

The extremal Sombor index of trees and unicyclic graphs with given matching number

Ting Zhou [†]

School of Mathematics

China University of Mining and Technology

Xuzhou 221116

Jiangsu

China

Zhen Lin ^{*}

School of Mathematics and Statistics

Qinghai Normal University

Xining 810001

Qinghai

China

Lianying Miao [§]

School of Mathematics

China University of Mining and Technology

Xuzhou 221116

Jiangsu

China

Abstract

In 2021, the Sombor index was introduced by Gutman, which is a new vertex-degree-based topological index. The Sombor index of a graph G is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}, \text{ where } d_G(v) \text{ is the degree of the vertex } v \text{ in } G. \text{ Let } \mathcal{T}_{n,m}$$

and $\mathcal{U}_{n,m}$ be the set of trees and unicyclic graphs on n vertices with fixed matching number

[†] E-mail: zhouting@cumt.edu.cn

^{*} E-mail: lnlinzhen@163.com (Corresponding Author)

[§] E-mail: miaolianying@cumt.edu.cn

m , respectively. In this paper, the trees and the unicyclic graphs with the maximum Sombor index among $\mathcal{T}_{n,m}$ and $\mathcal{U}_{n,m}$ are characterized, respectively.

Subject Classification: 05C05, 05C07, 05C35.

Keywords: Tree, Unicyclic graph, Sombor index, Matching number.

1. Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $N_G(v)$ denotes the set of all neighbors of v , and $d_G(v) = |N_G(v)|$ denotes the degree of vertex v in G . A pendant vertex of G is a vertex of degree 1. Let $\mathcal{P}_G$ be the set of all pendant vertices of G . Particularly, for $v \in V(G)$, denote $\mathcal{P}_G(v)$ the pendant neighbors of v and let $p_G(v) = |\mathcal{P}_G(v)|$. Let P_n and C_n denote the path and the cycle with n vertices, respectively. Let $T_{n,m}$, shown in Fig. 1.1, be the tree obtained by attaching a pendant vertex to $m-1$ non-central vertices of the star S_{n-m+1} , and let $U_{n,m}$, shown in Fig. 1.1, be the unicyclic graph obtained by attaching $n-2m+1$ pendant vertices and $m-2$ paths P_2 to one vertex of a cycle C_3 . It is easy to see that $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$ and both of $T_{n,m}$ and $U_{n,m}$ contain a matching with m edges. In particular, they have perfect matching for $n = 2m$.

The Sombor index of a graph G is $SO(G) = \sum_{uv \in E(G)} \sqrt{d_G^2(u) + d_G^2(v)}$, which is a new vertex-degree-based topological index proposed by Gutman [9].

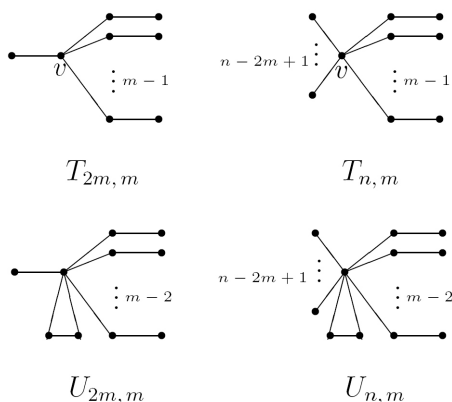


Fig 1.1

Graphs $T_{2m,m}$, $T_{n,m}$, $U_{2m,m}$, $U_{n,m}$.

The investigation of the Sombor index of graphs has quickly received much attention. Cruz et al. [4] studied the Sombor index of chemical graphs, and characterized the extremal graphs with respect to the Sombor index over the following sets: (connected) chemical graphs, chemical trees, and hexagonal systems. Das et al. [6] gave lower and upper bounds on the Sombor index of graphs with given some graph parameters. Moreover, they obtained several relations on Sombor index with the first and second Zagreb indices of graphs. Deng et al. [7] obtained a sharp upper bound for the Sombor index among all molecular trees with fixed numbers of vertices, and characterized those molecular trees achieving the extremal value. Réti et al. [22] characterized graphs with the maximum Sombor index in the classes of all connected unicyclic, bicyclic, tricyclic, tetracyclic, and pentacyclic graphs of a fixed order. Lin et al. [19] obtained lower and upper bounds on the spectral radius, energy and Estrada index of the Sombor matrix of graphs, and characterized the respective extremal graphs. Redžepović [21] showed that the Sombor index has good predictive potential. For other related results, one may refer to [10, 11, 15, 16, 23] and the references therein.

The following extremal value problem is of interest in graph theory and mathematical chemistry.

Question 1.1 : Given a set $\mathcal{G}$ of graphs, find an upper bound for the topological index over all graphs of $\mathcal{G}$, and characterize the respective extremal graphs.

Inspired by this problem, the topological index of special classes of graphs are well studied in the literature, such as the (general) sum-connectivity index of trees and unicyclic graphs with given matching number [8, 14], the general Randić index of trees with given matching number [18], the hyper-Zagreb index of trees, unicyclic and bicyclic graphs with given matching number [20], the Zagreb indices of graphs with given clique number [24], the ABC index of trees with given degree sequence [12], the Randić index of trees with given domination number [3], etc. Around the Question 1.1, we obtain upper bounds of the Sombor index of trees and unicyclic graphs with given matching number, and characterize the respective extremal graphs. The results as following:

Theorem 1.2 : If $T \in \mathcal{T}_{n,m}$, $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, then

$$SO(T) \leq (n-2m+1)\sqrt{(n-m)^2+1} + (m-1)\sqrt{(n-m)^2+4} + \sqrt{5}(m-1)$$

with equality if and only if $T \cong T_{n,m}$.

Theorem 1.3 : Let $U \in \mathcal{U}_{n,m}$, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Then

$$SO(U) \leq m\sqrt{(n-m+1)^2 + 4} + (n-2m+1)\sqrt{(n-m+1)^2 + 1} + \sqrt{5}(m-2) + \sqrt{8}$$

with equality if and only if $U \cong U_{n,m}$.

2. Preliminaries

The distance $d_G(u, v)$ between two vertices u, v of G is the length of one of the shortest (u, v) -paths in G . We call $M \subseteq E(G)$ a matching in G if its elements are edges and no two of them are adjacent in G . A matching M of G is maximum, if G has no matching M' with $|M'| > |M|$. The matching number of G is the number of edges in a maximum matching. If M is a matching of G such that vertex $v \in V(G)$ is incident with an edge of M , then v is said to be M -saturated, and if every vertex of G is M -saturated, the matching M is perfect. Obviously, a perfect matching must be maximum. We follow the notation and terminology in [2] except otherwise stated.

Lemma 2.1 : Let a and b be integers and $a, b \geq 1$.

- (i) The function $h_1(x) = \sqrt{x^2 + a} - \sqrt{(x-1)^2 + a}$ is increasing for $x \geq 2$.
- (ii) The function $h_2(x) = \sqrt{x^2 + b^2} - \sqrt{x^2 + (b-1)^2}$ is decreasing for $x \geq 2$.

Proof : Calculating the derivative of functions $h_1(x)$ and $h_2(x)$, we have the proof. $\square$

Lemma 2.2 ([17]) : Let $U \subseteq \mathbb{R}$ be an open interval and $f: U \rightarrow U$ a convex function. Let $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$ be such elements in U that inequalities $a_1 + a_2 + \dots + a_i \geq b_1 + b_2 + \dots + b_i$ hold for every $i \in \{1, 2, \dots, n\}$ and equality holds for $i = n$. Then, $f(a_1) + f(a_2) + \dots + f(a_n) \geq f(b_1) + f(b_2) + \dots + f(b_n)$.

Lemma 2.3 : Let $u_0 v_0 \in E(G)$, $N_G(u_0) \cap N_G(v_0) = \Phi$, and denote by $v_0, u_1, u_2, \dots, u_s$ and $u_0, v_1, v_2, \dots, v_t$ the neighbors of u_0 and v_0 , respectively, where $s, t \geq 1$. Let G' be a new graph with vertex set $V(G') = V(G)$ and edge set $E(G') = E(G) - \{v_0 v_i \mid 1 \leq i \leq t\} + \{u_0 v_i \mid 1 \leq i \leq t\}$. Then $SO(G') > SO(G)$.

Proof : According to the given conditions, we have $d_G(u_0) = s + 1$, $d_{G'}(u_0) = s + t + 1$, $d_G(v_0) = t + 1$, $d_{G'}(v_0) = 1$ and $d_G(u_i) = d_{G'}(u_i)$ for $1 \leq i \leq s$, $d_G(v_j) = d_{G'}(v_j)$ for $1 \leq j \leq t$. Since $f(x) = x^2$ is a convex function, by Lemma 2.2, $f(s + t + 1) + f(1) > f(s + 1) + f(t + 1)$. Thus we have

$$\begin{aligned}
 & SO(G') - SO(G) \\
 &= \sum_{uv \in E(G')} \sqrt{d_G^2(u) + d_G^2(v)} - \sum_{uv \in E(G)} \sqrt{d_G^2(u) + d_G^2(v)} \\
 &= \sum_{i=1}^s \sqrt{d_G^2(u_0) + d_G^2(u_i)} + \sum_{j=1}^t \sqrt{d_G^2(u_0) + d_G^2(v_j)} + \sqrt{d_G^2(u_0) + d_G^2(v_0)} \\
 &\quad - \left[\sum_{i=1}^s \sqrt{d_G^2(u_0) + d_G^2(u_i)} + \sum_{j=1}^t \sqrt{d_G^2(v_0) + d_G^2(v_j)} + \sqrt{d_G^2(u_0) + d_G^2(v_0)} \right] \\
 &= \sum_{i=1}^s \sqrt{(s+t+1)^2 + d_G^2(u_i)} + \sum_{j=1}^t \sqrt{(s+t+1)^2 + d_G^2(v_j)} + \sqrt{(s+t+1)^2 + 1} \\
 &\quad - \left[\sum_{i=1}^s \sqrt{(s+1)^2 + d_G^2(u_i)} + \sum_{j=1}^t \sqrt{(t+1)^2 + d_G^2(v_j)} + \sqrt{(s+1)^2 + (t+1)^2} \right] \\
 &> \sqrt{(s+t+1)^2 + 1} - \sqrt{(s+1)^2 + (t+1)^2} \\
 &> 0.
 \end{aligned}$$

This completes the proof. $\square$

3. The proof of Theorem 1.2

Lemma 3.1 ([13]) : Let $T \in \mathcal{T}_{2m, m}$, where $m \geq 2$. Then there exists $u \in \mathcal{P}_T$ such that the neighbor of u is of degree 2.

Lemma 3.2 ([1, 13]) : Let $T \in \mathcal{T}_{n, m}$, where $n > 2m$. Then there exists a maximum matching M and a vertex $u \in \mathcal{P}_T$ such that u is not M -saturated.

Lemma 3.3 : Let $m \geq 2$ and $T \in \mathcal{T}_{2m, m}$. Then

$$\sqrt{8(n-3)} + 2\sqrt{5} \leq SO(T) \leq \sqrt{m^2 + 1} + (m-1)\sqrt{m^2 + 4} + \sqrt{5}(m-1).$$

The equality in the left hand side holds if and only if $T \cong P_{2m}$, and the equality in the right hand side holds if and only if $T \cong T_{2m, m}$.

Proof : First, we want to explain that the left side inequality is obvious. Since $P_{2m} \in \mathcal{T}_{2m, m}$, from [9], the path P_n attains the minimum Sombor

index among all connected graphs of order n , so P_{2m} is also the minimum Sombor index among $\mathcal{T}_{2m,m}$.

Now, we consider the right side inequality. Let $f(m) = \sqrt{m^2 + 1} + (m-1)\sqrt{m^2 + 4} + \sqrt{5}(m-1)$. By direct calculation, we have $SO(T_{2m,m}) = f(m)$. Therefore, we next prove $SO(T) \leq SO(T_{2m,m})$ by induction on m . If $m = 2$, it is easy to check that $T = T_{4,2} = P_4$, the result holds.

Suppose that $m \geq 3$ and the result holds for trees in $\mathcal{T}_{2m-2,m-1}$. Let $T \in \mathcal{T}_{2m,m}$ with a perfect matching M . By Lemma 3.1, there exists $u_0 \in P_T$ adjacent to a vertex u of degree 2. Then $uu_0 \in M$ and $T - \{u, u_0\} \in \mathcal{T}_{2m-2,m-1}$. Let v be the neighbor of u different from u_0 . Suppose $N_T(v) = \{u, v_1, \dots, v_{s-1}\}$, that is to say $d_T(v) = s$. Here we have $p_T(v) = 0$ or $p_T(v) = 1$. Since T has a perfect matching, every pendant vertex of T is M -saturated, so $s \leq m$. By lemma 2.1 and the induction hypothesis, we have

$$\begin{aligned} SO(T) &= SO(T - \{u, u_0\}) + \sqrt{5} + \sqrt{d_T^2(v) + 4} \\ &\quad + p_T(v) \left[\sqrt{d_T^2(v) + 1} - \sqrt{(d_T(v) - 1)^2 + 1} \right] \\ &\quad + \sum_{i=p_T(v)+1}^{s-1} \left[\sqrt{d_T^2(v) + d_T^2(v_i)} - \sqrt{(d_T(v) - 1)^2 + d_T^2(v_i)} \right] \\ &= SO(T - \{u, u_0\}) + \sqrt{5} + \sqrt{s^2 + 4} + p_T(v) \left[\sqrt{s^2 + 1} - \sqrt{(s-1)^2 + 1} \right] \\ &\quad + \sum_{i=p_T(v)+1}^{s-1} \left[\sqrt{s^2 + d_T^2(v_i)} - \sqrt{(s-1)^2 + d_T^2(v_i)} \right] \\ &\leq f(m-1) + \sqrt{5} + \sqrt{m^2 + 4} + \sqrt{m^2 + 1} - \sqrt{(m-1)^2 + 1} \\ &\quad + (m-2) \left[\sqrt{m^2 + 4} - \sqrt{(m-1)^2 + 4} \right] \\ &= f(m) \end{aligned}$$

with equality if and only if $T - \{u, u_0\} \cong T_{2m-2,m-1}$, $d_T(v) = s = m$, $p_T(v) = 1$, $d_T(v_1) = 1$ and $d_T(v_i) = 2$ for $2 \leq i \leq s-1$, that is to say $T \cong T_{2m,m}$. This completes the proof. $\square$

Proof of Theorem 1.2 : Let $f(n, m) = (n - 2m + 1)\sqrt{(n - m)^2 + 1} + (m - 1)\sqrt{(n - m)^2 + 4} + (m - 1)$. By direct calculation, we have $SO(T_{n,m}) = f(n, m)$. Therefore, we next prove $SO(T) \leq SO(T_{n,m})$ by induction on n . If $n = 2m$, the result follows from Lemma 3.3.

Suppose that $n > 2m$ and the result holds for trees in $\mathcal{T}_{n-1,m}$. Let $T \in \mathcal{T}_{n,m}$. By Lemma 3.2, there is a maximum matching M and $u \in \mathcal{P}_T$ such that u is not M -saturated. Then $T - u \in \mathcal{T}_{n-1,m}$. If $uv \in E(T)$, since M is a maximum matching, v must be M -saturated. Note that there are $n-1-m$ edges in $E(T) - M$, so $d_T(v) - 1 \leq n-1-m$, that is $d_T(v) \leq n-m$. Suppose $N_T(v) = \{u, v_1, \dots, v_{s-1}\}$, that is to say $d_T(v) = s$. Here we have $1 \leq p_T(v) \leq s-1$. Note that at least $p_T(v) - 1$ pendant neighbors of v are not M -saturated, and there are $n-2m$ vertices are not M -saturated in T , then $p_T(v) - 1 \leq n-2m$. By Lemma 2.1 and induction hypothesis, we have

$$\begin{aligned}
 SO(T) &= SO(T-u) + \sqrt{d_T^2(v)+1} + (p_T(v)-1) \left[\sqrt{d_T^2(v)+1} - \sqrt{(d_T(v)-1)^2+1} \right] \\
 &\quad + \sum_{i=p_T(v)}^{s-1} \left[\sqrt{d_T^2(v)+d_T^2(v_i)} - \sqrt{(d_T(v)-1)^2+d_T^2(v_i)} \right] \\
 &= SO(T-u) + \sqrt{s^2+1} + (p_T(v)-1) \left[\sqrt{s^2+1} - \sqrt{(s-1)^2+1} \right] \\
 &\quad + \sum_{i=p_T(v)}^{s-1} \left[\sqrt{s^2+d_T^2(v_i)} - \sqrt{(s-1)^2+d_T^2(v_i)} \right] \\
 &\leq f(n-1, m) + \sqrt{(n-m)^2+1} \\
 &\quad + (n-2m) \left[\sqrt{(n-m)^2+1} - \sqrt{(n-m-1)^2+1} \right] \\
 &\quad + (m-1) \left[\sqrt{(n-m)^2+4} - \sqrt{(n-m-1)^2+4} \right] \\
 &= (n-2m+1) \sqrt{(n-m)^2+1} + (m-1) \sqrt{(n-m)^2+4} + (m-1) \\
 &= f(n, m)
 \end{aligned}$$

with equalities if and only if $G-u \cong T_{n-1,m}$, $d_T(v) = s = n-m$, $p_T(v) = n-2m+1$, $d_T(v_i) = 1$ for $1 \leq i \leq p_T(v)$ and $d_T(v_i) = 2$ for $p_T(v) \leq i \leq s-1$, that is to say $T \cong T_{n,m}$. This completes the proof. $\square$

4. The proof of Theorem 1.3

For a unicyclic graph U , we assume that the cycle $C_k = u_1 u_2 \dots u_k u_1$ is the unique cycle of U . Let $T_i = T_{u_i}$ be the tree component containing u_i in $U - E(C_k)$.

Lemma 4.1 ([5]) : Let $U \in \mathcal{U}_{2m,m}$, $m \geq 3$. If $u \in V(T_i) \cap \mathcal{P}_U$ is furthest from the root u_i such that $d_U(u, u_i) \geq 2$, then unique neighbor of u is of degree 2.

Lemma 4.2 ([25]) : Let $U \in \mathcal{U}_{n,m}$, where $n > 2m$, and $U \neq C_n$. Then there exists a maximum matching M and $u \in \mathcal{P}_U$ such that u is not M -saturated.

Lemma 4.3 : Let $U \in \mathcal{U}_{2m,m}$, where $m \geq 2$. Then

$$2\sqrt{8m} \leq SO(U) \leq m\sqrt{(m+1)^2 + 4} + \sqrt{(m+1)^2 + 1} + \sqrt{5(m-2)} + \sqrt{8}.$$

The equality in the left holds if and only if $U \cong C_{2m}$, and the equality in the right holds if and only if $U \cong U_{2m,m}$.

Proof : First, we want to explain that the left side inequality is obvious. Since $C_{2m} \in \mathcal{U}_{2m,m}$, from [22], the cycle C_n attains the minimum Sombor index among all unicyclic graphs of order n , so C_{2m} is also the minimum Sombor index among $\mathcal{U}_{2m,m}$.

Now, we consider the right side inequality. Let $g(m) = m\sqrt{(m+1)^2 + 4} + \sqrt{(m+1)^2 + 1} + \sqrt{5(m-2)} + \sqrt{8}$. By direct calculation, we have $SO(U_{2m,m}) = g(m)$. Therefore, we next prove $SO(U) < SO(U_{2m,m})$ for any $U \in \mathcal{U}_{2m,m} \setminus \{U_{2m,m}\}$. If $m = 2$, it is easy to check that $U \in \{U_{4,2}, C_4\}$, and $SO(C_4) < SO(U_{4,2})$, the result holds.

Next, suppose that $m \geq 3$ and U has the maximum Sombor index among $\mathcal{U}_{2m,m}$. Let $C_k = u_1u_2 \dots u_ku_1$ be the unique cycle of U , we consider the following two cases:

Case 1 : For $1 \leq i \leq k$, if $u \in \mathcal{P}_U \cap V(T_i)$, then $d_U(u, u_i) = 1$. In this case, U is a graph obtained from C_k by attaching some pendant vertices to u_i for $1 \leq i \leq k$, where $m \leq k < 2m$. Since every pendant vertex in U is M -saturated, we have $d_U(u_i) \leq 3$ for $1 \leq i \leq k$. If $m < k$, then there exists at least one edge $u_iu_{i+1(\bmod k)}$ of C_k such that $u_iu_{i+1(\bmod k)} \in M$. Let $U' = U - u_iu_{i+1(\bmod k)} + u_iu_{i+2(\bmod k)}$, by Lemma 2.3, $SO(U) < SO(U')$. This contradicts to the choice of U . Therefore, $m = k$, that is to say, U is the graph obtained by attaching a pendant vertex to each vertex of C_m . So $SO(U) = \sum_{uv \in E(U)} \sqrt{d_U^2(u) + d_U^2(v)} = (\sqrt{18} + \sqrt{10})m < g(m)$.

Case 2. There exists a vertex u and some $i_0 \in \{1, 2, \dots, k\}$ such that $u \in \mathcal{P}_U \cap V(T_{i_0})$ and $d_U(u, u_{i_0}) \geq 2$. In this case, let v be the unique neighbor of u , by Lemma 4.1, $d_U(v) = 2$. We prove our results by induction on m . Suppose that $m \geq 3$ and the result holds for unicyclic graphs in $\mathcal{U}_{2m-2, m-1}$. Since $uv \in M$ and $U - \{u, v\} \in \mathcal{U}_{2m-2, m-1}$. Let w be the neighbor of v different from u . Suppose $N_U(w) = \{v, w_1, \dots, w_{s-1}\}$, that is to say $d_U(w) = s$.

Similarly, we have $p_u(w) = 0$ or $p_u(w) = 1$. Since U has a perfect matching, every pendant vertex of U is M -saturated, we have $d_u(w) = s \leq m + 1$. By lemma 2.1 and the induction hypothesis, we have

$$\begin{aligned}
 SO(U) &= SO(U - \{u, v\}) + \sqrt{5} + \sqrt{d_u^2(w) + 4} \\
 &\quad + p_u(w) \left[\sqrt{d_u^2(w) + 1} - \sqrt{(d_u(w) - 1)^2 + 1} \right] \\
 &\quad + \sum_{i=p_u(w)+1}^{s-1} \left[\sqrt{d_u^2(w) + d_u^2(w_i)} - \sqrt{(d_u(w) - 1)^2 + d_u^2(w_i)} \right] \\
 &= SO(U - \{u, v\}) + \sqrt{5} + \sqrt{s^2 + 4} + p_u(w) \left[\sqrt{s^2 + 1} - \sqrt{(s-1)^2 + 1} \right] \\
 &\quad + \sum_{i=p_u(w)+1}^{s-1} \left[\sqrt{s^2 + d_u^2(w_i)} - \sqrt{(s-1)^2 + d_u^2(w_i)} \right] \\
 &\leq g(m-1) + \sqrt{5} + \sqrt{(m+1)^2 + 4} + \sqrt{(m+1)^2 + 1} - \sqrt{m^2 + 1} \\
 &\quad + (m-1) \left[\sqrt{(m+1)^2 + 4} - \sqrt{m^2 + 4} \right] \\
 &= g(m)
 \end{aligned}$$

with equalities if and only if $U - \{u, v\} \cong U_{2m-2, m-1}$, $p_u(w) = 1$, $d_u(w_1) = 1$, $d_u(w_i) = 2$ for $2 \leq i \leq s-1$ and $d_u(w) = s = m + 1$, that is to say $U \cong U_{2m, m}$. This completes the proof. $\square$

Proof of Theorem 1.3 : Let $g(n, m) = m\sqrt{(n-m+1)^2 + 4} + (n-2m+1)\sqrt{(n-m+1)^2 + 1} + \sqrt{5}(m-2) + \sqrt{8}$. By direct calculation, we have $SO(U_{n, m}) = g(n, m)$. Therefore, we next prove $SO(U) \leq SO(U_{n, m})$ by induction on n . If $n = 2m$, the result follows from Lemma 3.3.

Suppose that $n > 2m$ and the result holds for unicyclic graphs in $\mathcal{U}_{n-1, m}$. Let $U \in \mathcal{U}_{n, m}$. By Lemma 4.2, there is a maximum matching M and $u \in \mathcal{P}_U$ such that u is not M -saturated. Then $U - u \in \mathcal{U}_{n-1, m}$. If $uv \in E(U)$, $d_u(v) \leq n - m + 1$. Suppose $N_u(v) = \{u, v_1, \dots, v_{s-1}\}$, that is to say $d_u(v) = s$. Here we have $1 \leq p_u(v) \leq s-1$. Note that at least $p_u(v) - 1$ pendant vertices of v are not M -saturated, and there are $n - 2m$ vertices are not M -saturated in U , then $p_u(v) - 1 \leq n - 2m$. By Lemma 2.1 and induction hypothesis, we have

$$\begin{aligned}
 SO(U) &= SO(U - u) + (p_u(v) - 1) \left[\sqrt{d_u^2(v) + 1} - \sqrt{(d_u(v) - 1)^2 + 1} \right] \\
 &\quad + \sum_{i=p_u(v)}^{s-1} \left[\sqrt{d_u^2(v) + d_u^2(v_i)} - \sqrt{(d_u(v) - 1)^2 + d_u^2(v_i)} \right] + \sqrt{d_u^2(v) + 1}
 \end{aligned}$$

$$\begin{aligned}
&= SO(U-u) + (p_U(v)-1) \left[\sqrt{s^2+1} - \sqrt{(s-1)^2+1} \right] \\
&\quad + \sum_{i=p_U(v)}^{s-1} \left[\sqrt{s^2+d_U^2(v_i)} - \sqrt{(s-1)^2+d_U^2(v_i)} \right] + \sqrt{s^2+1} \\
&\leq g(n-1, m) + (n-2m) \left[\sqrt{(n-m+1)^2+1} - \sqrt{(n-m)^2+1} \right] \\
&\quad + m \left[\sqrt{(n-m+1)^2+4} - \sqrt{(n-m)^2+4} \right] + \sqrt{(n-m+1)^2+1} \\
&= g(n, m)
\end{aligned}$$

with equalities if and only if $U-u \cong U_{n-1, m}$, $p_U(v) = n-2m+1$, $d_U(v_i) = 2$ for $p_U(v) \leq i \leq s-1$ and $d_U(v) = s = n-m+1$, that is to say $U \cong U_{n, m}$. This completes the proof. $\square$

Acknowledgments

The authors are grateful to the anonymous referee for careful reading and valuable comments which result in an improvement of the original manuscript. This work was supported by the Postgraduate Scientific Research Innovation Project of Jiangsu Province under grant KYCX 21_2143.

References

- [1] R.A. Brualdi, J.L. Goldwasser, Permanent of the Laplacian matrix of trees and bipartite graphs, *Discrete Mathematics*, 48(1), (1984), 1-21.
- [2] J.A. Bondy, U.S.R. Murty, Graph Theory, in: Graduate Texts in Mathematics, vol. 244, *Springer*, 2008.
- [3] S. Bermudo, J.E. Nápoles, J. Rada, Extremal trees for the Randić index with given domination number, *Applied Mathematics and Computation*, 375, (2020), 125122.
- [4] R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs, *Applied Mathematics and Computation*, 399, (2021), 126018.
- [5] A. Chang, F. Tian, On the spectral radius of unicyclic graphs with perfect matching, *Linear Algebra and its Applications*, 370, (2003), 237-250.
- [6] K.Ch. Das, A.S. Çevik, I.N. Cangul, Y. Shang, On Sombor Index, *Symmetry*, 13(1), (2021), 140.

- [7] H. Deng, Z. Tang, R. Wu, Molecular trees with extremal values of Sombor indices, *International Journal of Quantum Chemistry*, <https://doi.org/10.1002/qua.26622>, (2021).
- [8] Z. Du, B. Zhou, N. Trinajstić, Minimum sum-connectivity indices of trees and unicyclic graphs of a given matching number, *J. Math. Chem.* 47, (2010), 842-855.
- [9] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Communications in Mathematical and in Computer Chemistry*, 86(1), (2021), 11-16.
- [10] I. Gutman, Some basic properties of Sombor indices, *Open Journal of Discrete Applied Mathematics*, 4(1), (2021), 1-3.
- [11] N. Ghanbari, S. Alikhan, Sombor index of certain graphs, *Iranian Journal of Mathematical Chemistry*, 12(1), (2021), 27-37.
- [12] L. Gan, B. Liu, Z. You, The ABC Index of Trees with given degree sequence, *MATCH Communications in Mathematical and in Computer Chemistry*, 68(1), (2012), 137-145.
- [13] Y. Hou, J. Li, Bounds on the largest eigenvalues of trees with a given size of matching, *Linear Algebra and its Applications*, 342, (2002), 203-217.
- [14] M.K. Jamil, I. Tomescu, Minimum general sum-connectivity index of trees and unicyclic graphs having a given matching number, *Discrete Applied Mathematics* 222, (2017), 143-150.
- [15] V.R. Kulli, Sombor indices of certain graph operators, *International Journal of Engineering Sciences & Research Technology*, 10, (2021), 127-134.
- [16] V.R. Kulli, I. Gutman, Computation of Sombor Indices of Certain Networks, *International Journal of Applied Chemistry*, 8, (2021), 1-5.
- [17] J. Karamata, Sur une inégalité relative aux fonctions convexes, *Publ. Math. Univ. Belgrade* 1, (1932), 145-148.
- [18] X. Li, J. Liu, L. Zhong, Trees with a given order and matching number that have maximum general Randić index, *Discrete Mathematics*, 310(17-18), (2010), 2249-2257.
- [19] Z. Lin, L. Miao, T. Zhou, On the spectral radius, energy and Estrada index of the Sombor matrix of graphs, *arXiv preprint arXiv:2102.03960v2* (2021).

- [20] H. Liu, Z. Tang, Maximal hyper-Zagreb index of trees, unicyclic and bicyclic graphs with a given order and matching number, *Discrete Mathematics Letters*, 4, (2020), 11-18.
- [21] I. Redžepović, Chemical applicability of Sombor indices, *Journal of the Serbian Chemical Society*, 86(5), (2021), 445-457.
- [22] T. Réti, T. Došlić, A. Ali, On the Sombor index of graphs, *Contributions to Mathematics*, 3, (2021), 11-18.
- [23] Z. Wang, Y. Mao, Y. Li, B. Furtula, On relations between Sombor and other degree-based indices, *Journal of Applied Mathematics and Computing*, <https://doi.org/10.1007/s12190-021-01516-x>, (2021).
- [24] K. Xu, The Zagreb indices of graphs with a given clique number, *Applied Mathematics Letters*, 24(6), (2011), 1026-1030.
- [25] A. Yu, F. Tian, On the spectral radius of unicyclic graphs, *MATCH Communications in Mathematical and in Computer Chemistry*, 51, (2004), 97-109.

Received April, 2021

Revised July, 2021