Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 26 (2023), No. 8, pp. 2257–2271
DOI : 10.47974/JDMSC-1568

On ideals of Sheffer stroke UP-algebras

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Abstract

The goal of the study is to introduce SUP-ideals on Sheffer Stroke UP-algebras (briefly, SUP-algebra) and its properties. We define SUP-ideals of SUP-algebras and then prove some properties. By describing a congruence relation on a SUP-algebra by the SUP-ideal, it is shown that the quotient set defined by the congruence relation is a SUP-algebra. Finally, SUP-homomorphisms on SUP-algebras are determined and the relationships between SUP-ideals of SUP-algebras are examined by means of the SUP-homomorphisms.

Subject Classification: *06F05, 03G25, 03G10.*

Keywords: *SUP-algebra, Sheffer stroke, SUP-ideal, congruence, SUP-homomorphism.*

1. Introduction

The Sheffer operation (or Sheffer stroke), introduced by H. M. Sheffer [16], is a single binary operation representing all Boolean operations

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or axioms on Boolean algebra [6] which is the basis of all modern programming languages. Thus, Sheffer operation provides some new developments in computer science. Besides, this operation is also used in algebraic structures such as orthoimplication algebras [1], ortholattices [2], Sheffer stroke UP-algebras [7], Sheffer stroke Hilbert algebras [8], fuzzy filters [9], strong Sheffer stroke non-associative MV-algebras [3] filters [10] and (fuzzy) filters of Sheffer stroke BL-algberas [11].

On the other side, UP-algebras which are generalizations of KU-algebras were introduced and developed by A. Iampan. Also, he studied ideals, subalgebras, a partially ordered relation, congruences, filters, homomorphisms, some properties on this algebraic structure and the relationship between them [4]-[5]. Then D. A. Romano deeply studied UP-ideals, UP-filters, a congruence relation and various properties of UP-algebras [12]-[14]. Moreover, he introduced UP-algebras with apartness [15].

We present basic definitions and notions of Sheffer stroke UP-algebras (in short, SUP-algebras) and Sheffer operation. Then we define a SUP-ideal of a SUP-algebra and give some properties. It is shown that the class of all SUP-ideals of a SUP-algebra forms a complete distributive lattice, and that every SUP-ideal of a SUP-algebra is its SUP-subalgebra but the inverse is generally not true. By describing a congruence relation on a SUP-algebra by its SUP-ideal, the quotient set on the algebraic structure is constructed. Also, we prove that the quotient set states a SUP-algebra, and build its SUP-ideal. Indeed, we determine a SUP-homomorphism on a SUP-algebra and indicate that the kernel of a SUP-homomorphism on a SUP-algebra is the SUP-ideal. Finally, it is demonstrated that features of SUP-ideals are preserved under the SUP-homomorphisms.

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke UP-algebra are given.

Definition 2.1 : [6] Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:

$(S1)$ $x \mid y = y \mid x$,

$(S2)$ $(x \mid x) \mid (x \mid y) = x$,

$(S3)$ $x \mid ((y \mid z) \mid (y \mid z)) = ((x \mid y) \mid (x \mid y)) \mid z$,

$(S4)\ (x\,|\,((x\,|\,x)\,|\,(y\,|\,y)))\,|\,(x\,|\,((x\,|\,x)\,|\,(y\,|\,y))) = x.$

Definition 2.2 : [7] A Sheffer stroke UP-algebra (briefly, SUP-algebras) is a structure $\langle A,|\rangle$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $x, y, z \in A$:

$(SUP-1)$ $(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(((y\,|\,(x\,|\,x))\,|\,(z\,|\,(y\,|\,y)))\,|\,((y\,|\,(x\,|\,x))\,|\,(z\,|\,(y\,|\,y)))))\,|\,(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(((y\,|\,(x\,|\,x))\,|\,(z\,|\,(y\,|\,y)))\,|\,((y\,|\,(x\,|\,x))\,|\,(z\,|\,(y\,|\,y))))) = 0,$

$(SUP-2)$ $x\,|\,x = x\,|\,(0\,|\,0)$, and

$(SUP-3)$ $(x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)) = 0$ and $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) = 0$ imply $x = y$.

Lemma 2.1 : [7] In a SUP-algebra $\langle A,|\rangle$, the following hold for all $x, y, z \in A$:

(1) $(x\,|\,(x\,|\,x))\,|\,(x\,|\,(x\,|\,x)) = 0.$

(2) $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) = 0$ *and* $(z\,|\,(y\,|\,y))\,|\,(z\,|\,(y\,|\,y)) = 0$ *imply* $(z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)) = 0.$

(3) $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) = 0$ *imply* $(((y\,|\,(z\,|\,z))\,|\,(y\,|\,(z\,|\,z)))\,|\,(x\,|\,(z\,|\,z)))\,|\,(((y\,|\,(z\,|\,z))\,|\,(y\,|\,(z\,|\,z)))\,|\,(x\,|\,(z\,|\,z))) = 0.$

(4) $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) = 0$ *imply* $(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(z\,|\,(y\,|\,y)))\,|\,(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(z\,|\,(y\,|\,y))) = 0.$

(5) $(((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))\,|\,(x\,|\,x))\,|\,(((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))\,|\,(x\,|\,x)) = 0.$

(6) $x\,|\,(x\,|\,(y\,|\,y)) = 0\,|\,0$ *if and only if* $x\,|\,x = x\,|\,(y\,|\,y).$

(7) $(((y\,|\,(y\,|\,y))\,|\,(y\,|\,(y\,|\,y)))\,|\,(x\,|\,x))\,|\,(((y\,|\,(y\,|\,y))\,|\,(y\,|\,(y\,|\,y)))\,|\,(x\,|\,x)) = 0.$

(8) $x\,|\,0 = 0\,|\,0.$

Proposition 2.1 : [7] Let $\langle A,|\rangle$ be a SUP-algebra. Then the binary relation $x \le y$ if and only if $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) = 0$ is a partial order on A.

Lemma 2.2 : *Let* $\langle A,|\rangle$ *be a SUP-algebra. Then*

(1) $x \le y$ *implies*

(*a*) $y\,|\,(z\,|\,z) \le x\,|\,(z\,|\,z),$

(b) $z\,|\,(x\,|\,x)\leq z\,|\,(y\,|\,y)$,

(2) $x\leq y$ *if and only if* $y\,|\,y\leq x\,|\,x$,

(3) $y\,|\,(x\,|\,x)\leq x$,

(4) $y\leq (y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x))$,

(5) $x\leq y$ *implies* $x\leq (y\,|\,(z\,|\,z))\,|\,(y\,|\,(z\,|\,z))$,

(6) $z\,|\,(y\,|\,y)\leq z\,|\,(y\,|\,(x\,|\,x))$,

(7) $((z\,|\,(y\,|\,y))\,|\,(z\,|\,(y\,|\,y)))\,|\,(x\,|\,x)\leq z\,|\,(y\,|\,(x\,|\,x))$ *and*

(8) $x\,|\,((y\,|\,(z\,|\,z))\,|\,(y\,|\,(z\,|\,z)))\leq (x\,|\,(y\,|\,y))\,|\,((x\,|\,(z\,|\,z))\,|\,(x\,|\,(z\,|\,z)))$,

for all $x,y,z\in A$.

3. Ideals of SUP-algebras

In this section, we give some definitions and notions about SUP-ideals of a SUP-algebra. Let A be a SUP-algebra, unless otherwise is stated.

Definition 3.1: A subset $J\subseteq A$ is called a SUP-ideal of A if it satisfies the following properties:

$(SUPi-1)$ $y\in J$ imply $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x))\in J$,

$(SUPi-2)$ $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x))\in J$ and $x\in J$ imply $y\in J$,

for all $x,y\in A$

Example 3.1 : Consider the SUP-algebra $\langle A,|\rangle$, where $A\{0,a,b,c\}$ with Cayley table as below [7]:

$\vert$	0	a	b	c
0	a	a	a	a
a	a	0	c	b
b	a	c	c	a
c	a	b	a	b

Then it is clear that A itself and $\{0\}$ are SUP-ideals of A. Also, $\{0,b\}$ and $\{0,c\}$ are SUP-ideals of A.

Lemma 3.1 : *If J is a SUP-ideal of A, then $x\leq y$ and $x\in J$ imply $y\in J$, for all $x,y\in A$.*

Proof: Let J be a SUP-ideal of A. Then we obtain from Lemma 2.1 (1) and $(SUPi-1)$ that $0=(x|(x|x))|(x|(x|x))\in J$, for all $x\in A$. Let $x\leq y$ and $x\in J$. Since $(y|(x|x))|(y|(x|x))=0\in J$ from Proposition 2.1, it follows from $(SUPi-2)$ that $y\in J$. □

Theorem 3.1: *Let J be a subset of A such that $0\in J$. Then J is a SUP-ideal of A if and only if $(z|(x|x))|(z|(x|x))\notin J$ and $y\in J$ imply $(((z|(y|y))|(z|(y|y)))|(x|x))|(((z|(y|y))|(z|(y|y)))|(x|x))\notin J$, for all $x,y,z\in A$.*

Proof: Let J be a SUP-ideal of A, $(z|(x|x))|(z|(x|x))\notin J$ and $y\in J$. Assume that $(((z|(y|y))|(z|(y|y)))|(x|x))|(((z|(y|y))|(z|(y|y)))|(x|x))\in J$. Since $(((z|(x|x))|(z|(x|x)))|(y|y))|(((z|(x|x))|(z|(x|x)))|(y|y))=(((z|(y|y))|(z|(y|y)))|(x|x))|(((z|(y|y))|(z|(y|y)))|(x|x))\in J$ from (S1) and (S3), we get from $(SUPi-2)$ that $(z|(x|x))|(z|(x|x))\in J$ which contradicts with the hypothesis. Thus, it has to be $(((z|(y|y))|(z|(y|y)))|(x|x))|(((z|(y|y))|(z|(y|y)))|(x|x))\notin J$.

Conversely, let J be a subset of A such that $0\in J$, and $(z|(x|x))|(z|(x|x))\notin J$ and $y\in J$ imply $(((z|(y|y))|(z|(y|y)))|(x|x))|(((z|(y|y))|(z|(y|y)))|(x|x))\notin J$, for all $x,y,z\in A$. Suppose that $(y|(x|x))|(y|(x|x))\notin J$ and $y\in J$. Then $0=(0|0)|(0|0)=(0|(x|x))|(0|(x|x))=(((y|(y|y))|(y|(y|y)))|(x|x))|(((y|(y|y))|(y|(y|y)))|(x|x))\notin J$ from (S1), (S2), Lemma 2.1 (1) and (8) which is a contradiction. Thus, $(y|(x|x))|(y|(x|x))\in J$, i.e., $(SUPi-1)$ is satisfied. Assume that $(y|(x|x))|(y|(x|x))\in J$ and $x\in J$ but $y\notin J$. Since $y=(y|(0|0))|(y|(0|0))\notin J$ from (S2) and $(SUP-2)$, we have $(y|(x|x))|(y|(x|x))=(((y|(x|x))|(y|(x|x)))|(0|0))|(((y|(x|x))|(y|(x|x)))|(0|0))\notin J$ which is a contradiction. So, $y\in J$ which means that $(SUPi-2)$ is proved. Thereby, J is a SUP-ideal of A. □

Theorem 3.2 : *Let J be a subset of A such that $0\in J$. Then J is a SUP-ideal of A if and only if $x\in J$ and $y\notin J$ imply $(y|(x|x))|(y|(x|x))\notin J$, for all $x,y\in A$.*

Proof: Let J be a SUP-ideal of A. Suppose that $x\in J$ and $(y|(x|x))|(y|(x|x))\in J$ but $y\notin J$. Then it contradicts with $(SUPi-2)$. Thus, $(y|(x|x))|(y|(x|x))\notin J$.

Conversely, let J be a subset of A such that $0\in J$, and $x\in J$ and $y\notin J$ imply $(y|(x|x))|(y|(x|x))\notin J$, for all $x,y\in A$. Assume that $y\in J$ and $(y|(x|x))|(y|(x|x))\notin J$. So, $0=(0|0)|(0|0)=(0|(x|x))|(0|(x|x))=(((y|(y|y))|(y|(y|y)))|(x|x))|(((y|(y|y))|(y|(y|y)))|(x|x))=(((y|(x|x))|(y|(x|x)))|(y|y))|(((y|(x|x))|(y|(x|x)))|(y|y))\notin J$

which is a contradiction. Thus, $(y|(x|x))|(y|(x|x)) \in J$ which means that $(SUPi-1)$ holds. Also, $(SUPi-2)$ is clear from the hypothesis. Hence, J is a SUP-ideal of A. □

Theorem 3.3 : *Let J be a subset of A such that $0 \in J$. Then J is a SUP-ideal of A if and only if $(z|(x|x))|(z|(x|x)) \notin J$ and $(((z|(y|y))|(z|(y|y)))|(x|x)) | (((z|(y|y))|(z|(y|y)))|(x|x)) \in J$ imply $y \notin J$, for all $x, y, z \in A$.*

Proof: Let J be a SUP-ideal of A. Assume that $(z|(x|x))|(z|(x|x)) \notin J$ and $(((z|(y|y))|(z|(y|y)))|(x|x)) | (((z|(y|y))|(z|(y|y)))|(x|x)) \in J$ but $y \in J$. Then it is obtained from Theorem 3.1 that $(((z|(y|y))|(z|(y|y))) | (x|x)) | (((z|(y|y))|(z|(y|y)))|(x|x)) \notin J$ which is a contradiction. Hence, it must be $y \notin J$.

Conversely, let J be a subset of A such that $0 \in J$, and $(z|(x|x)) | (z|(x|x)) \notin J$ and $(((z|(y|y))|(z|(y|y)))|(x|x)) | (((z|(y|y)) | (z|(y|y)))|(x|x)) \in J$ imply $y \notin J$, for all $x, y, z \in A$. Suppose that $x \in J$, $y \notin J$ and $(y|(x|x))|(y|(x|x)) \in J$. Since $(y|(0|0))|(y|(0|0)) = y \notin J$ and $(((y|(x|x))|(y|(x|x)))|(0|0)) | (((y|(x|x))|(y|(x|x)))|(0|0)) = (y|(x|x))|(y|(x|x)) \in J$ from $(SUP-2)$ and (S2), it follows that $x \notin J$ which is a contradiction. Thus, $(y|(x|x))|(y|(x|x)) \notin J$. Therefore, J is a SUP-ideal of A by Theorem 3.2. □

Corollary 3.1 : *Let J be a subset of A such that $0 \in J$. Then J is a SUP-ideal of A if and only if $(y|(x|x))|(y|(x|x)) \in J$ and $y \notin J$ imply $x \notin J$, for all $x, y \in A$.*

Theorem 3.4 : *The family $\Im_A$ of all SUP-ideals of A forms a complete distributive lattice.*

Proof: Let $\{J_i\}_{i\in I}$ be a family of SUP-ideals of a SUP-algebra $\langle A, |\rangle$. Assume that $y \in \bigcap_{i\in I} J_i$. Then $y \in J_i$, for all $i \in I$. Since J_i is a SUP-ideal of A, for all $i \in I$, we get $(y|(x|x))|(y|(x|x)) \in J_i$, for all $i \in I$ which implies that $(y|(x|x))|(y|(x|x)) \in \bigcap_{i\in I} J_i$. Also, let $(y|(x|x))|(y|(x|x)) \in \bigcap_{i\in I} J_i$ and $x \in \bigcap_{i\in I} J_i$. So, $(y|(x|x))|(y|(x|x)) \in J_i$ and $x \in J_i$, for all $i \in I$. Since J_i is a SUP-ideal of A, for all $i \in I$, we have $y \in J_i$, for all $i \in I$ which means that $y \in \bigcap_{i\in I} J_i$. Thus, $\bigcap_{i\in I} J_i$ is a SUP-ideal of A. Let Ω be the family of all SUP-ideals of A containing the union $\bigcup_{i\in I} J_i$. Then $\bigcap \Omega$ is a SUP-ideal of A. If $\wedge_{i\in I} J_i = \bigcap_{i\in I} J_i$ and $\vee_{i\in I} J_i = \bigcap \Omega$, then $(\Im, \wedge, \vee)$ is a complete lattice. Also, it is distributive from the definitions of $\vee$ and $\wedge$. □

Corollary 3.2 : *Let* χ *be a subset of a SUP-algebra* $\langle A, | \rangle$. *Then there is the minimal SUP-ideal* $\langle \chi \rangle$ *containing the subset* χ.

Proof: Let $M = \{J : J \text{ is a } SUP- \text{ ideal } of\ A \text{ containing } \chi \subseteq A\}$. Then $\bigcap M$ is the minimal SUP-ideal of A containing $\chi \subseteq A$. □

Definition 3.2 : [7] Let $\mathcal{A} = \langle A, | \rangle$ be a SUP-algebra. A subset B of A is called a SUP-subalgebra of A if the fixed element 0 of A is in B and $\langle B, | \rangle$ forms a SUP-algebra. Clearly, A and $\{0\}$ are SUP-subalgebras of A.

Lemma 3.2 : *Any SUP-ideal of a SUP-algebra is s SUP-subalgebra of this SUP-algebra.*

Proof: Let $\langle A, | \rangle$ be a SUP-algebra and J be a SUP-ideal of A. We have already known $0 \in J$ since J is a SUP-ideal of A. Then J satisfies $(SUP-1)-(SUP-3)$ for all $x, y, z \in J$ because $J \subseteq A$ and A is a SUP-algebra. Thus, $\langle J, | \rangle$ be a SUP-subalgebra of A. □

However, the inverse of Lemma 3.2 is not true.

Example 3.2 : Consider the SUP-algebra A in Example 3.1. A subset $\{0, a\}$ of A is a SUP-subalgebra of A but it is not SUP-ideal of A.

Definition 3.3 : Let J be a SUP-ideal of A. Define the binary relation $\sim_J$ on A as follows:

$x \sim_J y$ if and only if $(y|(x|x))|(y|(x|x)) \in J$ and $(x|(y|y))|(x|(y|y)) \in J$, for all $x, y \in A$.

Example 3.3 : Consider the SUP-ideal $J = \{0, c\}$ in Example 3.1. Then

$$\sim_J = \{(0,0),(a,a),(b,b),(c,c),(0,c),(c,0),(a,b),(b,a)\}$$

is a binary relation on A. In fact, it can be seen easily that $\sim_J$ is an equivalence relation on A.

Definition 3.4: If xRy implies $x|zRy|z$, for all $x, y, z \in A$, then an equivalence relation R on A is called a congruence.

Example 3.4 : Consider the SUP-algebra in Example 3.1. Then the equivalence relation $\theta = \{(0,0),(a,a),(b,b),(c,c),(0,b),(b,0),(a,c),(c,a)\}$ is a congruence on A.

Lemma 3.3 : An equivalence relation R on A is a congruence relation if and only if xRy and zRt implies $x|zRy|t$, for any $x, y, z, t \in A$.

Proof: Let R be a congruence on A, and let x,y,z,t be arbitrary elements in A such that xRy and zRt. Then, it follows from $(S1)$ that $x|zRy|z$ and $y|zRy|t$. Thus, we have $x|zRy|t$ from transitivity of R.

Conversely, assume that xRy and zRt imply $x|zRy|t,$, for any $x,y,z,t \in A$. Let x,y,z be arbitrary elements in A such that xRy. Since zRz, we have $x|zRy|z$ from the assumption. Then, R is a congruence relation on A. □

Lemma 3.4 : *Let J be a SUP-ideal of A and the relation $\sim_J$ is defined as Definition 3.3. Then, $\sim_J$ is a congruence on A.*

Proof : We first show that the binary relation $\sim_J$ is an equivalence relation on A.

- Reflexive: We have already known $(x|(x|x))|(x|(x|x)) = 0$ from Lemma 3.1 (1). Because J is a SUP-ideal of A, we have $(x|(x|x))|(x|(x|x)) = 0 \in J$, i.e., $x \sim_J x$ for all $x \in A$.
- Symmetric: Let x,y be any elements in A such that $x \sim_J y$. Then $(y|(x|x))|(y|(x|x)) \in J$ and $(x|(y|y))|(x|(y|y)) \in J$. From here, we obtain $(x|(y|y))|(x|(y|y)) \in J$ and $(y|(x|x))|(y|(x|x)) \in J$, i.e., $y \sim_J x$.
- Transitive: Let x,y,z be any elements in A such that $x \sim_J y$ and $y \sim_J z$. Then $(y|(x|x))|(y|(x|x)) \in J$, $(x|(y|y))|(x|(y|y)) \in J$ and $(z|(y|y))|(z|(y|y)) \in J$, $(y|(z|z))|(y|(z|z)) \in J$. Since

$$
\begin{aligned}
&(((((z|(x|x))|(z|(x|x)))|(y|(x|x)))|(((z|(x|x))|(z|(x|x)))|(y|(x|x))))|\\
&(((z|(y|y))|(z|(y|y)))|((z|(y|y))|(z|(y|y)))))|(((((z|(x|x))|(z|(x|x)))|\\
&(y|(x|x)))|(((z|(x|x))|(z|(x|x)))|(y|(x|x))))|(((z|(y|y))|(z|(y|y)))|\\
&((z|(y|y))|(z|(y|y)))))\\
= \ &(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|\\
&((y|(x|x))|(z|(y|y)))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|\\
&(z|(y|y)))|((y|(x|x))|(z|(y|y))))) \qquad ((S2)\ and\ (S3))\\
= \ &0 \in J. \qquad (SUP-1)
\end{aligned}
$$

and $(z|(y|y))|(z|(y|y)) \in J$, it follows that $(((z|(x|x))|(z|(x|x)))|(y|(x|x)))|(((z|(x|x))|(z|(x|x)))|(y|(x|x))) \in J$ from $(SUPi-2)$. Since

$$(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(((y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)))\,|\,((y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)))))\,|$$
$$(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(((y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)))\,|\,((y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)))))$$
$$= (((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(y\,|\,(x\,|\,x)))\,|\,(((z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)))\,|\,(y\,|\,(x\,|\,x))) \in J \qquad (S2)$$

and $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) \in J,$ we obtain $(z\,|\,(x\,|\,x))\,|\,(z\,|\,(x\,|\,x)) \in J$ from $(SUPi-2)$. Similarly, we have $(x\,|\,(z\,|\,z))\,|\,(x\,|\,(z\,|\,z)) \in J$. Thus, $x \sim_J z$. Therefore, $\sim_J$ is an equivalence relation on A.

Finally, it is shown that the equivalence relation $\sim_J$ is a congruence on A. Let x,y,z,t be arbitrary elements in A such that $x \sim_J y$ and $z \sim_J t$. Then $(y\,|\,(x\,|\,x))\,|\,(y\,|\,(x\,|\,x)) \in J,$ $(x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)) \in J,$ and $(t\,|\,(z\,|\,z))\,|\,(t\,|\,(z\,|\,z)) \in J,$ $(z\,|\,(t\,|\,t))\,|\,(z\,|\,(t\,|\,t)) \in J.$

(a_1) By $(S1)-(S3)$, $(SUP-1)$ and the fact that J is a SUP-ideal of A, we have

$$(((((x\,|\,z)\,|\,(x\,|\,z))\,|\,(y\,|\,z))\,|\,(((x\,|\,z)\,|\,(x\,|\,z))\,|\,(y\,|\,z)))\,|\,(((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))\,|$$
$$((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))))\,|\,(((((x\,|\,z)\,|\,(x\,|\,z))\,|\,(y\,|\,z))\,|\,(((x\,|\,z)\,|\,(x\,|\,z))\,|\,(y\,|\,z)))\,|$$
$$(((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))\,|\,((x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)))))$$
$$= \quad (((z\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|\,(z\,|\,((x\,|\,x)\,|\,(x\,|\,x))))\,|\,((((y\,|\,y)\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|$$
$$(z\,|\,((y\,|\,y)\,|\,(y\,|\,y))))\,|\,(((y\,|\,y)\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|\,(z\,|\,((y\,|\,y)\,|\,(y\,|\,y))))))\,|$$
$$(((z\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|\,(z\,|\,((x\,|\,x)\,|\,(x\,|\,x))))\,|\,((((y\,|\,y)\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|$$
$$(z\,|\,((y\,|\,y)\,|\,(y\,|\,y))))\,|\,(((y\,|\,y)\,|\,((x\,|\,x)\,|\,(x\,|\,x)))\,|\,(z\,|\,((y\,|\,y)\,|\,(y\,|\,y))))))$$
$$= \quad 0 \in J.$$

Since $(x\,|\,(y\,|\,y))\,|\,(x\,|\,(y\,|\,y)) \in J,$ we obtain from $(SUPi-2)$ and (S1) that $((y\,|\,z)\,|\,((x\,|\,z)\,|\,(x\,|\,z)))\,|\,((y\,|\,z)\,|\,((x\,|\,z)\,|\,(x\,|\,z))) \in J$. Similarly, $((x\,|\,z)\,|\,((y\,|\,z)\,|\,(y\,|\,z)))\,|\,((x\,|\,z)\,|\,((y\,|\,z)\,|\,(y\,|\,z))) \in J$. Thus, $x\,|\,z \sim_J y\,|\,z$.

(a_2) By substituting $[x := z]$, $[y := t]$ and $[z := y]$ in (a_1), simultaneously, it is obtained from (S1) that $y\,|\,z \sim_J y\,|\,t$.

From the transitivity of $\sim_J$ and $(S1)$, we obtain $x\,|\,z \sim_J y\,|\,t$. □

Theorem 3.5 : *Let J be a SUP-ideal of A and $\sim$ be the congruence relation on A described by J. Then $\langle A/\sim, |_\sim \rangle$ is also a SUP-algebra where $A/J \equiv A/\sim = \{[x]_\sim : x \in A\}$ and the Sheffer stroke $|_\sim$ on A/J is defined by $[x]_\sim \,|_\sim\, [y]_\sim = [x\,|\,y]_\sim$, for all $x,y \in A$ and $J = [0]_\sim$.*

Proof: Assume that J is a SUP-ideal of A and $\sim$ is the congruence relation on A described by J. Let $A/J \equiv A/\sim = \{[x]_\sim : x \in A\}$ be a structure and a binary operation $|_\sim$ on A/J be defined by $[x]_\sim |_\sim [y]_\sim = [x|y]_\sim$, for all $x, y \in A$.

It is shown that $J = [0]_\sim$. Let $x \in [0]_\sim$. Then $x \sim 0$ which means that $0 = (0|(x|x)) | (0|(x|x)) \in J$ and $x = (x|(0|0)) | (x|(0|0)) \in J$ from (S2), Lemma 2.1 (8) and $(SUP-2)$. Thus, $[0]_\sim \subseteq J$. Since it is known from (S2), Lemma 2.1 (8) and $(SUP-2)$ that $0 = (0|(x|x)) | (0|(x|x)) \in J$ and $x = (x|(0|0)) | (x|(0|0)) \in J$, for all $x \in J$, it follows that $x \sim 0$ which implies that $x \in [0]_\sim$, for all $x \in J$. So, $J \subseteq [0]_\sim$.

It is clear from the definition of the binary operation $|_\sim$ that $|_\sim$ is the Sheffer stroke on A/J and satisfies $(SUP-1)$–$(SUP-2)$. Now, we indicate that $|_\sim$ satisfies $(SUP-3)$. Let $([x]_\sim |_\sim ([y]_\sim |_\sim [y]_\sim)) |_\sim ([x]_\sim |_\sim ([y]_\sim |_\sim [y]_\sim)) = [0]_\sim$ and $([y]_\sim |_\sim ([x]_\sim |_\sim [x]_\sim)) |_\sim ([y]_\sim |_\sim ([x]_\sim |_\sim [x]_\sim)) = [0]_\sim$. Then $[(x|(y|y))|$

$(x|(y|y))]_\sim = ([x]_\sim |_\sim ([y]_\sim |_\sim [y]_\sim)) |_\sim ([x]_\sim |_\sim ([y]_\sim |_\sim [y]_\sim)) = [0]_\sim$ and $[(y|(x|x)) \mid (y|(x|x))]_\sim = ([y]_\sim |_\sim ([x]_\sim |_\sim [x]_\sim)) |_\sim ([y]_\sim |_\sim ([x]_\sim |_\sim [x]_\sim)) = [0]_\sim$ which implies that $(x|(y|y)) | (x|(y|y)) \sim 0$ and $(y|(x|x)) | (y|(x|x)) \sim 0$. Since $0 = (0|(x|(y|y))) | (0|(x|(y|y))) \in J$, $(x|(y|y)) | (x|(y|y)) = (((x|(y|y)) | (x|(y|y))) | (0|0)) | (((x|(y|y)) | (x|(y|y))) | (0|0)) \in J$ and $0 = (0|(y|(x|x))) | (0|(y|(x|x))) \in J$, $(y|(x|x)) | (y|(x|x)) = (((y|(x|x)) | (y|(x|x))) | (0|0)) | (((y|(x|x)) | (y|(x|x))) | (0|0)) \in J$ from (S2), Lemma 2.1 (8) and $(SUP-2)$, it is obtained that $x \sim y$ which means that $[x]_\sim = [y]_\sim$.

Therefore, $A/J \equiv A/\sim = \{[x]_\sim : x \in A\}$ is a SUP-algebra. □

Theorem 3.6 : *Let J be a SUP-ideal of A. Then G is a SUP-ideal of A such that $J \subseteq G$ if and only if the set $G/J = \{[x]_J \in A/J : x \in G\}$ is a SUP-ideal of a SUP-algebra A/J.*

Proof: It is known from Theorem 3.5 that $\langle A/J, |_J \rangle$ is a SUP-algebra. It follows that $J = [0]_J \in G/J$. Let $[y]_J \in G/J$. Then $y \in G$. Since G is a SUP-ideal of A, we get $(y|(x|x)) | (y|(x|x)) \in G$. Thus, $([y]_J |_J ([x]_J |_J [x]_J)) |_J ([y]_J |_J ([x]_J |_J [x]_J)) = [(y|(x|x)) | (y|(x|x))]_J \in G/J$. Let $([y]_J |_J ([x]_J |_J [x]_J)) |_J ([y]_J |_J ([x]_J |_J [x]_J)) \in G/J$ and $[x]_J \in G/J$. So, $(y|(x|x)) | (y|(x|x)) \in G$ and $x \in G$. Since G is a SUP-ideal of A, it follows that $y \in G$ which implies that $[y]_J \in G/J$.

Conversely, it is proved from the definition of G/J. □

4. SUP-homomorphisms on SUP-algebras

In this section, we introduce some definitions and notions about SUP-homomorphisms on SUP-algebras.

Definition 4.1 : [7] Let $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ be SUP-algebras. A mapping $f : A \to B$ is called a SUP-homomorphism if

$$f(x \,|_A\, y) = f(x) \,|_B\, f(y)$$

for all $x, y \in A$.

Lemma 4.1 : [7] *Let $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ be SUP-algebras, and let the mapping $f : A \to B$ be a SUP-homomorphism. Then $f(A)$ is a SUP-subalgebra of B.*

Lemma 4.2 : *Let $\langle A, |_A \rangle$ and $\langle B, |_B \rangle$ be SUP-algebras, and let the mapping $f : A \to B$ be a SUP-homomorphism. Then $Kerf$ is a SUP-ideal of A.*

Proof : Let the mapping $f : A \to B$ be a SUP-homomorphism and $Kerf$ be defined as $\{x \in A : f(x) = 0_B\}$.

- We have $0_A \in Kerf$ since $f(0_A) = 0_B$.
- Let $y \in Kerf$. Then $f(y) = 0_B$. Since

$$\begin{aligned} 0_B &= (0_B \,|_B\, 0_B) \,|_B\, (0_B \,|_B\, 0_B) \\ &= (0_B \,|_B\, (f(x) \,|_B\, f(x))) \,|_B\, (0_B \,|_B\, (f(x) \,|_B\, f(x))) \\ &= (f(y) \,|_B\, (f(x) \,|_B\, f(x))) \,|_B\, (f(y) \,|_B\, (f(x) \,|_B\, f(x))) \\ &= f((y \,|_A\, (x \,|_A\, x)) \,|_A\, (y \,|_A\, (x \,|_A\, x))) \end{aligned}$$

  from (S1)-(S2) and Lemma 2.1 (8), we get $(y \,|_A\, (x \,|_A\, x)) \,|_A\, (y \,|_A\, (x \,|_A\, x)) \in Kerf$.
- Let $(y \,|_A\, (x \,|_A\, x)) \,|_A\, (y \,|_A\, (x \,|_A\, x)) \in Kerf$ and $x \in Kerf$. Then $f((y \,|_A\, (x \,|_A\, x)) \,|_A\, (y \,|_A\, (x \,|_A\, x))) = 0_B$ and $f(x) = 0_B$. Since

$$\begin{aligned} 0_B &= f((y \,|_A\, (x \,|_A\, x)) \,|_A\, (y \,|_A\, (x \,|_A\, x))) \\ &= (f(y) \,|_B\, (f(x) \,|_B\, f(x))) \,|_B\, (f(y) \,|_B\, (f(x) \,|_B\, f(x))) \\ &= (f(y) \,|_B\, (0_B \,|_B\, 0_B)) \,|_B\, (f(y) \,|_B\, (0_B \,|_B\, 0_B)) \\ &= (f(y) \,|_B\, f(y)) \,|_B\, (f(y) \,|_B\, f(y)) \\ &= f(y) \end{aligned}$$

  from $(SUP-2)$ and (S2), it follows that $y \in Kerf$.

Thus, $Kerf$ is a SUP-ideal of A. □

Theorem 4.1 : *Let* $\langle A,|_A\rangle$ *and* $\langle B,|_B\rangle$ *be SUP-algebras, and let the mapping* $f : A \to B$ *be a SUP-homomorphism. Then, the followings are satisfied:*

- If G is a SUP-ideal of A, then $f(G)$ is a SUP-ideal of $f(A)$.
- If H is a SUP-ideal of B, then $f^{-1}(H)$ is a SUP-ideal of A.

Proof:

- Suppose that G is a SUP-ideal of A. Let $v \in f(G) \subseteq f(A)$. Then there exists $y \in G$ such that $v = f(y)$. Since G is a SUP-ideal of A, it is obtained from $(SUPi-1)$ that $(y|_A(x|_A x))|_A(y|_A(x|_A x)) \in G$. So,

$$\begin{aligned}(v|_B(u|_B u))|_B(v|_B(u|_B u)) &= (f(y)|_B(f(x)|_B f(x)))|_B(f(y)|_B(f(x)|_B f(x))) \\ &= f((y|_A(x|_A x))|_A(y|_A(x|_A x))) \in f(G)\end{aligned}$$

  where $f(x) = u \in f(A)$. Let $(v|_B(u|_B u))|_B(v|_B(u|_B u)) \in f(G)$ and $u \in f(G)$. Since $f((y|_A(x|_A x))|_A(y|_A(x|_A x))) = (f(y)|_B(f(x)|_B f(x)))|_B(f(y)|_B(f(x)|_B f(x))) = (v|_B(u|_B u))|_B(v|_B(u|_B u)) \in f(G)$ and $u = f(x) \in f(G)$, we have $(y|_A(x|_A x))|_A(y|_A(x|_A x)) \in G$ and $x \in G$ such that $u = f(x)$ and $v = f(y)$. Since G is a SUP-ideal of A, it follows that $y \in G$ which implies that $v = f(y) \in f(G)$. Thus, $f(G)$ is a SUP-ideal of $f(A)$.
- Assume that H is a SUP-ideal of B. Let $y \in f^{-1}(H)$. Then $f(y) \in ff^{-1}(H) \subseteq H$. Since H is a SUP-ideal of B, we have $f((y|_A(x|_A x))|_A(y|_A(x|_A x))) = (f(y)|_B(f(x)|_B f(x)))|_B(f(y)|_B(f(x)|_B f(x))) \in H$ which means that $(y|_A(x|_A x))|_A(y|_A(x|_A x)) \in f^{-1}(H)$. Let $(y|_A(x|_A x))|_A(y|_A(x|_A x)) \in f^{-1}(H)$ and $x \in f^{-1}(H)$. Then $(f(y)|_B(f(x)|_B f(x)))|_B(f(y)|_B(f(x)|_B f(x))) = f((y|_A(x|_A x))|_A(y|_A(x|_A x))) \in ff^{-1}(H) \subseteq H$ and $f(x) \in ff^{-1}(H) \subseteq H$. Since H is a SUP-ideal of B, it follows that $f(y) \in H$ which implies that $y \in f^{-1}(H)$. Hence, $f^{-1}(H)$ is a SUP-ideal of A. □

Theorem 4.2 : *Let G and J be SUP-ideals of A such that $J \subseteq G$. Then* $(A/J)/(G/J) \cong A/G$.

Proof: By Theorem 3.6, $G/J = \{[x]_J \in A/J : x \in G\}$ is a SUP-ideal of a SUP-algebra $A/J = \{[x]_J : x \in A\}$. Moreover, the factor-set $(A/J)/(G/J) = \{[[x]_J]_{G/J} : [x]_J \in A/J\}$ is a SUP-algebra defined by its SUP-ideal G/J of the SUP-algebra A/J and $A/G = \{[x]_G : x \in A\}$ is a SUP-algebra defined by its SUP-ideal G of the SUP-algebra A from Theorem 3.5.

Let a mapping $\psi : (A/J)/(G/J) \to A/G$ be defined by $[[x]_J]_{G/J} \mapsto [x]_G$, for any $[[x]_J]_{G/J} \in (A/J)/(G/J)$.

Since

$$\begin{aligned} d[[x]_J]_{G/J} = [[y]_J]_{G/J} &\Leftrightarrow [x]_J \sim_{G/J} [y]_J \\ &\Leftrightarrow [(y \mid (x \mid x)) \mid (y \mid (x \mid x))]_J = ([y]_J \mid_J ([x]_J \mid_J [x]_J)) \mid_J ([y]_J \mid_J \\ &\quad ([x]_J \mid_J [x]_J)) \in G/J \end{aligned}$$

and

$$\begin{aligned} &[(x \mid (y \mid y)) \mid (x \mid (y \mid y))]_J = ([x]_J \mid_J ([y]_J \mid_J [y]_J)) \mid_J \\ &([x]_J \mid_J ([y]_J \mid_J [y]_J)) \in G/J \\ &\Leftrightarrow (y \mid (x \mid x)) \mid (y \mid (x \mid x)) \in G \text{ and } (x \mid (y \mid y)) \mid (x \mid (y \mid y)) \in G \\ &\Leftrightarrow x \sim_G y \\ &\Leftrightarrow \psi([[x]_J]_{G/J}) = [x]_G = [y]_G = \psi([[y]_J]_{G/J}), \end{aligned}$$

it is obtained that ψ is well-defined and one-to-one. Since

$$\begin{aligned} \psi([[x]_J]_{G/J} \mid_{G/J} [[y]_J]_{G/J}) &= \psi([[x]_J \mid_J [y]_J]_{G/J}) \\ &= \psi([[x \mid y]_J]_{G/J}) \\ &= [x \mid y]_G \\ &= [x]_G \mid_G [y]_G \\ &= \psi([[x]_J]_{G/J}) \mid_G \psi([[y]_J]_{G/J}), \end{aligned}$$

for any $[[x]_J]_{G/J}, [[y]_J]_{G/J} \in (A/J)/(G/J)$ and $[0]_G = \psi([[0]_J]_{G/J})$, it follows that ψ is a SUP-homomorphism.

It is obvious from the definition of ψ that it is onto.

Thereby, ψ is a SUP-isomorphism which means that $(A/J)/(G/J) \cong A/G$. □

5. Conclusion

In this paper, we give fundamental definitions and notions of Sheffer stroke and SUP-algebras, and then define a SUP-ideal of this algebraic structure. The necessary properties are proved so that a nonempty subset of a SUP-algebra is its SUP-ideal. Also we show that the class of all SUP-ideals of a SUP-algebra forms a complete distributive lattice and that there exists the minimal SUP-ideal containing a subset of this algebraic structure. After presenting a SUP-subalgebra of a SUP-algebra, it is indicated that any SUP-ideal of this algebra is its SUP-subalgebra but the inverse does not usually hold. Also, we define a congruence relation on a SUP-algebra by its SUP-ideal. Thus, we construct the quotient set by means of the congruence relation on a SUP-algebra and prove that the quotient set is a SUP-algebra. Indeed, we determine a SUP-ideal of the quotient set. By describing a SUP-homomorphism on SUP-algebras, it is demonstrated that the image of a SUP-algebra under the SUP-homomorphism is also a SUP-subalgebra of the other SUP-algebra and that the kernel of a SUP-homomorphism on a SUP-algebra is the SUP-ideal. Finally, it is shown that the image and inverse image of a SUP-ideal of a SUP-algebra under the SUP-homomorphism are also SUP-ideals of the other SUP-algebra.

In future work, we plan to study the neutrosophic structures on Sheffer sroke UP-algebras.

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Received July, 2021

Revised Februrary, 2022