$\begin{aligned} C_{(n+1,m)} &= 6(C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i) \\ &\quad -(C_{n-1}(B_m - B_{m-1}) + (B_{n-1} - B_{n-2})B_m i) \\ &= 6C_n(B_m - B_{m-1}) + 6(B_n - B_{n-1})B_m i - C_{n-1}(B_m - B_{m-1}) \\ &\quad -(B_{n-1} - B_{n-2})B_m i \\ &= 6C_n(B_m - B_{m-1}) + 6B_nB_m i - 6B_{n-1}B_m i - C_{n-1}(B_m - B_{m-1}) \\ &\quad -B_{n-1}B_m i + B_{n-2}B_m i \\ &= 6C_n(B_m - B_{m-1}) - C_{n-1}(B_m - B_{m-1}) + 6B_nB_m i - B_{n-1}B_m i \\ &\quad -6B_{n-1}B_m i + B_{n-2}B_m i \\ &= (6C_n - C_{n-1})(B_m - B_{m-1}) + (6B_n - B_{n-1})B_m i \\ &\quad -(6B_{n-1} - B_{n-2})B_m i \\ &= C_{n+1}(B_m - B_{m-1}) + B_{n+1}B_m i - B_nB_m i \\ &= C_{n+1}(B_m - B_{m-1}) + (B_{n+1} - B_n)B_m i, \end{aligned}$$

as we wanted to show. □

6. Some identities of bidimensional Lucas-balancing numbers in complex form

From now on, we will explore some bidimensional identities of this sequence.

Proposition 6.1: The sum of the numbers $C(p, m)$ of odd index p can be described by:

$$\sum_{l=1}^{n} C_{(2l-1,m)} = \left(\frac{4-C_n+C_{n+1}+2ST_n-2ST_{n+1}}{4}\right)(B_m - B_{m-1}) \\ +\left(3(B_{n-1}B_n)+\frac{1}{2}(B_{2n-1}+1)-B_{n-1}B_n\right)B_m i.$$

***Proof*:** By Theorem 5.3, we have

$$\sum_{l=1}^{n} C_{(2l-1,m)} = \sum_{l=1}^{n}(C_{2l-1}(B_m - B_{m-1}) + (B_{2l-1} - B_{2l-2})B_m i).$$

Thus,

$$\begin{aligned}\sum_{l=1}^{n} C_{(2l-1,m)} &= (C_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i) + \cdots \\ &\quad + (C_{2n-1}(B_m - B_{m-1}) + (B_{2n-1} - B_{2n-2})B_m i) \\ &= (C_1 + C_3 + \cdots + C_{2n-1})(B_m - B_{m-1}) \\ &\quad + ((B_1 - B_0) + (B_3 - B_2) + \cdots + (B_{2n-1} - B_{2n-2}))B_m i \\ &= \sum_{l=1}^{n} C_{2l-1}(B_m - B_{m-1}) + \sum_{l=1}^{n} B_{2l-1}B_m i - \sum_{l=1}^{n-1} B_{2l}B_m i,\end{aligned}$$

and the result follows immediately applying items 2, 3 and 5 of Proposition 2.3. □

Proposition 6.2: The sum of the numbers $C(p, m)$ of even and not null index p can be described by:

$$\sum_{l=1}^{n} C_{(2l,m)} = \left(\frac{-1-ST_n+ST_{n+1}}{2}\right)(B_m - B_{m-1}) \\ +\left(B_nB_{n+1}-3(B_{n-1}B_n)-\frac{1}{2}(B_{2n-1}+1)\right)B_m i.$$

***Proof*:** By Theorem 5.3, we have

$$\sum_{l=1}^{n} C_{(2l,m)} = \sum_{l=1}^{n}(C_{2l}(B_m - B_{m-1}) + (B_{2l} - B_{2l-1})B_m i).$$

Thus, the result follows analogously to the previous proposition from

$$\sum_{l=1}^{n} C_{(2l,m)} = \sum_{l=1}^{n} C_{2l}(B_m - B_{m-1}) + \sum_{l=1}^{n} B_{2l}B_m i - \sum_{l=1}^{n} B_{2l-1}B_m i,$$

and applying the items 2, 3 and 4 of Proposition 2.3. □

Proposition 6.3: The sum of the first p numbers $C(p, m)$, can be described by:

$$\sum_{l=1}^{n} C_{(l,m)} = \left(\frac{2-C_n+C_{n+1}}{4}\right)(B_m - B_{m-1}) + B_n B_m i.$$

***Proof*:** In an analogous way to what was done in the previous propositions, by Theorem 5.3, we obtain:

$$\sum_{l=1}^{n} C_{(l,m)} = \sum_{l=1}^{n} (C_l(B_m - B_{m-1}) + (B_l - B_{l-1})B_m i).$$

So, we obtain:

$$\begin{aligned}\sum_{l=1}^{n} C_{(l,m)} &= (C_1(B_m - B_{m-1}) + (B_1 - B_0)B_m i) + \cdots \\ &\quad + (C_n(B_m - B_{m-1}) + (B_n - B_{n-1})B_m i) \\ &= (C_1 + C_2 + \cdots + C_n)(B_m - B_{m-1}) \\ &\quad + ((B_1 - B_0) + (B_2 - B_1) + \cdots + (B_n - B_{n-1}))B_m i \\ &= \sum_{l=1}^{n} C_l(B_m - B_{m-1}) + B_n B_m i,\end{aligned}$$

that from item 7 of Proposition 2.6 in [7], the result follows. □

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