

A recursion relation for the number of Goldbach partitions of an even integer

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Abstract

The contour integral representation of the number of Goldbach partitions of an even integer, $G(n)$, is extended to an integral with a support function that equals a linear combination of integers $\{G(m)\}$. A support function is found such that there is a nontrivial integral relation relating number of Goldbach partitions of n and $m < n$. The proof of the existence of a partition of any even integer greater than or equal to four into the sum of two primes follows from a recursion relation, resulting from an integral identity, that yields a non-zero lower bound for $G(n)$. A partition of n , given a partition of $\frac{n}{2}$, is derived from an algorithm based on solutions to congruence relations related to the roots of unity. This result provides a method for generating a prime partition of every even integer greater than or equal to 6 in addition to the above demonstration of its existence.

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1. Introduction

The Hardy-Littlewood method [9] has been developed to estimate the number of partitions of an integer into sums of powers. When the exponent is one and the addends are primes, a formula for the number of Goldbach partitions is obtained. The number of ordered binary Goldbach partitions is given by

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$$G(n) = \frac{1}{2\pi} \int_0^1 R(\theta)^2 e^{-in\theta} d\theta \quad (1.1)$$

where $R(\theta) = \sum_p e^{ip\theta}$. The computation of this integral for the different partitions of an integer is based on the division of the circle into major arcs, determining the asymptotic growth, and minor arcs, contributing to the variation in the approximation [13]. This technique is adequate for establishing that every large odd integer above 10^{7194} [3], and every odd integer greater than 5 if the generalized Riemann hypothesis holds [7] [18], is the sum of three primes [17]. The existence of ternary Goldbach partitions of odd integers, beginning with 7, has been proven recently with refined estimates of the contributions of the major and minor arcs [10]. This technique may be adapted to the binary Goldbach problem to show that the density of exceptional even integers, which cannot be expressed as the sum of two primes, is less than $\frac{1}{(\log n)^A}$ [4][5][8] and even $n^{\delta-1}$ for some $\delta < 1$ [12], while even integers larger than e^{10^6} can be partitioned into sums of one prime and one almost prime [2].

The proof of the non-existence of exceptional even integers greater than or equal to four cannot be developed with integrals over major and minor arcs because of the relative magnitude of the error term and the traditional estimate of the singular integral

$$J(n) = \frac{1}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} u(\theta)^2 e^{-in\theta} d\theta \quad (1.2)$$

$$u(\theta) = \sum_{m=1}^n e^{im\theta}$$

Since the proof does not require an accurate estimate of the number of partitions of an even integer into the sum of two primes, and further, by definition, $G(n)$ is positive, it is sufficient to establish a non-zero lower bound. Regarding $G(n)$ to be a contour integral $\frac{1}{2\pi} \oint \vec{V}^{(n)} \cdot d\vec{l}$, where $\vec{V}^{(n)}(r=1) = R(\theta)^2 e^{-in\theta}$, $d\vec{l} = (0, d\theta)$, the non-vanishing of this integral depends on inequality of $\vec{V}^{(n)}$ with a gradient of a nonsingular function $\vec{\nabla}\psi$. Elaborating on results of an earlier paper [6], the existence of a non-zero lower bound for $n \geq 4$, shall be proven by equating dual representations of $G(n)$, including an expansion based a generalization of the Cauchy integral formula for nonanalytic functions. A recursion relation will be derived from an integral identity and, based on an asymptotic estimate of the Goldbach partitions of integers less than n , the estimate will be

shown in §7 to be valid for n because the leading terms are dominant and angular limits of the coefficients of the remaining terms do not affect the magnitude significantly. The use of recursive arithmetic for the proof of Goldbach-type conjectures has been considered previously [14].

Given a cyclotomic number $e^{2\pi i \frac{p_1}{n}}$, every other root of unity can be equated to a power of this number. Extending it to a product of $e^{2\pi i \frac{p_1}{n}}$ and $e^{2\pi i \frac{p_2}{n}}$, a congruence relation is derived. The solutions to the equality of this congruence with n modulo $2n$ in §8 yield prime partition of n , given the expressions of $\frac{n}{2}$ as the sums of two primes.

2. Singular Gradients and the Nonvanishing Number of Goldbach Partitions of Even Integers Greater Than or Equal to Four

Let $\vec{V}^{(n-2)}$ be a vector field for $G(n-2)$ and $\vec{V}^{(n)} = \bar{z}^2 \vec{V}^{(n-2)}$, as $z^{-2} \vec{V}^{(n-2)}$ immediately gives a non-zero value for $G(n)$. Since the line integral $G(n-2) = \frac{1}{2\pi} \oint \vec{V}^{(n-2)} \cdot \overrightarrow{d\ell}$ would vanish only if $\vec{V}^{(n-2)}$ is the gradient of a nonsingular function, the value of the integral $\frac{1}{2\pi} \oint \vec{V}^{(n)} \cdot \overrightarrow{d\ell}$ when the vector field $\vec{V}^{(n)}$ is the product of a polynomial of degree -2 in z and $\bar{z}$ and the gradient of a singular function or a vector field that is not a gradient, shall be considered.

If $\vec{V}^{(n-2)}$ is not a gradient, it is clear that the line integral $\frac{1}{2\pi} \oint \vec{V}^{(n)} \cdot \overrightarrow{d\ell}$ will not be zero. It will be supposed then that the vector field $\vec{V}^{(n-2)}$ is equal to a gradient.

It follows that

$$\frac{1}{2\pi} \oint \vec{V}^{(n)} \cdot \overrightarrow{d\ell} = \frac{1}{2\pi} \iint_D e^{-2i\theta} \vec{V}^{(n-2)} \cdot \overrightarrow{d\ell}.$$

Since the θ component of the vector field in the contour integral $G(n-2)$ is

$$\begin{aligned} \vec{V}_\theta^{(n-2)}(r=1) &= R(\theta)^2 e^{-i(n-2)\theta} \\ &= \sum_{p_1, p_2} e^{i(p_1 + p_2 - (n-2))\theta} = \frac{1}{r} \frac{\partial \Xi}{\partial \theta} \Big|_{r=1} \end{aligned} \quad (2.1)$$

$$\begin{aligned} \Xi(r=1, \theta) = & \sum_{\substack{p_1, p_2 \\ p_1 + p_2 \neq n-2}} \frac{1}{i(p_1 + p_2 - (n-2))} e^{i(p_1 + p_2 - (n-2))\theta} \\ & + \sum_{p_1 + p_2 = n-2} \theta \end{aligned} \quad (2.2)$$

A nonsingular extrapolation of this function to the entire disk [6] is

$$\begin{aligned} \Xi(r, \theta) = & i \sum_{\substack{p_1, p_2 \\ p_1 + p_2 < n-2}} \frac{1}{(n-2) - (p_1 + p_2)} r^{(n-2) - (p_1 + p_2)} e^{-i((n-2) - (p_1 + p_2))\theta} \\ & + \sum_{p_1 + p_2 = n-2} \theta \\ & - i \sum_{p_1 + p_2 > n-2} \frac{1}{p_1 + p_2 - (n-2)} r^{p-1+p-2-(n-2)} e^{i(p_1 + p_2 - (n-2))\theta} \end{aligned} \quad (2.3)$$

The use of logarithmic terms in the method based on major and minor arcs leads to sums of arithmetic functions which can be estimated with an error term that is sufficiently small. In the direct complex analytic evaluation of the contour integral, it is preferable to consider the sum without logarithmic factors to obtain a non-zero lower bound for $G(n)$, $n \geq 4$.

It has been verified that the integral

$$\frac{1}{2\pi} \int \int_D e^{-2i\theta} \left(4 \frac{\partial \Xi}{\partial \theta} + r \frac{\partial^2 \Xi}{\partial r \partial \theta} + i \frac{\partial^2 \Xi}{\partial \theta^2} \right) r \, dr \, d\theta$$

equals $G(n)$ [6].

The expression for $G(n)$ in $(z, \bar{z})$ and (r, θ) coordinates have been found to be equivalent, even for a singular extrapolation of the function (2.2) into the disk

$$\begin{aligned} \Xi(z, \bar{z}) = & i \sum_{p_1 + p_2 < n-2} z^{-((n-2) - (p_1 + p_2))} \\ & - \frac{i}{2} \sum_{p_1 + p_2 = n-2} \ln \left(\frac{z}{\bar{z}} \right) \\ & - i \sum_{p_1 + p_2 > n-2} z^{p_1 + p_2 - (n-2)} \\ \Xi(r, \theta) = & i \sum_{p_1 + p_2 < n-2} r^{-((n-2) - (p_1 + p_2))} e^{-i((n-2) - (p_1 + p_2))\theta} \\ & + \sum_{p_1 + p_2 = n-2} \theta \end{aligned} \quad (2.4)$$

$$- i \sum_{p_1 + p_2 > n-2} r^{p_1 + p_2 - (n-2)} e^{i((n-2) - (p_1 + p_2))\theta}.$$

Nontrivial constraints on $G(n)$ follow if a support function is included in the singular $z, \bar{z}$ integral. The equivalence of the dual representations of $G(n)$ can be used to demonstrate that it is non-zero.

Theorem 1 : *There exist support functions such that, if these functions are included in the singular integrals, it is not possible to set $G(n)$ equal to zero.*

Proof : If the support function is $\bar{z}^{-1} \frac{\partial F(z, \bar{z})}{\partial \bar{z}}$,

$$\begin{aligned} & - \frac{i}{\pi} \sum_{p_1 + p_2 < n-2} \int_D \int \bar{z} z^{-(n-1) - (p_1 + p_2)} \left(\bar{z}^{-1} \frac{\partial F(z, \bar{z})}{\partial \bar{z}} \right) d\bar{z} \wedge dz \quad (2.5) \\ & = - \frac{i}{\pi} \sum_{p_1 + p_2 < n-2} \left[\int_C \frac{F(z, \bar{z})}{z^{(n-1) - p_1 + p_2 - 2}} d\bar{z} \wedge dz \right. \\ & \quad \left. + \frac{i}{\pi} \frac{2\pi i}{[(n-2) - (p_1 + p_2)]!} \frac{\partial^{(n-2) - (p_1 + p_2)} F(z, \bar{z})}{\partial z^{(n-2) - (p_1 + p_2)}} \Big|_{z=0} \right] \\ & = - \frac{i}{\pi} \sum_{p_1 + p_2 < n-2} \int_C \frac{F(z, \bar{z})}{z^{(n-1) - p_1 + p_2 - 2}} d\bar{z} \wedge dz \\ & \quad - \sum_{m=4}^{n-3} \frac{2}{[n-2-m]!} \frac{\partial^{(n-2-m)} F(z, \bar{z})}{\partial z^{(n-2-m)}} \Big|_{z=0} G(m) \end{aligned}$$

The limits $\lim_{z \rightarrow 0}$ can be taken in different angular directions for the functions with parameters $\{\delta_i\}$ and $\{\delta'_i\}$, leading to the factors $e^{-2i\theta}$ and $e^{-2i\theta'}$ for $\frac{\bar{z}}{z}$. The real and imaginary parts of the equality may be considered separately. The imaginary part vanishes as $G(n)$ is real, and this condition may be imposed at each power of $\frac{1}{z}$ leading to relations between the angles at which the limits $z \rightarrow 0$ are evaluated.

The r, θ integral would be

$$\frac{1}{2\pi} \int_D \int e^{-2i\theta} F(r, \theta) \left(4 \frac{\partial \Xi}{\partial \theta} + r \frac{\partial^2 \Xi}{\partial r \partial \theta} + i \frac{\partial^2 \Xi}{\partial \theta^2} \right) r dr d\theta$$

It will be verified in §4 that the support function $\delta_0 \sinh\left(\frac{z}{\bar{z}}\right) + \dots + \delta_{N-1} \left(\sinh\left(\frac{z}{\bar{z}}\right)\right)^N$ yields a truncated series, which is required for analysis of the relation between $G(n)$ and $\{G(m), m \leq n-1\}$. It has been demonstrated that, under certain conditions, singular

integral equations can be regularized [11], and the previously divergent parts then can be cancelled leaving a finite remainder. The equation $J\varphi(x) = \xi(x) \cdot \varphi(x) + \int_M G(x, g) \wedge^* \varphi(y)$, with $x, y \in A$ for some bundle $A, \varphi \in \Gamma(A, M)$, $\xi \in \Gamma(A \otimes A, M)$ and $G \in \Gamma(A \otimes A, M \times M - \text{diagonal})$, on a Riemannian manifold of real dimension m is regularizable if the operator $H_x J$ is invertible for all $x \in M$, where

$$H_x J \sigma(u) = \xi(x) \cdot \sigma(u) + \int_{T_x} \frac{\tilde{F}_x(u, v)}{\|u - v\|^m} \wedge^* \sigma(v) \quad (2.6)$$

where T_x is the tangent space at x , for each $u \in T_x - \{0\}$, $\lim_{x \rightarrow y} r^m G(x, y) = F_x(u)$, $F|_{S_x} = F_x$ is a C^∞ bundle map, with $S_x = T_x \cap S$ and S being the sphere bundle, $\int_S F_x = 0$, and $r^{m-h} G(x, y) - r^{-h} F_x(x - y) \in \Gamma(A \otimes A, 0 \times 0)$ and $(\tilde{F}_x(u, v)(a, b) = F_x(u - v)(a - u, b - v))$, $a, b \in T_x$, $\tilde{F}_x(u, v) \in \Gamma(A \otimes A, T_x \times T_x - \text{diagonal})$. For Eq.(3.8), $M = \mathbb{C}$, $\dim M = 2$, $A = \mathbb{C}$, and if $G(x, y) \sim \frac{1}{r^k}$, $r^k G(x, y)$ is finite as $x \rightarrow y$. Consequently,

$$\begin{aligned} \|u - v\|^m \wedge^* \sigma(v) &\rightarrow \int_{T_x} dz \wedge d\bar{z} \frac{\text{finite}}{\|u - v\|^2} \\ &= \int_{\mathbb{C}/N_\varepsilon(z=0)} dz \wedge d\bar{z} \frac{\text{finite}}{\|u - v\|^2} + 2\pi \int_{|z|=\varepsilon} d|z| \frac{\text{finite}}{|z|} \\ &= \int_{\mathbb{C}/N_\varepsilon(z=0)} dz \wedge d\bar{z} \frac{\text{finite}}{\|u - v\|^2} + 2\pi \varepsilon \frac{\text{finite}}{\varepsilon} \\ z &= u - v \end{aligned}$$

Since the ε is cancelled in the last term, it is finite in the limit $\varepsilon \rightarrow 0$, $H_x J$ is an invertible operator and the integral equation is regularizable.

The r, θ integral would be

$$\frac{1}{2\pi} \int \int_D e^{-2i\theta} F(r, \theta) \left(4 \frac{\partial \Xi}{\partial \theta} + r \frac{\partial^2 \Xi}{\partial r \partial \theta} + i \frac{\partial^2 \Xi}{\partial \theta^2} \right) r dr d\theta$$

It will be verified in §4 that the support function $\delta_0 \sinh\left(\frac{z}{z}\right) + \dots + \delta_{N-1} \left(\sinh\left(\frac{z}{z}\right)\right)^N$ yields a truncated series, which is required for analysis of the relation between $G(n)$ and $\{G(m)\}$, $m \leq n - 1$.

There exist values of the parameters $\delta_0, \delta_1, \dots, \delta_{N-1}$ which represent a zero of the coefficient of the singular expression at each positive order in $\frac{1}{z}$. Since the terms proportional to $\frac{1}{z^m}$, $m \geq 1$, must cancel, the resulting singular expressions vanish. However, after fixing the angles and the parameters $\{\delta_i\}$ such that the singular part of the integral expression

vanishes, linear independence of the nonsingular combination follows for an appropriate choice of the function $F(z, \bar{z})$ and its derivatives $\left. \frac{\partial^{(n-2-m)} F(z, \bar{z})}{\partial z^{(n-2-m)}} \right|_{z=0}$. Since the vanishing of the nonsingular combination, when the singular coefficients are zero, implies linear dependence of the expressions, it follows also that, if the expressions are linearly independent, the nonlinear combination must be non-zero. When the nonsingular combination is linearly independent, then it contains parameters which can be varied without affecting the vanishing the coefficients of $\frac{1}{z^m}$, $m \geq 1$. Since the differences between the nonsingular combinations are clearly non-zero for variations of the unconstrained parameters, the formula implies that $G(n)$ must be non-zero. The property of linear independence again holds for the function $\delta_0 \sinh\left(\frac{z}{\bar{z}}\right) + \dots + \delta_{N-1} \left(\sinh\left(\frac{z}{\bar{z}}\right)\right)^N$.

Therefore, while it is possible to equate the sum to a finite result, it is not feasible to set $G(n)$ equal to zero when there are negative integer powers of z in $\Xi(z, \bar{z})$. If an extrapolation of $\Xi(r=1, \theta)$ into the disk which contains fractional negative powers of z such as

$$\begin{aligned} \Xi(z, \bar{z}) = & i \sum_{p_1+p_2 < n-2} \frac{1}{(n-2)-(p_1+p_2)} \bar{z}^{(1-\alpha)((n-2)-(p_1+p_2))} z^{-\alpha((n-2)-(p_1+p_2))} \\ & - \frac{i}{2} \sum_{p_1+p_2 = n-2} \ln\left(\frac{z}{\bar{z}}\right) \\ & - i \sum_{p_1+p_2 > n-2} \frac{1}{(p_1+p_2)-(n-2)} z^{p_1+p_2-(n-2)} \end{aligned} \quad (2.7)$$

where $0 < \alpha < 1$, integrals of the form

$$- \frac{i}{\pi} \sum_{p_1+p_2 < n-2} \iint_D \bar{z} \bar{z}^{(1-\alpha)((n-2)-(p_1+p_2))+1} z^{-\alpha((n-2)-(p_1+p_2))-1} \left(f(z, \bar{z}) \frac{\partial F(z, \bar{z})}{\partial \bar{z}} \right) d\bar{z} \wedge dz \quad (2.8)$$

occur. If $\alpha((n-2)-(p_1+p_2)) < 1$, the factors of $\bar{z}^{(1-\alpha)((n-2)-(p_1+p_2))+1}$ and $z^{-\alpha((n-2)-(p_1+p_2))-1}$ may be absorbed into $f(z, \bar{z})$ and the Cauchy integral formula can be used to obtain again a linear combination of $\{G(m)\}$.

The general formula for $\Xi(z, \bar{z})$ is

$$\begin{aligned} \Xi(z, \bar{z}) = & i \sum_{s=p_1+p_2 < n-2} \frac{1}{n-2-(p_1+p_2)} \left[\sum_m a_{ms} z^{\alpha_{ms}} \bar{z}^{-\beta_{ms}} \right]^{p_1+p-2-(n-2)} \\ & - \frac{i}{2} \sum_{p_1+p_2=n-2} \ln \left(\sum_m b_{ms} z^{\gamma_{ms}} \bar{z}^{-\varepsilon_{ms}} \right) \\ & - i \sum_{p_1+p_2 > n-2} \left[\sum_m c_{ms} z^{\eta_{ms}} \bar{z}^{-\iota_{ms}} \right]^{p_1+p_2-(n-2)} \end{aligned} \quad (2.9)$$

where $\alpha_{ms} + \beta_{ms} = 1$, $\sum_m a_{ms} = 1$, $\gamma_{ms} + \varepsilon_{ms} = 2$, $\sum_m b_{ms} = 1$, $\eta_{ms} + \iota_{ms} = 1$, $\sum_m c_{ms} = 1$. If $\alpha_{ms} < 0, \beta_{ms} > 1$, $\gamma_{ms} = \varepsilon_{ms} = 1$, $\eta_{ms} > 1, \iota_{ms} < 0$ for all m, s , there are no nontrivial constraints on $\{G(m)\}$, implying that $G(n)$ can be set equal to zero. However, since Ξ is a nonsingular function of $z, \bar{z}$, $G(n-2)$ would vanish, which eventually leads to a contradiction after sufficient number of iterations. Based on the relation $V^{(n)} = \bar{z}^2 \vec{\nabla} \Xi$, if any of these conditions on $\alpha_m, \beta_m, \gamma_m, \varepsilon_m, \eta_m, \iota_m$ are not satisfied, both $G(n-2) \neq 0$ and a nontrivial relation for $\{G(m)\}$ are obtained.

Given $\vec{V}^{(n-2)}$, the vector field $V^{(n)} = z^{-\mu_n} \bar{z}^{\nu_n} \vec{\nabla} \Sigma(z, \bar{z})$ is equivalent to the gradient of a scalar field if

$$\vec{\nabla} \Phi(z, \bar{z}) = z^{-\nu_n} \bar{z}^{\mu_n} \vec{\nabla} \Sigma(z, \bar{z}) \quad (2.10)$$

for some function $\Phi(z, \bar{z})$. The integrability condition for this equation is

$$\vec{\nabla}(z^{-\mu_n} \bar{z}^{\nu_n}) \times \vec{\nabla} \Sigma(z, \bar{z}) = 0. \quad (2.11)$$

Since $\vec{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta} \right)$ in (r, θ) coordinates,

$$\begin{aligned} \vec{\nabla}(z^{-\mu_n} \bar{z}^{\nu_n}) &= \vec{\nabla}(r^{-\mu_n + \nu_n} e^{-i(\mu_n + \nu_n)\theta}) \\ &= ((-\mu_n + \nu_n) r^{-\mu_n + \nu_n - 1} e^{-i(\mu_n + \nu_n)\theta}, -i(\mu_n + \nu_n) r^{-\mu_n + \nu_n - 1} e^{-i(\mu_n + \nu_n)\theta}) \\ &= \left((-\mu_n + \nu_n) z^{-\frac{\mu_n}{2} - \frac{1}{2}} \bar{z}^{\frac{\nu_n}{2} - \frac{1}{2}} \cdot -i(\mu_n + \nu_n) r^{-\mu_n + \nu_n - 1} e^{-i(\mu_n + \nu_n)\theta} \right) \end{aligned} \quad (2.12)$$

Many of the extrapolations of $\Sigma(r, \theta)$ to the interior of the disk have gradients which are not proportional to $\vec{\nabla}(z^{-\mu_n} \bar{z}^{\nu_n})$. The vanishing of the cross product, and therefore, $G(n)$ can occur if $\vec{\nabla}(z^{-\mu_n} \bar{z}^{\nu_n}) = (1, -i) \cdot z^{-\frac{1}{2}} \bar{z}^{\nu_n - \frac{1}{2}}$ and $\mu_n = 0$.

Consider the series expansion

$$\begin{aligned} \Xi(r, \theta) = & i \sum_{p_1+p_2 < n-2} \frac{r^{(n-2)-(p_1+p_2)}}{(n-2)-(p_1+p_2)} e^{-i((n-2)-(p_1+p_2))\theta} + \sum_{p_1+p_2=n-2} \theta \\ & -i \sum_{p_1+p_2 > n-2} \frac{r^{(n-2)-(p_1+p_2)}}{p_1+p_2-(n-2)} e^{i((p_1+p_2)-(n-2))\theta}. \end{aligned} \quad (2.13)$$

Since

$$\begin{aligned} \frac{\partial}{\partial r} \Xi(r, \theta) = & i \sum_{p_1+p_2 < n-2} r^{(n-2)-(p_1+p_2)-1} e^{-i((n-2)-(p_1+p_2))\theta} \\ & + i \sum_{p_1+p_2 > n-2} r^{(n-2)-(p_1+p_2)-1} e^{i((p_1+p_2)-(n-2))\theta} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \Xi(r, \theta) = & \sum_{p_1+p_2 < n-2} r^{(n-2)-(p_1+p_2)-1} e^{-i((n-2)-(p_1+p_2))\theta} + \frac{1}{r} \sum_{p_1+p_2=n-2} 1 \\ & + \sum_{p_1+p_2 > n-2} r^{(n-2)-(p_1+p_2)-1} e^{i((p_1+p_2)-(n-2))\theta} \end{aligned} \quad (2.14)$$

it follows that

$$\hat{\nabla} \Sigma(r, \theta) = (1, -i) \cdot \frac{\partial \Xi(r, \theta)}{\partial r} \quad (2.15)$$

if $G(n-2)=0$. With this choice of $\Sigma(z, \bar{z})$, $\mu_n=0$ and $\nu_n=2$, $\bar{\nabla}(z^{-\mu_n} \bar{z}^{\nu_n}) \times \bar{\nabla} \Sigma(z, \bar{z}) = 0$, and $\vec{V}^{(n)} = \bar{\nabla} \Phi(z, \bar{z})$.

A more general extrapolation of $\Sigma(z, \bar{a})$ from $|z|=1$ to the interior of the disk is

$$i \sum_{\substack{p_1+p_2 \\ \neq n-2}} z^{\alpha_n(p_1, p_2)} \bar{z}^{\beta_n(p_1, p_2)} + \sum_{p_1+p_2=n-2} \theta \quad (2.16)$$

where $\alpha_n(p_1, p_2) - \beta_n(p_1, p_2) = p_1 + p_2 - (n-2)$. The derivatives in the gradient are

$$\begin{aligned} & \frac{\partial}{\partial r} \left\{ i \sum_{p_1+p_2 \neq n-2} \frac{r^{\beta_n(p_1, p_2)} e^{i(p_1+p_2-(n-2))\theta}}{(n-2)-(p_1+p_2)} + \sum_{p_1+p_2=n-2} \theta \right\} \\ & = i \sum_{\substack{p_1+p_2 \\ \neq n-2}} \frac{\beta_n(p_1, p_2) r^{\beta_n(p_1, p_2)} e^{i(p_1+p_2-(n-2))\theta}}{(n-2)-(p_1+p_2)} + \sum_{p_1+p_2=n-2} \theta \end{aligned} \quad (2.17)$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left\{ i \sum_{\substack{p_1+p_2 \\ \neq n-2}} \frac{r^{\beta_n(p_1, p_2)} e^{i(p_1+p_2-(n-2))\theta}}{(n-2)-(p_1+p_2)} + \sum_{p_1+p_2=n-2} \theta \right\}$$

$$\sum_{\substack{p_1+p_2 \\ \neq n-2}} r^{\beta_n(p_1, p_2)-1} e^{i(p_1+p_2-(n-2))\theta} + \frac{1}{r} G(n-2)$$

The ratio of the θ component to the r component of the gradient is independent of r , θ only if $G(n-2)=0$ and $\beta_n(p_1, p_2) \propto n-2-(p_1+p_2)$. Let

$$\frac{1}{i \frac{\beta_n(p_1, p_2)}{n-2-(p_1+p_2)}} = -i \frac{n-2-(p_1+p_2)}{\beta_n(p_1, p_2)} = \frac{-i(\mu_n + \nu_n)}{-\mu_n + \nu_n} \quad (2.18)$$

$$\beta_n(p_1, p_2) = \frac{\mu_n - \nu_n}{\mu_n + \nu_n} (p_1 + p_2 - (n-2)).$$

Requiring $\mu_n + \nu_n = 2$,

$$\beta_n(p_1, p_2) = \frac{\mu_n - \nu_n}{2} (p_1 + p_2 - (n-1)) = (1 - \nu_n)(p_1 + p_2 - (n-2)) \quad (2.19)$$

Unless $\nu_n = 1$, it changes the sign in the sum $\sum_{p_1+p_2 \neq n-2}$.

Let $\nu_n < 1$. Then $\beta_n(p_1, p_2) < 0$ for $p_1 + p_2 < n-2$ and

$$i z^{-\mu_n} \bar{z}^{\nu_n} \sum_{\substack{p_1+p_2 \\ < n-2}} \frac{r^{\beta_n(p_1, p_2)}}{n-2-(p_1+p_2)} e^{i(p_1+p_2-(n-2))\theta} \quad (2.20)$$

$$= i \sum_{\substack{p_1+p_2 \\ < n-2}} \frac{r^{(1-\nu_n)(p_1+p_2-(n-2))-\mu_n+\nu_n}}{n-2-(p_1+p_2)} e^{i(p_1+p_2-(n-2)-\mu_n-\nu_n)\theta}$$

which remains singular at the origin since $\nu_n > 1$. If $\nu_n > 1$, $\beta_n(p_1, p_2) < 0$ for $p_1 + p_2 > n-2$, and the negative exponents of the singular terms in the third sum are unbounded.

Although $G(n) = \frac{1}{2\pi} \oint \vec{\nabla} \Phi \cdot d\vec{\ell}$ vanishes if $\Phi(z, \bar{z})$ is nonsingular, the sum, singular at $z=0$, in $\Sigma(z, \bar{z})$, appears to be necessary when $\vec{\nabla}(z^{-\mu_n} \bar{z}^{\nu_n}) = 0$. The multiplicative factor $z^{-\mu_n} \bar{z}^{\nu_n}$ in $\vec{\nabla} \Phi(z, \bar{z})$ cannot remove this singularity. Nevertheless, $G(n)$ can be represented by a surface integral, $\frac{1}{2\pi} \oint \vec{V}^{(n)} \cdot d\vec{\ell} = \frac{1}{2\pi} \int_D \vec{\nabla} \times \vec{V}^{(n)} \cdot d\vec{S}$.

The vector field $\bar{V}^{(n)}$ also could be extrapolated into the disk with the vector field $[\sum_m d_m z^{-\mu_m} \bar{z}^{\nu_m}] \bar{V} \Xi$, $\mu_m + \nu_m = 2$, $\sum_m d_m = 1$. Suppose that $\bar{V} \Psi_2 = z^{-\mu_m} \bar{z}^{\nu_m} \bar{V} \Xi$. Then the contour integral $\frac{1}{2\pi} \oint \bar{V} \Psi_2 \cdot d\bar{z}$ equals

$$\oint -\frac{i}{r} e^{-i\theta} \frac{\partial \Psi_2}{\partial \theta} dz = -i \int \int_D \frac{\partial}{\partial z} \left(\frac{z^{-\mu_m} \bar{z}^{\nu_m}}{r} e^{-i\theta} \frac{\partial \Xi}{\partial \theta} \right) d\bar{z} \wedge dz \quad (2.21)$$

If a negative power of z arises in the sum over p_1, p_2 such that $p_1 + p_2 < n-2$, it is necessary to consider the term

$$\begin{aligned} & i \sum_{p_1 + p_2 = s} \frac{1}{n-2-(p_1+p_2)} \left[\sum_m z^{\alpha_{ms}} \bar{z}^{-\beta_{ms}} \right]^{p_1+p_2-(n-2)} \\ &= i \sum_{p_1 + p_2 = s} \frac{1}{n-2-s} \left\{ \left[\sum_m z^{\alpha_{ms}} \bar{z}^{-\beta_{ms}} \right]^{-1} \right\}^{n-2-s} \end{aligned} \quad (2.22)$$

Setting $a_{ms} = 0$ for $m > 1$,

$$\begin{aligned} & i \sum_{p_1 + p_2 = s} \frac{1}{n-2-s} \left\{ \left[\sum_m z^{\alpha_{ms}} \bar{z}^{-\beta_{ms}} \right]^{-1} \right\}^{n-2-s} \\ &= i \sum_{p_1 + p_2 = s} \frac{1}{n-2-s} \left[(z^{\alpha_{1s}} \bar{z}^{-\beta_{1s}})^{-1} \right]^{n-2-s} \\ &= i \sum_{p_1 + p_2 = s} \frac{1}{n-2-s} (z^{-\alpha_{1s}} \bar{z}^{\beta_{1s}})^{n-2-s} = \frac{i}{n-2-s} (z^{-\alpha_{1s}} \bar{z}^{\beta_{1s}})^{n-2-s} G(s) \end{aligned} \quad (2.23)$$

The integral over the disk is then

$$\begin{aligned} & -\frac{i}{2\pi} \int \int_D \frac{\partial}{\partial \bar{z}} \left[z^{-\mu_m-1} \bar{z}^{\nu_m} i \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) \right. \\ & \quad \left. \left(\frac{i}{n-2-s} (z^{-\alpha_{1s}} \bar{z}^{\beta_{1s}})^{n-2-s} G(s) \right) \right] d\bar{z} \wedge dz \\ &= \frac{i}{2\pi} (\beta_{1s} - \alpha_{1s}) G(s) \int \int_D \frac{\partial}{\partial \bar{z}} \left(z^{-\mu_m-1} \bar{z}^{\nu_m} (z^{-\alpha_{1s}} \bar{z}^{\beta_{1s}})^{n-2-s} \right) d\bar{z} \wedge dz \\ &= \frac{i}{2\pi} (\beta_{1s} - \alpha_{1s}) (\nu_m + \beta_{1s} (n-2-s)) \\ & \quad \cdot \int \int_D \bar{z}^{\nu_m + \beta_{1s}(n-2-s)-1} z^{-\mu_m-1-\alpha_{1s}(n-2-s)} d\bar{z} \wedge dz \end{aligned} \quad (2.24)$$

With the inclusion of a support function such that integer exponents are not obtained, a nontrivial constraint is obtained for $\{G(m)\}$ when $-\mu_m - 1 - \alpha_{1s}(n-2-s)$ is less than zero or fractional. If the summed exponent of z positive and integer, an identity results. Nevertheless, since it has been established through a different extrapolation, $V^{(n)} = \bar{z}^2 \bar{\nabla} \Xi$, that a negative power of z in Ξ leads to a nontrivial constraint on $\{G(m)\}$ [6], it is not possible to set $G(n)$ equal to zero in a relation derived though the extension into the disk, $z^{-\mu_m} \bar{z}^{\nu_m} \bar{\nabla} \Xi$. A similar result occurs if the negative power of z arises in the sum over p_1, p_2 , $p_1 + p_2 > n-2$ or when the logarithmic term in the sum over p_1, p_2 , $p_1 + p_2 = n-2$ is dependent on r .

It is therefore not possible to allow $G(n)$ immediately to be zero if $\Xi(z, \bar{z})$ contains negative powers of z . A nontrivial relation, with both singular and nonsingular parts, always results for any extrapolation of the vector field $\bar{V}^{(n)}$ into the disk.

If $G(n)$ is directly set equal to zero, $\Xi(z, \bar{z})$ must be a nonsingular function. The vector field $\bar{V}^{(n-2)} = \bar{\nabla} \Xi$ is a gradient of a nonsingular function of $z, \bar{z}$ and $\frac{1}{2\pi} \oint \bar{V}^{(n-2)} \cdot d\vec{l} = \frac{1}{2\pi} \iint_D \bar{\nabla} \times \bar{V} \Xi \cdot d\vec{S} = 0$ and $G(n-2)$ would vanish, which is contrary to the nonvanishing of $G(m)$ for $m < 4 \times 10^{14}$ [15]. Consequently, if $G(n-2)$ is not zero, $G(n)$ also does not vanish and by induction, there exists a Goldbach partition of any even integer greater than or equal to four. $\square$

Although it may appear that a non-zero value can be obtained when the even integer is 2, the result would be formalistic and the number of Goldbach partitions of this even integer equals zero, however, as the partition is eliminated once 1 is not counted as a prime. The non-zero bound for $G(n)$, therefore, would be valid when n is an even integer greater than or equal to 4. The Goldbach conjecture then is proven, since it must be possible to express all even integers greater than or equal to 6 as a sum of two odd primes, with the integer 2 being the only even prime.

3. The Support Function

The formula for $G(n)$ follows from the identity

$$G(n) = \frac{1}{2\pi} \oint_C F(z, \bar{z}) dz = \iint_D \frac{\partial F(z, \bar{z})}{\partial \bar{z}} d\bar{z} \wedge dz \quad (3.1)$$

where $F(z, \bar{z}) = -\frac{i}{r} e^{-i\theta} \frac{\partial \Psi_2}{\partial \theta}$ where Ψ_2 is defined by $\bar{\nabla} \Psi_2(r, \theta) = \bar{z}^2 \bar{\nabla} \Xi(r, \theta)$ and $\Xi(r, \theta)$ generates $G(n-2)$. The formula

$$\iint_D \frac{\frac{\partial f}{\partial \bar{z}}}{(z-z_0)^k} d\bar{z} \wedge dz = \int_C \frac{f(z, \bar{z})}{(z-z_0)^k} dz - \frac{2\pi i}{(k-1)!} \frac{\partial^{(k-1)} f}{\partial z^{(k-1)}}(z_0) a \quad (3.2)$$

will be used for integrating higher derivatives of the support function.

With the inclusion of a support function, the $z, \bar{z}$ integral equals a linear combination of $\{G(m)\}$. A support function is required such that $F(\delta_0, \delta_1, \dots, \delta_{N-1}, 0)$, $F'(\delta_0, \delta_1, \dots, \delta_{N-1}, 0)$, ..., $F^{(n-3)}(\delta_0, \delta_1, \dots, \delta_{N-1}, 0)$ are independent and do not vanish simultaneously.

Non-zero coefficients of $G(m)$, $m \leq n$, occur when the support function is equals $\delta_0 \sinh\left(\frac{z}{\bar{z}}\right) + \dots + \left(\delta_{N-1} \sinh\left(\frac{z}{\bar{z}}\right)\right)^N$. If $\frac{\partial F_{31}^{(1)}}{\partial \bar{z}} = \bar{z} \sinh\left(\frac{z}{\bar{z}}\right)$,

$$F_{31}^{(1)}(z, \bar{z}) = \int \bar{z} \sinh\left(\frac{z}{\bar{z}}\right) d\bar{z} = \frac{1}{2} \bar{z}^2 \sinh\left(\frac{z}{\bar{z}}\right) + \frac{z}{\bar{z}} \cosh\left(\frac{z}{\bar{z}}\right) - \frac{1}{2} z^2 \text{shi}\left(\frac{z}{\bar{z}}\right) \quad (3.3)$$

where $\text{shi}(x) = \int^x \frac{\sinh(x')}{x'} dx'$ and

$$\begin{aligned} \iint_D \bar{z} \sinh\left(\frac{z}{\bar{z}}\right) d\bar{z} \wedge dz &= \frac{1}{2} \int_C \left[\frac{\bar{z}^2}{z} \sinh\left(\frac{z}{\bar{z}}\right) dz + \frac{1}{z} \cosh\left(\frac{z}{\bar{z}}\right) - \frac{1}{2} \frac{z}{\bar{z}} \text{shi}\left(\frac{z}{\bar{z}}\right) \right] \\ &\quad - \pi i \left[\bar{z}^2 \sinh\left(\frac{z}{\bar{z}}\right) + z \bar{z} \cosh\left(\frac{z}{\bar{z}}\right) - z^2 \text{shi}\left(\frac{z}{\bar{z}}\right) \right]_{z=0} = 2\pi i \end{aligned}$$

Similarly, $\iint_D \frac{\bar{z}}{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 d\bar{z} \wedge dz = -\pi i$, $\iint_D \frac{\bar{z}}{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 d\bar{z} \wedge dz = 0$, $\iint_D \frac{\bar{z}}{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^4 d\bar{z} \wedge dz = 0$ and the integrals of $\frac{\bar{z}}{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^m$, m odd, $m \geq 5$, vanish.

The sum over the primes p_1, p_2 such that $p_1 + p_2 > n-2$ includes the integrals $\iint_D \bar{z} z^{p_1+p_2-(n-1)} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^m d\bar{z} \wedge dz$, which equals zero for $m = 1, 3, 4$ and m odd, $m \geq 5$, while

$$\begin{aligned} &\iint_D \bar{z} z^{p_1+p_2-(n-1)} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 d\bar{z} \wedge dz \quad (3.4) \\ &= \int_C z^{p_1+p_2-(n-1)} \left[\frac{\bar{z}^2}{4} \cosh\left(2\frac{z}{\bar{z}}\right) + \frac{z\bar{z}}{2} \sinh\left(2\frac{z}{\bar{z}}\right) - z^2 \text{chi}\left(2\frac{z}{\bar{z}}\right) - \frac{1}{4} \bar{z}^2 \right] dz \\ &\quad - 2\pi i z^{p_1+p_2-(n-2)} \left[\frac{\bar{z}^2}{4} \cosh\left(2\frac{z}{\bar{z}}\right) + \frac{z\bar{z}}{2} \sinh\left(2\frac{z}{\bar{z}}\right) - z^2 \text{chi}\left(2\frac{z}{\bar{z}}\right) - \frac{1}{4} \bar{z}^2 \right]_{z=0} \end{aligned}$$

$$= \frac{4\pi}{p_1 + p_2 - (n-4)}.$$

The following integrals arise in the sum over the primes p_1, p_2 such that $p_1 + p_2 < n-2$

$$\begin{aligned} & \int \int_D \frac{\bar{z}}{z^{(n-1)-(p_1+p_2)}} \sinh\left(\frac{z}{\bar{z}}\right) d\bar{z} \wedge dz \quad (3.5) \\ &= \int_C \bar{z} z^{-((n-1)-(p_1+p_2))} \left[\frac{\bar{z}^2}{2} \sinh\left(\frac{z}{\bar{z}}\right) + \frac{z\bar{z}}{2} \cosh\left(\frac{z}{\bar{z}}\right) - \frac{z^2}{2} \text{shi}\left(\frac{z}{\bar{z}}\right) \right] dz \\ & - \frac{2\pi i}{((n-2)-(p_1+p_2))!} \frac{\partial^{((n-2)-(p_1+p_2))}}{\partial z^{((n-2)-(p_1+p_2))}} \left[\frac{\bar{z}^2}{2} \sinh\left(\frac{z}{\bar{z}}\right) + \frac{z\bar{z}}{2} \cosh\left(\frac{z}{\bar{z}}\right) - \frac{z^2}{2} \text{shi}\left(\frac{z}{\bar{z}}\right) \right]_{z=0} \\ &= \pi i (2\delta_{p_1+p_2, n-2} - \frac{1}{6}\delta_{p-1+p_2, n-6} + -\frac{1}{180}\delta_{p_1+p_2, n-10} + \dots) \\ & - \frac{2\pi i}{((n-2)-(p_1+p_2))!} \left[\frac{(-1)^{n-6}}{z^{n-6}} \sinh\left(\frac{z}{\bar{z}}\right) + \dots + \frac{1}{z\bar{z}^{n-7}} \sinh\left(\frac{z}{\bar{z}}\right) \delta_{n-5(\bmod 2), 0} \right. \\ & \quad \left. + \frac{1}{z\bar{z}^{n-7}} \cosh\left(\frac{z}{\bar{z}}\right) \delta_{n-5(\bmod 2), 1} \right] \end{aligned}$$

$$\begin{aligned} & \int \int_D \frac{\bar{z}}{z^{(n-1)-(p_1+p_2)}} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 d\bar{z} \wedge dz \quad (3.6) \\ &= \int_C \bar{z} z^{-((n-1)-(p_1+p_2))} \left[\frac{\bar{z}^2}{4} \cosh\left(2\frac{z}{\bar{z}}\right) + \frac{1}{2} \sinh\left(2\frac{z}{\bar{z}}\right) - z^2 \text{chi}\left(2\frac{z}{\bar{z}}\right) - \frac{z^2}{4} \right] dz \\ & - \frac{2\pi i}{((n-2)-(p_1+p_2))!} \frac{\partial^{((n-2)-(p_1+p_2))}}{\partial z^{((n-2)-(p_1+p_2))}} \left[\frac{\bar{z}^2}{4} \cosh\left(2\frac{z}{\bar{z}}\right) + \frac{1}{2} \sinh\left(2\frac{z}{\bar{z}}\right) \right. \\ & \quad \left. - z^2 \text{chi}\left(2\frac{z}{\bar{z}}\right) - \frac{\bar{z}^2}{4} \right]_{z=0} \\ &= 3\pi i \left[\left(1 + \frac{2}{3(n-4)-(p_1+p_2)} \right) \delta_{p_1+p_2, n-4} + \frac{2}{9} \delta_{p_1+p_2, n-8} + \dots \right] \\ & + \frac{2\pi i}{(n-4)-(p_1+p_2)} \end{aligned}$$

$$\begin{aligned}
& -\frac{2\pi i}{((n-2)-(p_1+p_2))!} \frac{\partial^{((n-2)-(p_1+p_2))}}{\partial z^{((n-2)-(p_1+p_2))}} \left[\frac{\bar{z}^2}{4} \cosh\left(2\frac{z}{\bar{z}}\right) + \frac{1}{2} \sinh\left(2\frac{z}{\bar{z}}\right) \right. \\
& \quad \left. - z^2 \operatorname{chi}\left(\frac{z}{\bar{z}}\right) - \frac{\bar{z}^2}{4} \right] \Big|_{z=0} \\
& \iint_D \frac{\bar{z}}{z^{(n-1)-(p_1+p_2)}} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 d\bar{z} \wedge dz \tag{3.7} \\
& = \int_C \bar{z}^{-((n-1)-(p_1+p_2))} \left[\frac{\bar{z}^2}{2} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 - \frac{3}{2} z\bar{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 \cosh\left(\frac{z}{\bar{z}}\right) \right. \\
& \quad \left. - \frac{3}{8} z^2 \operatorname{shi}\left(\frac{z}{\bar{z}}\right) + \frac{9}{8} z^2 \operatorname{shi}\left(3\frac{z}{\bar{z}}\right) \right] dz \\
& - \frac{2\pi i}{((n-2)-(p_1+p_2))!} \frac{\partial^{((n-2)-(p_1+p_2))}}{\partial z^{((n-2)-(p_1+p_2))}} \left[\frac{\bar{z}^2}{2} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 \right. \\
& \quad \left. - \frac{3}{2} z\bar{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 \cosh\left(\frac{z}{\bar{z}}\right) - \frac{3}{8} z^2 \operatorname{shi}\left(\frac{z}{\bar{z}}\right) + \frac{9}{8} z^2 \operatorname{shi}\left(3\frac{z}{\bar{z}}\right) \right] \Big|_{z=0} \\
& = \pi i \left(4\delta_{p_1+p_2, n-6} + \frac{4}{3}\delta_{p_1+p_2, n-10} + \dots \right) \\
& - \frac{2\pi i}{((n-2)-(p_1+p_2))!} \frac{\partial^{((n-2)-(p_1+p_2))}}{\partial z^{((n-2)-(p_1+p_2))}} \left[\frac{\bar{z}^2}{2} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 \right. \\
& \quad \left. - \frac{3}{2} z\bar{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 \cosh\left(\frac{z}{\bar{z}}\right) - \frac{3}{8} z^2 \operatorname{shi}\left(\frac{z}{\bar{z}}\right) + \frac{9}{8} z^2 \operatorname{shi}\left(3\frac{z}{\bar{z}}\right) \right] \Big|_{z=0} \\
& \vdots
\end{aligned}$$

After multiplication by $-\frac{i}{\pi}$, the imaginary part is eliminated only when $\theta = 0, \pi$. If the derivatives are evaluated at $\theta \neq 0, \pi$, another equation is required for the vanishing of the imaginary part, and an additional parameter δ_{n-4} should be introduced.

The singularities of the $z, \bar{z}$ integral are proportional to $\frac{1}{z}, \dots, \frac{1}{z^{n-6}}$, given that the maximum order of the derivatives is $n-4$, which follows if the range of the $p_1 + p_2$ in this sum is 2 to $n-3$. Since the Goldbach partitions are defined for integers greater than or equal to four, the range of $p_1 + p_2$ in this sum may be chosen to be 4 to $n-3$. However, the enlargement of this range is more general and will be used in the evaluation of the $z, \bar{z}$ integral.

The divergence in the sum over $p_1, p_2, p_1 + p_2 > n-2$ arising from $-4i \sum_{p_1+p_2>n-2} \frac{1}{p_1+p_2-(n-4)}$ can be removed by setting δ_1 equal to 0.

The r, θ integral is

$$\begin{aligned} & \frac{2}{\pi} \int_D \int e^{-2i\theta} \left\{ \sum_{p_1+p_2 < n-2} r^{-((n-2)-(p_1+p_2))} e^{-i((n-2)-(p_1+p_2))\theta} \right. \\ & + \sum_{p_1+p_2=n-2} 1 + \sum_{p_1+p_2 > n-2} r^{p_1+p_2-(n-2)} e^{i(p_1+p_2-(n-2))\theta} \left. \right\} \\ & [\delta_0 \sinh(e^{2i\theta}) + \dots + \delta_{N-1} (\sinh(e^{2i\theta}))^N] r dr d\theta \end{aligned} \quad (3.8)$$

The singularities of the r, θ integral are proportional to $\frac{1}{r^2}, \dots, \frac{1}{r^{n-6}}$.

It will be sufficient therefore to select parameters $\delta_0, \delta_2, \dots, \delta_{n-5}$. Equality of the r, θ and $z, \bar{z}$ integrals gives rise to equations for the parameters, which will be used to provide an estimate for magnitude of the linear combination of $G(m)$, $m \geq 4$, arising in the sum over the primes.

4. Linear Independence of the Equations for Removal of the Singularities

The equations for the removal of the singularities would be

$$\begin{aligned} \delta_0 \frac{\partial^3}{\partial z^3} & \left\{ \left[\frac{\bar{z}^2}{2} \sinh\left(\frac{z}{\bar{z}}\right) + \frac{z\bar{z}}{2} \cosh\left(\frac{z}{\bar{z}}\right) - \frac{z^2}{2} \text{shi}\left(\frac{z}{\bar{z}}\right) \right] \right. \\ & + \delta_2 \left[\frac{\bar{z}^2}{2} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 - \frac{3}{2} z\bar{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 \cosh\left(\frac{z}{\bar{z}}\right) \right. \\ & \left. \left. - \frac{3}{8} z^2 \text{shi}\left(\frac{z}{\bar{z}}\right) + \frac{9}{8} z^2 \text{shi}\left(3\frac{z}{\bar{z}}\right) \right] + \dots \right\} \Big|_{z=0} = 0 \\ & \vdots \\ \frac{\partial^{(n-4)}}{\partial z^{(n-4)}} & \left\{ \left[\frac{\bar{z}^2}{2} \sinh\left(\frac{z}{\bar{z}}\right) + \frac{z\bar{z}}{2} \cosh\left(\frac{z}{\bar{z}}\right) - \frac{z^2}{2} \text{shi}\left(\frac{z}{\bar{z}}\right) \right] \right. \\ & + \delta_2 \left[\frac{\bar{z}^2}{2} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 - \frac{3}{2} z\bar{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^2 \cosh\left(\frac{z}{\bar{z}}\right) \right. \end{aligned} \quad (4.1)$$

$$\begin{aligned}
& - \frac{3}{8} z^2 \operatorname{shi} \left(\frac{z}{\bar{z}} \right) + \frac{9}{8} z^2 \operatorname{shi} \left(3 \frac{z}{\bar{z}} \right) + \dots \Bigg|_{z=0} \\
& = h_{0,n-4} \delta_0 + h_{2,n-4} \delta_2 + \dots + h_{n-5,n-4} \delta_{n-5}
\end{aligned}$$

where $h_{i,j}$ are coefficients derived from the r, θ integral.

Defining

$$\begin{aligned}
f_0 &= \frac{\partial^3}{\partial z^3} \left[\frac{\bar{z}^2}{2} \sinh \left(\frac{z}{\bar{z}} \right) + \frac{z\bar{z}}{2} \cosh \left(\frac{z}{\bar{z}} \right) - \frac{z^2}{2} \operatorname{shi} \left(\frac{z}{\bar{z}} \right) \right] \\
f_2 &= \frac{\partial^3}{\partial z^3} \left[\frac{\bar{z}^2}{2} \left(\sinh \left(\frac{z}{\bar{z}} \right) \right)^3 - \frac{3}{2} z\bar{z} \left(\sinh \left(\frac{z}{\bar{z}} \right) \right)^2 \cosh \left(\frac{z}{\bar{z}} \right) - \frac{3}{8} z^2 \operatorname{shi} \left(\frac{z}{\bar{z}} \right) \right. \\
& \quad \left. + \frac{9}{8} z^2 \operatorname{shi} \left(3 \frac{z}{\bar{z}} \right) \right] \\
& \vdots
\end{aligned} \tag{4.2}$$

the set of $n-6$ equations is

$$(zf_0)(0)\delta_0 + (zf_2)(0)\delta_2 + \dots + (zf_{n-5})(0)\delta_{n-5} = 0 \tag{4.3}$$

$$\begin{aligned}
(z^2 f'_0)(0)\delta_0 + (z^2 f'_2)(0)\delta_2 + \dots + (z^2 f'_{n-5})(0)\delta_{n-5} &= h_{0,4}\delta_0 + h_{2,4}\delta_2 + \dots \\
&+ h_{n-5,4}\delta_{n-5} \\
&\vdots
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
(z^{n-6} f_0^{(n-7)})(0)\delta_0 + (z^{n-6} f_2^{(n-7)})(0)\delta_2 + \dots \\
+ (z^{n-6} f_{n-5}^{(n-7)})(0)\delta_{n-5} &= h_{0,n-4}\delta_0 + h_{2,4}\delta_2 + \dots + h_{n-5,n-4}\delta_{n-4}
\end{aligned}$$

the equations for the elimination of the singularities are

$$\begin{aligned}
(zf_2)(0)\delta_2 + \dots + (zf_{n-5})(0)\delta_{n-5} &= -(zf_0)(0)\delta_0 \\
[(z^2 f'_2)(0) - h_{2,4}]\delta_2 + \dots + [(z^2 f'_{n-5})(0) - h_{n-5,4}]\delta_{n-5} &= [h_{0,4} - (z^2 f'_0)(0)]\delta_0 \\
&\vdots \\
[(z^{n-6} f_2^{(n-7)})(0) - h_{2,n-4}]\delta_2 + \dots + [z^{n-6} f_{n-5}^{(n-7)}(0) - h_{n-5,n-4}]\delta_{n-5} \\
&= [h_{0,n-4} - (z^{n-6} f_0^{(n-7)})(0)]\delta_0
\end{aligned} \tag{4.5}$$

Then δ_ℓ can be expressed as a ratio of the determinants

$$\begin{vmatrix} (zf_2)(0) & \dots & -(zf_0)(0)\delta_0 & \dots & (zf_{n-5})(0) \\ & \dots & [h_{0,4} - (z^2 f'_0)(0)]\delta_0 & \dots & [(z^2 f'_{n-5})(0) - h_{n-5,4}] \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad (4.6)$$

and

$$\begin{vmatrix} (zf_2)(0) & \dots & (zf_\ell)(0)\delta_0 & \dots & (zf_{n-5})(0) \\ & \dots & [(z^2 f'_0)(0) - h_{\ell,4}]\delta_0 & \dots & [(z^2 f'_{n-5})(0) - h_{n-5,4}] \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad (4.7)$$

An estimate of δ_ℓ depends on $(z^{s+1} f_\ell^{(s)})(0)$ and $h_{i,j}$. At $r = \theta = 0$, the $z^s f_\ell^{(s)}$ increases approximately as $\max_k (\ell - 1 - 2k)^{s-2} \binom{\ell-1}{k} \text{shi}(\ell - 1 - 2k)$.

The magnitude of the term is not affected by the direction of the limit $z \rightarrow 0$. The singular terms in the r, θ integral have the form

$$\frac{2}{\pi} \sum_{p_1+p_2 < n-2} \iint_D r^{-(n-3)-(p_1+p_2)} e^{-i(n-(p_1+p_2))\theta} \delta_\ell (\sinh(e^{2i\theta}))^\ell dr d\theta \quad (4.8)$$

If $n \geq 4 + 2(\ell - 1)$, this sum is bounded by

$$\frac{2}{\pi} \sum_{p_1+p_2 < n-2} \iint_D r^{-(n-3)-(p_1+p_2)} 2\pi \delta_\ell \frac{\ell^2}{[(n-4) - 2(\ell-1)]!} \quad (4.9)$$

implying that $h_{i,j}$ is a perturbation in the matrix.

Consequently, δ_ℓ will be approximately equal to the negative of the ratio of two Wronskians, since the factors of $z^{\frac{1}{2}(n-6)(n-5)}$ cancel, and the linear relations would be independent, as the determinant (4.6) does not vanish and $(z^{s+1} f_\ell^{(s)})(0)$ increases rapidly with ℓ and s .

5. The Magnitude of the Nonsingular Linear Combination

The relation for the nonsingular part of the integral has the form $g_0 \delta_0 + g_2 \delta_2 + \dots + g_{n-5} \delta_{n-5} = h_0 \delta_0 + \dots + h_{n-5} \delta_{n-5}$, where $\{g_\ell\}$ is derived from the $z, \bar{z}$ integral and $\{h_\ell\}$ is obtained from the r, θ integral.

The nonsingular part of the $z, \bar{z}$ integral is the contour integral. For the ℓ^{th} term, the integrand is determined by the integral

$$\int \bar{z} \left(\sinh \left(\frac{z}{\bar{z}} \right) \right)^{\ell+1} d\bar{z} = -z^2 \int \frac{1}{w^3} (\sinh w)^\ell dw \quad (5.1)$$

Since

$$\begin{aligned} \int \frac{(\sinh w)^q}{w^p} dw &= \frac{-(p-2)(\sinh w)^q + qw(\sinh w)^{q-1} \cosh w}{(p-1)(p-2)w^{p-1}} \\ &+ \frac{q(q-1)}{(p-1)(p-2)} \int \frac{(\sinh w)^{q-2}}{w^{p-2}} dw + \frac{q^2}{(p-1)(p-2)} \int \frac{(\sinh w)^q}{w^{p-2}} dw \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} \int \frac{(\sinh w)^{2m+1}}{w} dw &= \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} shi((2m+1-2k)w) \\ \int \frac{(\sinh w)^{2m}}{w} dw &= \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} chi((2m-2k)w), \end{aligned} \quad (5.3)$$

the following cancellations occur for odd powers

$$\begin{aligned} &\frac{(2m+1)2m}{2} \int \frac{(\sinh w)^{2m-1}}{w} dw + \frac{(2m+1)^2}{2} \int \frac{(\sinh w)^{2m+1}}{w} dw \\ &= m(2m+1) \left[\frac{1}{2^{2m-2}} \sum_{k=0}^{m-1} (-1)^k \binom{2m-1}{k} shi((2m-1-2k)w) \right] \\ &+ \frac{(2m+1)^2}{2} \left[\frac{1}{2^{2m}} \sum_{k=0}^{m-1} (-1)^k \binom{2m+1}{k} shi((2m+1-2k)w) \right] \\ &= \frac{1}{2^{2m-2}} \left[m(2m+1) - \frac{(2m+1)^2}{8} \frac{(2m+1)2m}{(k+1)(2m-k)} \right] \binom{2m-1}{k} shi((2m-1-2k)w) \end{aligned} \quad (5.4)$$

and, for even powers,

$$\begin{aligned} &\frac{2m(2m-1)}{2} \int \frac{(\sinh w)^{2m-2}}{w} dw + \frac{(2m)^2}{2} \int \frac{(\sinh w)^{2m}}{w} dw \\ &= \frac{1}{2^{2m-3}} \left[m(2m-1) - \frac{m(2m-1)}{(k+1)(2m+1-k)} \right] \binom{2m-2}{k} chi((2m-2-2k)w) \end{aligned} \quad (5.5)$$

The term in the sum over k with the largest magnitude is determined by the extremum of $\binom{2m-1}{k} shi((2m-1-2k)w)$ and $\binom{2m-2}{k} chi((2m-2-2k)w)$ respectively. From Stirling's approximation, this is equivalent to the

maximum value of $\frac{e^{-2\kappa}}{\kappa^\kappa(1-\kappa)^{1-\kappa}}, 0 < \kappa < 1$, where k equals $\langle \kappa(2m-1) \rangle$ for odd powers and $\langle \kappa(2m-2) \rangle$ for even powers.

The critical value of κ is $\frac{1}{1+e^2}$, which implies that the maximum value of $\binom{2m-1}{k} \text{shi}((2m-1-2k)w)$ increases in magnitude approximately as $e^{2m-1}(1.135322615)^{2m-1}$ and the maximum absolute value of $\binom{2m-1}{k} \text{chi}((2m-2-2k)w)$ increases approximately as $e^{2m-2}(1.135322615)^{2m-2}$. The summation over p_1, p_2 introduces a factor for each hyperbolic function $\text{shi}((2m-1-2k)w)$ or $\text{chi}((2m-2-2k)w)$ which is a linear combination of the number of Goldbach partitions, $G(m)$, for $m \leq n-2$, which may be denoted as $N_{\ell+1,k}(\{G(m)\})$. Let $N_{\ell+1,k_c}(\{G(m)\})$ be the factor corresponding to the critical value k_c .

An expansion of the determinants (4.5) and (4.6) into minors implies that the ratio will depend on $\frac{-(z^{\ell+1}f_0^\ell)(0)}{(z^{\ell+1}f_\ell^\ell)(0)}$. The magnitude of this ratio decreases as $[(\ell-1-2k_c)^{\ell-1}e^{\ell-1}(1.135322615)^{\ell-1}]^{-1}$.

Given this estimate of $|\delta_\ell|$, it follows that

$$|g_\ell \delta_\ell| \sim N_{\ell+1,k_c}(\{G(m)\})e^{\ell-1}(1.135322615)^{\ell-1} [(\ell-1-2k_c)^{\ell-1}e^{\ell-1}(1.135322615)^{\ell-1}]^{-1} \quad (5.6)$$

$$= N_{\ell+1,k_c}(\{G(m)\})(\ell-2)^{-(\ell-2)} \left(1 - \frac{2}{1+e^2}\right)^{-(\ell-2)} \quad (5.7)$$

$$\doteq (1.3131035285)^{\ell-1} N_{\ell+1,k_c}(\ell-1)^{-(\ell-1)} \quad \ell \gg 1$$

and the magnitude of $g_\ell \delta_\ell$ will be decreasing rapidly with ℓ after the initial terms in the sequence.

The nonsingular part of the r, θ integral is

$$\frac{2}{\pi} \sum_{p_1+p_2=n-3} \int_0^1 dr \int_0^{2\pi} e^{-3i\theta} d\theta + \frac{2}{\pi} \sum_{p_1+p_2=n-2} \int_0^1 r dr \int_0^{2\pi} e^{-2i\theta} [\delta_0 \sinh(e^{2i\theta}) + \dots + \delta_{N-1} (\sinh(e^{2i\theta}))^N] d\theta \quad (5.8)$$

$$\begin{aligned}
& + \frac{2}{\pi} \sum_{p_1+p_2 > n-2} \int_0^1 r^{p_1+p_2-(n-3)} dr \int_0^{2\pi} e^{i(p_1+p_2-n)\theta} [\delta_0 \sinh(e^{2i\theta}) + \dots \\
& \qquad \qquad \qquad + \delta_{N-1} (\sinh(e^{2i\theta}))^N] d\theta \\
& = \frac{2}{\pi} \frac{1}{2} \delta_0 G(n-2) \cdot 2\pi = 2\delta_0 G(n-2)
\end{aligned}$$

and therefore, $h_\ell = 0$ for $\ell \geq 2$.

6. A Recursion Relation for $G(n)$

The Hardy-Littlewood method has been used to obtain an asymptotic formula for the number of Goldbach partitions of an even integer n given the validity of the generalized Riemann hypothesis.

Theorem 2 : *There exists a non-zero lower bound for the number of Goldbach partitions of any even integer greater than or equal to four.*

Proof : It has been verified that all even integers greater than two and less than 4×10^{14} can be partitioned into the sum of two primes. The estimate [9]

$$G(n) \sim 2 \left(1 - \frac{1}{\lambda}\right) \Pi_2 \frac{n}{(\log n)(\log(n-2))} \prod_{\substack{p \text{ odd prime} \\ \text{divisors of } n}} \frac{p-1}{p-2} \quad (6.1)$$

for $\lambda > 1$, where $\Pi_2 = 0.660161858\dots$, is valid also for all large even integers less than 4×10^{14} when the relation is defined to be equivalence in order [1].

Suppose that the formula is valid up to $n-2$

$$G(n-2) \sim 2 \left(1 - \frac{1}{\lambda}\right) \Pi_2 \frac{n-2}{(\log(n-2))(\log(n-4))} \prod_{\substack{p \text{ odd prime} \\ \text{divisors of } n-2}} \frac{p-1}{p-2} \quad (6.2)$$

Given that

$$\begin{aligned}
& \int_C z^{-((n-1)-(p_1+p_2))\theta} \left[\frac{\bar{z}^2}{2} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^4 + 2z\bar{z} \left(\sinh\left(\frac{z}{\bar{z}}\right) \right)^3 \cosh\left(\frac{z}{\bar{z}}\right) + z^2 \text{chi}\left(2\frac{z}{\bar{z}}\right) \right. \\
& \qquad \qquad \qquad \left. - z^2 \text{chi}\left(4\frac{z}{\bar{z}}\right) \right] dz \quad (6.3) \\
& = \pi i (-\delta_{p_1+p_2, n-8} + \dots)
\end{aligned}$$

the recursion relation derived from §2 and Eq. (5.7) is

$$\begin{aligned}
 2\delta_0 G(n-2) + \delta_0 \left(-\frac{1}{6}G(n-6) - \frac{1}{180}G(n-10) + \frac{49}{150}G(n-14) + \dots \right) \\
 + \delta_2 \left(4G(n-6) + \frac{4}{3}G(n-10) + \dots \right) + \delta_3 (-G(n-8) + \dots) + \dots \\
 = 2\delta_0 G(n-2)
 \end{aligned} \tag{6.4}$$

(6.5)

implying that

$$\begin{aligned}
 \left(\frac{1}{6} - 4\frac{\delta_2}{\delta_0} + \dots \right) G(n) = & \left(-\frac{1}{180}G(n-4) + \frac{49}{150}G(n-8) + \dots \right) \\
 & + \frac{\delta_2}{\delta_0} \left(\frac{4}{3}G(n-4) + \dots \right) + \frac{\delta_3}{\delta_0} (-G(n-2) + \dots) + \dots
 \end{aligned} \tag{6.6}$$

If, for example, θ is set equal to zero, $\frac{-(zf_0)(0)}{(zf_2)(0)} = 2.259440952 \times 10^{-3}$ and $\frac{-(z^2 f'_0)(0)}{(z^2 f'_3)(0)} = -9.380367687 \times 10^{-4}$.

It can be verified that, if $\theta = 0$, subsequent terms in the series $-\frac{1}{180}G(n-4) + \frac{49}{150}G(n-8) + \dots$ are $0.066306374G(n-12) + 2.463623672 \times 10^{-3}G(n-16) + 5.864504992 \times 10^{-5}G(n-20) + \dots$.

The angles $\{\theta_{G,n,\ell}\}$, at which the derivatives of $f_2, f_3, \dots, f_{n-5}$ are evaluated, must be chosen such that each term is integer. The equations for $G(m)$, $m \leq n$, and the asymptotic formula (6.2), will indicate that there exist values of $\{\theta_{G,m,\ell}\}$ yielding an expression $\frac{\delta_2}{\delta_0} \left(\frac{4}{3}G(n-4) + \dots \right) + \frac{\delta_3}{\delta_0} (-G(n-2) + \dots) + \dots$ with a sufficiently large magnitude.

Since the determinant (4.6) depends on $z^{\ell+1} f_\ell^\ell(\theta_{G,n,\ell})$ and within the range of $[\theta_{G,n,\ell} - \varepsilon, \theta_{G,n,\ell} + \varepsilon]$, where $\varepsilon \sim \frac{1}{n}$, each entry in the matrix would be multiplied by at most a factor of $\left(\frac{1}{1-\frac{1}{n}} \right)$ and the determinant would be altered by at most a factor of $\left(1 + \frac{2}{n} \right)^{(n-6)^2} \rightarrow e^{2(n-6)}$ for $n \gg 1$. This factor is not sufficient to change significantly the rapid decrease of δ_ℓ with respect

to ℓ . Overlapping of the intervals for each n covers the entire angular range since $\sum_{n=8}^N \frac{1}{n} > 2\pi$ for $N > 4820$.

Provided that the initial ratios $\frac{\delta_2}{\delta_0}, \frac{\delta_3}{\delta_0}, \dots$ are not too large and the angles $\theta_{G,n,\ell}$, $\ell \leq n_0$, where n_0 is an integer defining the dominant contributions to $G(n)$ in the nonsingular combination, are chosen such that the $\cos(\theta_{G,n,0}) \not\approx 0$, it follows that, if the recursion relation (6.5) holds for $G(m)$, $m \leq n-2$, with the asymptotic formula valid for $G(m)$, $G(n) = O\left(\frac{n}{(\log n)(\log n-2)}\right)$ for angles $\{\theta_{G,n,\ell}\}$ in a neighbourhood of $\{\theta_{G,m,\ell}\}$ for any $m \leq n-2$. The values of $\theta_{G,n,\ell}$, $\ell \leq n_0$ always can be selected to have this property by the asymptotic formula for $G(n)$ [9], with the remaining terms determining the accuracy of the estimate.

Therefore, if the estimates of the number of Goldbach partitions of even integers less than n are valid, it may be demonstrated that there is an expression of n as the sum of two primes, and indeed, the number of such binary partitions of n can be bounded below by a fraction of the asymptotic estimate for $n-2$. A potential problem stems from the use of the fraction of the asymptotic estimate of $G(n-2)$ for the inductive proof of the lower bound for the number of Goldbach partitions of $n+2$. Continuing the inductive argument indefinitely, the number of binary partitions of $n+k$ would be bounded below only by $f_1 f_2 \dots f_k G(n-2)$ which could tend to a number arbitrarily close to zero as $k \rightarrow \infty$. However, it is sufficient to establish a bound of the form (6.1) with $\lambda > 1 + \varepsilon$, $\varepsilon > 0$, for $G(n)$, given the estimate (6.2) for $G(m)$, $m \leq n-2$, because this bound increases with n . The possibility of factor $1 - \frac{1}{\lambda}$ decreasing with n is vitiated by the coefficients in the recursion relation. $\square$

It is clear that $G(n)$ will vanish for some odd integers and shall be non-zero for other odd integers, depending on the occurrence of twin primes. The inductive proof for even integers does not rely on this property. From the recursion relation (6.4) and the estimates of $\frac{\delta_\ell}{\delta_0}$ and $\theta_{G,m,\ell}$, it follows that the nonvanishing of $\{G(m)\}$, $4 \leq m \leq n-2$, m even, is sufficient for the derivation of the non-zero lower bound for $G(n)$, $n \geq 6$, n even.

7. Congruence Relations for Goldbach Prime Pairs and the Generation of Partitions of Even Integers

If n is an even integer, for any prime p_1 , there always exists k_1 such that

$$e^{\frac{2\pi i m}{n}} = (e^{\frac{2\pi i p_1}{n}})^{k_1} m < \left\lceil \frac{n}{2} \right\rceil \quad (7.1)$$

$$m \equiv p_1 k_1 \pmod{n}$$

Given another prime p_2 , there exists an integer k_2 such that

$$e^{\frac{2\pi i}{n} \left(\frac{2m}{n} \right)} = e^{\frac{2\pi i}{n} \left(\frac{p_1 k_1 + p_2 k_2}{n} \right)} \quad (7.2)$$

Suppose that the Goldbach conjecture is valid for all even integers

$2m \leq n-2$. Then there exists primes p_1, p_2 such that $e^{\frac{2\pi i}{n} \left(\frac{2m}{n} \right)} = e^{\frac{2\pi i}{n} \left(\frac{p_1 + p_2}{n} \right)}$. It follows that the two congruence relations are satisfied

$$\begin{aligned} a_1 k_1 + a_2 k_2 &\equiv 2m \pmod{n} \\ a_1(k_1 - 1) + a_2(k_2 - 1) &\equiv 0 \pmod{n} \end{aligned} \quad (7.3)$$

for some prime pair (p_1, p_2) . Conversely, if the two congruences are satisfied, a Goldbach prime pair can be obtained from $p_1 + p_2 \equiv 2m \pmod{n}$ after imposing the condition $p_1, p_2 < n$.

For any two distinct primes p_1, p_2 , there exist integers $\varepsilon_1, \varepsilon_2$ such that $p_1 \varepsilon_1 + p_2 \varepsilon_2 = 1$. The relation also implies that $\gcd(\varepsilon_1, \varepsilon_2) = 1$. If k_1, k_2 are integers defined by $p_1 k_1 + p_2 k_2 = 2m$ for a pair of primes p_1, p_2 , then they must equal $2m\varepsilon_1 - \delta_1 n$, $2m\varepsilon_2 - \delta_2 n$, where δ_1, δ_2 are maximum integers such that $2m\varepsilon_1 - \delta n > 0$, $2m\varepsilon_2 - \delta n > 0$. Then $\gcd(k_1, k_2, n) = \gcd(2m\varepsilon_1 - \delta_1 n, 2m\varepsilon_2 - \delta_2 n, n) = \gcd(2m, n) \mid 2m$. The number of incongruent solutions to $a_1 k_1 + a_2 k_2 \equiv 2m \pmod{n}$ would be $n \gcd(k_1, k_2, n)$.

Theorem 3 : *There exists a prime solution less than n , to the congruence conditions*

$$\begin{aligned} a'_1 k'_1 + a'_2 k'_2 &\equiv n \pmod{2n} \\ a'_1(k'_1 - 1) + a'_2(k'_2 - 1) &\equiv 0 \pmod{2n} \end{aligned} \quad (7.4)$$

with one of the following forms:

- (i) $p'_1 = \tilde{p}_1 + C_3 \frac{n}{2^b}$
- (ii) $p'_1 = \frac{\tilde{p}_1 + \tilde{p}'_1}{2} + C_3 \frac{n}{2^b}$
- (iii) $p'_1 = \frac{\frac{n}{2} - 1 + \tilde{p}_1}{2} + C_3 \frac{n}{2^b}$

if $\frac{n}{2}$ is an even integer.

Proof : If k'_1, k'_2 are even, they can be chosen such that $\gcd(k'_2 - 1, 2n) = 1$ and $k'_2 - 1$ has an inverse. More generally, if $2^c \parallel k'_2 - 1$, from the second congruence condition,

$$2^c a'_2 \equiv -a'_1(k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \pmod{2n} \quad (7.5)$$

and

$$a'_1 \left[k'_1 - \frac{1}{2^c} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right] \equiv n \pmod{2n} \quad (7.6)$$

$$\text{When } 2^b \parallel \left[k'_1 - \frac{1}{2^c} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right],$$

$$2^b a'_1 \equiv \left[\frac{1}{2^b} k'_1 - \frac{1}{2^{b+c}} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right]^{-1} n \pmod{2n} \quad (7.7)$$

However,

$$\begin{aligned} & \left[\frac{1}{2^b} k'_1 - \frac{1}{2^{b+c}} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right]^{-1} n \\ &= \left[\frac{1}{2^b} k'_1 - \frac{1}{2^{b+c}} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right]^{-1} \frac{n}{2} \\ &+ \left[\frac{1}{2^b} k'_1 - \frac{1}{2^{b+c}} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right]^{-1} \frac{n}{2} \end{aligned} \quad (7.8)$$

Congruence relations modulo n ,

$$\tilde{p}_1 k''_1 + \tilde{p}_2 k''_2 \equiv \frac{n}{2} \pmod{n} \quad (7.9)$$

$$\tilde{p}_1 (k''_1 - 1) + \tilde{p}_2 (k''_2 - 1) \equiv 0 \pmod{n}$$

can be satisfied by primes if there exists an even integer $2s_n$ such that $2s_n(\tilde{p}_1 k_1'' + 2s_n k_2'' \equiv \tilde{p}_1(k_1'' - 1) + \tilde{p}_2(k_2'' - 1) \pmod{n}$ or

$$\tilde{p}_1((2s_n - 1)k_1'' + 1) + \tilde{p}_2((2s_n - 1)k_2'' + 1) \equiv 0 \pmod{n} \quad (7.10)$$

This equation can be solved if the primes are equal. When $\tilde{p}_1 \neq \tilde{p}_2$, it must be established that there exists a value of s_n such that $(2s_n - 1)k_1'' \equiv (2s_n - 1)k_2'' \pmod{n}$ is congruent to -1 or 0 . In the first instance, $k_1' \equiv k_2' \pmod{n}$, while there is a solution to the second congruence only if $k_1' \equiv k_2' \equiv 0 \pmod{n}$.

By Eq.(7.9), there is a prime $\tilde{p}_1$ which equals

$$\left[\frac{1}{2^{b'}} k_1'' - \frac{1}{2^{b'+c'}} k_2'' (k_1'' - 1) \left(\frac{k_2'' - 1}{2^{c'}} \right)^{-1} \right]^{-1} \frac{n}{2} \quad (7.11)$$

where $2^{c'} \parallel k_2'' - 1$ and $2^{b'} \parallel \left[k_1'' - \frac{1}{2^{c'}} k_2'' (k_1'' - 1) \left(\frac{k_2'' - 1}{2^{c'}} \right)^{-1} \right]$. Since

$$\begin{aligned} & \left[\frac{1}{2^b} k_1' - \frac{1}{2^{b+c}} k_2' (k_1' - 1) \left(\frac{k_2' - 1}{2^c} \right)^{-1} \right]^{-1} \pmod{n} \\ &= \left\{ \left[\frac{1}{2^b} k_1' - \frac{1}{2^{b+c}} k_2' (k_1' - 1) \left(\frac{k_2' - 1}{2^c} \right)^{-1} \right] \pmod{n} \right\}^{-1}, \end{aligned} \quad (7.12)$$

it follows that

$$\begin{aligned} & \left[\frac{1}{2^b} k_1' - \frac{1}{2^{b+c}} k_2' (k_1' - 1) \left(\frac{k_2' - 1}{2^c} \right)^{-1} \right]^{-1} \\ &= \left\{ \left[\frac{1}{2^b} k_1' - \frac{1}{2^{b+c}} k_2' (k_1' - 1) \left(\frac{k_2' - 1}{2^c} \right)^{-1} \right] \pmod{n} \right\}^{-1} + C_1 n \\ & \pmod{2n} \end{aligned} \quad (7.13)$$

Then

$$\left[\frac{1}{2^b} k'_1 - \frac{1}{2^{b+c}} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right] (\text{mod } n) \quad (7.14)$$

$$= \left[\frac{1}{2^{b'}} k''_1 - \frac{1}{2^{b'+c'}} k''_2 (k''_1 - 1) \left(\frac{k''_2 - 1}{2^{c'}} \right)^{-1} \right]$$

for some $b', c', k''_1, k''_2 < n$, and

$$\left[\frac{1}{2^b} k'_1 - \frac{1}{2^{b+c}} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right]^{-1} \frac{n}{2} \quad (7.15)$$

$$\equiv \left[\frac{1}{2^{b'}} k''_1 - \frac{1}{2^{b'+c'}} k''_2 (k''_1 - 1) \left(\frac{k''_2 - 1}{2^{c'}} \right)^{-1} \right]^{-1} \frac{n}{2} + C_1 \frac{n^2}{2} \pmod{2n}$$

$$\equiv \tilde{p}_1 + C_1 \frac{n^2}{2} + C_2 n \pmod{2n}$$

for some prime $\tilde{p}_1$ since there exists a Goldbach partition of $\frac{n}{2}$. From Eq.(8.7),

$$2^b a'_1 \equiv 2^{b'+1} \tilde{p}_1 + C_1 n^2 + 2C_2 n \pmod{2n} \quad (7.16)$$

$$a'_1 \equiv 2^{b'+1-b} \tilde{p}_1 + \frac{(C_1 n^2 + 2C_2 n)}{2^b} \pmod{2n}$$

As $\tilde{p}_1$ is a prime, Eq (8.15) is consistent only if $b' \geq b-1$. Since $b' \leq b$, this implies that b' equals $b-1$ or b . The expression $\left[k'_1 - \frac{1}{2^c} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1} \right]$ is derived from $\tilde{p}_1 k''_1 + \tilde{p}_2 k''_2 \equiv \frac{n}{2} \pmod{n}$, which has one factor of two less than $a'_1 k'_1 + a'_2 k'_2$. This factor of two may be contained in a'_1 or $k'_1 - \frac{1}{2^c} k'_2 (k'_1 - 1) \left(\frac{k'_2 - 1}{2^c} \right)^{-1}$. For a prime solution $a'_1 = p'_1$, the factor of two must arise in the latter factor, implying that $b' = b-1$.

Switching the roles of a'_1 and a'_2 in the congruence relations, a similar formula for a'_2 is obtained. By Theorem 2, consider a Goldbach prime pair p'_1, p'_2 , and let $p'_1 > p'_2$. Then, $p'_2 - (2^{b-1} - C_3) \frac{n}{2^b} < \frac{n}{2}$ and $2p'_2 - 2(2^{b-1} - C_3) \frac{n}{2^b}$

equals either $2\tilde{p}_2$, $\tilde{p}_2 + \tilde{p}'_2$ or $1 + \tilde{p}_2$, the first two expressions following from existence of Goldbach partitions of even integers less than n and the last from the possibility of including one in the solution to the congruence conditions.

Then, one of the three equations holds

$$\begin{aligned} p'_2 &= \tilde{p}_2 + (2^{b-1} - C_3) \frac{n}{2^b} \\ p'_2 &= \frac{\tilde{p}_2 + \tilde{p}'_2}{2} + (2^{b-1} - C_3) \frac{n}{2^b} \\ p'_2 &= \frac{1 + \tilde{p}_2}{2} + (2^{b-1} - C_3) \frac{n}{2^b}. \end{aligned} \tag{7.17}$$

Since $p'_1 + p'_2 = n$, the complementary relations are

$$\begin{aligned} p'_1 &= \tilde{p}_1 + C_3 \frac{n}{2^b} \\ p'_1 &= \frac{\tilde{p}_1 + \tilde{p}'_1}{2} + C_3 \frac{n}{2^b} \\ p'_1 &= \frac{\frac{n}{2} - 1 + \tilde{p}_1}{2} + C_3 \frac{n}{2^b}, \end{aligned} \tag{7.18}$$

where $\tilde{p}_1 + \tilde{p}_2 = \frac{n}{2}$ and $\tilde{p}'_1 + \tilde{p}'_2 = \frac{n}{2}$. The existence of a prime $\tilde{p}'_1$ satisfying one of these equations is a consequence of the congruence relation (7.15) and the existence of a Goldbach partition of $\frac{n}{2}$. $\square$

For any Goldbach pair $(\tilde{p}_1, \tilde{p}_2)$, it remains to be demonstrated that there exist primes of the form given in Eq.(7.17). Clearly, there are an infinite number of primes of the form $\tilde{p}_1 + kn2^b$ by Dirichlet's theorem. While the magnitudes of these primes have not been established, there must be a prime $\tilde{p}_1$ and $2\tilde{p}_1$, which is required by the above theorem.

8. Conclusion

The recursion relation for $G(n)$ derived from the integral identity is shown to yield a lower bound for $G(n)$, given that the asymptotic Hardy-Littlewood estimate holds for integers less than $n-2$. Since angular limits of $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ may be chosen such that the estimate is valid, the coefficients of the leading terms will be significantly greater than the remaining terms.

the evaluation of the limits for a different angular parameter does not alter the positivity of the linear combination of $\{G(m)\}$. This result is confirmed by an explicit formula for the primes in a partition of n , given the prime partitions of $\frac{n}{2}$. It follows that any even integer $n \geq 6$ is the sum of two odd primes.

The support function must be selected to have no effect on the general result regarding the lower limit for $G(n)$. The Skolem-Mahler-Lech theorem states that the measure of the set of zeros of an arithmetic function satisfying a linear recurrence relation would be vanishing within $\mathbb{N}$ if no periodicity can be established [16]. Since periodicity would require vanishing of $G(n)$ for n less than a given bound, it would be contrary the nonvanishing of the number of Goldbach partitions of even integers less than this upper limit. Consequently, it is evident this condition will characterize any exceptional set of values for which $G(n)$ equals zero with a generic support function yielding nontrivial relations derived from the equality of the integrals over the unit disk in the $(z, \bar{z})$ and (r, θ) coordinates. The class of support functions yielding positive values of $G(n)$ would be related by a differentiable mapping that preserves the conditions compatible with a nontrivial set of independent linear equations with a solution for the number of Goldbach partitions.

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