

$L(1, d)$ -labeling number of direct product of two cycles

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Abstract

For positive number d , an $L(1, d)$ -labeling f of G is an assignment of numbers to vertices of G such that $|f(u) - f(v)| \geq 1$ if $d(u, v) = 1$, and $|f(u) - f(v)| \geq d$ if $d(u, v) = 2$. Then the span of f is the difference between the maximum and the minimum numbers assigned by f . The $L(1, d)$ -labeling number of G , denoted by $\lambda_{1,d}(G)$, is the minimum span over all $L(1, d)$ -labelings of G . In this paper, the authors give the $L(1, d)$ -labeling numbers of the

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direct product of two cycles C_m and C_n for $d \geq 2$ for some values of m and n . Some cases for $1 \leq d < 2$ are also discussed.

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1. Introduction

In recent years, the $L(1, d)$ -labeling problem for $d \geq 1$ has been paid attention to study since it can be applied in computer network to solve the problem of the scarcity of available codes for communication with minimum interference. For example, the *Packet Radio Network* (PRN) is a computer network that used radio frequencies to transmit data among computers [1]. It is formed by the computer stations which consist of computers and transceivers. If two stations can transmit to each other directly, they are called *adjacent*. Two stations are called *at distance two* if they are nonadjacent but they are adjacent to the same one station. A PRN using CDMA has two types of interference: *Direct* interference and *Secondary* interference. If direct interference is so weak that we can ignore it, Bertossi and Bonuccelli [2] introduced the code assignment for the network that two distance-two apart stations transmit on different codes. In 2005, Jin and Yeh [5] generalized the code assignment problem to $L(1, d)$ -labeling problem, that is, a vertex and its label will correspond to a station and its code, respectively. If two stations are adjacent, its vertices are linked by an edge. In order to avoid two types of interferences, labels are assigned to all vertices of a graph so that adjacent vertices receive labels whose difference is at least 1, and vertices at distance two receive labels whose difference is at least d , where $d \geq 1$.

For positive number d , an $L(1, d)$ -labeling f of G is a mapping of numbers to vertices of G such that $|f(u) - f(v)| \geq 1$ if u, v are adjacent, and $|f(u) - f(v)| \geq d$ if u, v are distance two apart. The *span* of f is the maximum difference of all pairs of numbers assigned by f . The $L(1, d)$ -labeling number of G , denoted by $\lambda_{1,d}(G)$, is the minimum span over all $L(1, d)$ -labeling of G . All notation not defined in this article can be found in the book [3].

$L(1, d)$ -numbers of graphs for $d \geq 1$ have been studied in some literatures. Interested readers are referred to the articles [2, 4, 5, 6, 7, 8, 9, 10, 11, 12].

The following result is not difficult to see.

Lemma 1.1 : Let d be a positive number with $d \geq 1$. Suppose G is a graph and H is an induced subgraph of G . Then $\lambda_{1,d}(G) \geq \lambda_{1,d}(H)$.

Note that Lemma 1.1 is not true if H is not an induced subgraph.

Given two graphs G and H , the *direct product* of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (x, y) and (x', y') are adjacent if and only if $xx' \in E(G)$ and $yy' \in E(H)$. Product of graphs have been extensively studied and used as models for many real-world networks of computer. In this paper, we only study on the $L(1, d)$ -labeling number of direct product of two cycles for $d \geq 1$.

Let the cycle $C_m = x_0 x_1 \cdots x_{m-1} x_0$. For convenience, we denote the vertex $(x_s, x_t) \in V(C_n \times C_m)$ by $v_{s,t}$, $0 \leq s \leq n-1$ and $0 \leq t \leq m-1$, throughout this paper. Also, since $C_n \times C_m \cong C_m \times C_n$, we always assume that $n \leq m$.

Let f be a labeling of graph $C_n \times C_m$ and its corresponding *labeling matrix* be denoted by $M_f(C_n \times C_m) = (f_{s,t})_{n \times m}$, where $f_{s,t} = f(v_{s,t})$.

For any $v_{s,t} \in V(C_n \times C_m)$, it is adjacent to four vertices $v_{s \pm 1, t \pm 1}$ and at distance two from eight (when $m \geq 4$) vertices $v_{s \pm 2, t \pm 2}$, $v_{s, t \pm 2}$ and $v_{s \pm 2, t}$, where $s \pm 1$, $s \pm 2$ should be taken under modulo n and $t \pm 1$, $t \pm 2$ should be taken under modulo m .

For $m \geq n \geq 3$, according to the labeling matrix $M_f = M_f(C_n \times C_m)$, for easy to check, we let an augmented matrix $A_f = A_f(C_n \times C_m)$ by inserting the first two rows of M_f at the bottom of M_f , after that inserting the first two columns of the resulting matrix at the right most of it, and inserting the last two columns of the resulting matrix at the left most of it. Hence A_f is an $(n+2) \times (m+4)$ matrix. That is,

$$A_f = \left[\begin{array}{cc|ccc} f_{0,m-2} & f_{0,m-1} & & M_f & & f_{0,0} & f_{0,1} \\ \vdots & \vdots & & & & \vdots & \vdots \\ f_{n-1,m-2} & f_{n-1,m-1} & & & & f_{n-1,0} & f_{n-1,1} \\ \hline f_{0,m-2} & f_{0,m-1} & f_{0,0} \cdots f_{0,m-1} & & f_{0,0} & f_{0,1} \\ f_{1,m-2} & f_{1,m-1} & f_{1,0} \cdots f_{1,m-1} & & f_{1,0} & f_{1,1} \end{array} \right].$$

By the symmetry of distance, it suffices to check the labels $f_{s,t}$, $f_{s+1,t \pm 1}$, $f_{s,t+2}$, $f_{s+2,t}$ and $f_{s+2,t \pm 2}$ such that they satisfy the conditions of $L(1, d)$ -labeling.

A set of labels is said to be d -separated if the distance of any two labels from this set is greater than or equal to d .

2. $L(1, d)$ -labeling number of $C_3 \times C_m$

When $n = 3$ and $m \geq 4$. In this case, the vertex $v_{s,t}$ is at distance two from eight vertices $v_{s\pm 1, t\pm 2}$, $v_{s, t\pm 2}$, $v_{s\pm 1, t}$ and adjacent to four vertices $v_{s\pm 1, t\pm 1}$. Note that the first indices are taken under modulo 3 and the second indices are taken under modulo m . By the symmetry of distance, it suffices to check the labels $f_{s,t}$, $f_{s+1, t\pm 1}$, $f_{s, t+2}$, $f_{s+1, t}$ and $f_{s+1, t\pm 2}$ such that they satisfy the conditions of $L(1, d)$ -labeling. Since the related entries are involved at the same row and the next row, the augmented matrix may be revised by deleting the last row.

Let m be a positive integer. We let $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$.

Lemma 2.1 : Let M be a labeling matrix corresponding to a λ - $L(1, d)$ -labeling of $C_3 \times C_m$, where $d \geq 1$, $m \geq 3$. Let J be a subinterval of $[0, \lambda]$ such that $\ell(J) < d$. For a fixed $t \in \mathbb{Z}_m$, J contains at most 1 entries come from the t -th and $(t+2)$ -nd columns of M , where $t+2$ is taken under modulo m .

Proof : The lemma follows from $v_{0,t}$, $v_{1,t}$, $v_{2,t}$, $v_{0,t+2}$, $v_{1,t+2}$, $v_{2,t+2}$ being of distance 2 pairwise. $\square$

Corollary 2.2 : Let $I_i = [id, (i+1)d)$ for $0 \leq i \leq k$, where $k = \lfloor \lambda/d \rfloor$. Keeping the assumption of Lemma 2.1, we have $k \geq 5$.

Theorem 2.3 : For $d \geq 1$, $\lambda_{1,d}(C_3 \times C_{4p}) = 5d$.

Proof : From Corollary 2.2, we have $\lambda_{1,d}(C_3 \times C_{4p}) \geq 5d$. Let $A = d \begin{bmatrix} 0 & 0 & 3 & 3 \\ 1 & 1 & 4 & 4 \\ 2 & 2 & 5 & 5 \end{bmatrix}$.

Define the labeling matrix of a labeling f for $C_3 \times C_{4p}$ by

$M_f(C_3 \times C_{4p}) = M = [A \cdots A]$, which is a $1 \times p$ block matrix with each block being A .

As an example, let us provide the corresponding augmented matrix

$$A_f(C_3 \times C_{4p}) = \begin{bmatrix} 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 & \cdots & 0 & 0 & 3 & 3 & 0 & 0 \\ 4 & 4 & 1 & 1 & 4 & 4 & 1 & 1 & 4 & 4 & \cdots & 1 & 1 & 4 & 4 & 1 & 1 \\ 5 & 5 & 2 & 2 & 5 & 5 & 2 & 2 & 5 & 5 & \cdots & 2 & 2 & 5 & 5 & 2 & 2 \\ 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 & \cdots & 0 & 0 & 3 & 3 & 0 & 0 \end{bmatrix}$$

Since the columns of the matrix is periodic with period 4, we only need to check one block. That is, we only need to check the related entries of the each entry in the main block (the 4×4 block)

$$d \begin{bmatrix} 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 \\ 4 & 4 & 1 & 1 & 4 & 4 & 1 & 1 \\ 5 & 5 & 2 & 2 & 5 & 5 & 2 & 2 \\ 3 & 3 & 0 & 0 & 3 & 3 & 0 & 0 \end{bmatrix}$$

It is easy to check that f is a $5d$ - $L(1, d)$ -labeling of $C_3 \times C_m$. Then, $\lambda_{1,d}(C_3 \times C_m) \leq 5d$.

Hence, $\lambda_{1,d}(C_3 \times C_m) = 5d$. $\square$

Lemma 2.4 : For $d \geq 1$, $\lambda_{1,d}(C_3 \times C_m) \geq 6d$, where $m \geq 5$ and $m \not\equiv 0 \pmod{4}$.

Proof : Suppose $\lambda_{1,d}(C_3 \times C_m) = \lambda < 6d$. Let $M = (M_{i,j})$ be the labeling matrix corresponding to a λ - $L(1, d)$ -labeling of $C_3 \times C_m$. Let $[0, \lambda] = \cup_{i=0}^{i=5} I_i$, where $I_i = [id, (i+1)d)$ for $0 \leq i \leq 4$ and $I_5 = [5d, \lambda]$. By Lemma 2.1, without loss of generality, we may assume that $M_{0,0}, M_{1,0}, M_{2,0} \in I_x \cup I_y \cup I_z$ and $M_{0,2}, M_{1,2}, M_{2,2} \in I_a \cup I_b \cup I_c$, where $\{x, y, z, a, b, c\} = \{0, 1, 2, 3, 4, 5\}$. Applying Lemma 2.1 repeatedly, we shall obtain the general results as follows.

$$\begin{aligned} M_{0,4l}, M_{1,4l}, M_{2,4l} &\in I_x \cup I_y \cup I_z, \\ M_{0,4l+2}, M_{1,4l+2}, M_{2,4l+2} &\in I_a \cup I_b \cup I_c, \end{aligned} \quad (2.1)$$

where $l \geq 0$ and $4l, 4l+2$ are taken under modulo m . When $m = 2l+1$ or $m = 4l+2$ for some $l \geq 1$, we always get that $4l+2 \equiv 0 \pmod{m}$. According to (2.1), we have $M_{0,0}, M_{1,0}, M_{2,0} \in I_a \cup I_b \cup I_c$, which contradicts our assumption. Thus, $\lambda_{1,d}(C_3 \times C_m) \geq 6d$ for $m \not\equiv 0 \pmod{4}$ and $d \geq 1$. $\square$

Theorem 2.5 : For $d \geq 1$, $\lambda_{1,d}(C_3 \times C_m) = 6d$ if $m \not\equiv 0 \pmod{4}$ and $m \notin \{3, 5, 6, 9, 10\}$.

Proof : From Lemma 2.4, we know that $\lambda_{1,d}(C_3 \times C_m) \geq 6d$ for $d \geq 1$ and $m \not\equiv 0 \pmod{4}$.

On the other hand, we are going to provide a $6d$ - $L(1, d)$ -labeling f each case.

Case 1 : Suppose $m = 4p+9$, where $p \geq 1$. Let M be the labeling matrix of $C_3 \times C_{4p}$ defined in Theorem 2.3.

$$\text{Let } A = d \begin{bmatrix} 0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 & 6 \\ 1 & 1 & 4 & 4 & 6 & 1 & 1 & 4 & 4 \\ 6 & 2 & 2 & 5 & 5 & 2 & 2 & 5 & 5 \end{bmatrix}. \text{ Let } M_f(C_3 \times C_m) = [M \ A].$$

Case 2 : Suppose $m = 4p + 10$, where $p \geq 1$.

Let $B = d \begin{bmatrix} 6 & 6 & 2 & 2 & 5 & 5 & 1 & 1 & 4 & 4 \\ 0 & 0 & 3 & 3 & 6 & 6 & 2 & 2 & 5 & 5 \\ 1 & 1 & 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 \end{bmatrix}$. Let $M_f(C_3 \times C_m) = [M \ B]$, where

M is the same matrix as Case 1.

Case 3 : Suppose $m = 4p + 3$, where $p \geq 1$.

Let $C = d \begin{bmatrix} 4 & 4 & 0 & 0 \\ 6 & 3 & 1 & 6 \\ 5 & 2 & 2 & 5 \end{bmatrix}$ and $D = d \begin{bmatrix} 3 & 3 & 0 \\ 4 & 2 & 6 \\ 5 & 1 & 1 \end{bmatrix}$. We define $M_f(C_3 \times C_m) =$

$\overbrace{[C \cdots CD]}^p$.

□

It is not difficult to verify that each f defined above is a $6d - L(1, d)$ -labeling. Thus, $\lambda_{1,d}(C_3 \times C_m) \leq 6d$, for each of the above cases. This completes the proof.

Now, we are going to consider the remaining cases, i.e., $m \in \{3, 5, 6, 9, 10\}$.

Theorem 2.6 : For $d \geq 1$, $\lambda_{1,d}(C_3 \times C_6) = 8d$.

Proof : Since nine vertices $v_{0,0}, v_{1,0}, v_{2,0}, v_{0,2}, v_{1,2}, v_{2,2}, v_{0,4}, v_{1,4}, v_{2,4}$ are of distance two apart each other, their labels form a d -separated set. This implies that $\lambda_{1,d}(C_3 \times C_6) \geq 8d$.

On the other hand, define a labeling f for graph $C_3 \times C_6$ by the following labeling matrix.

$$M_f(C_3 \times C_6) = d \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 3 & 4 & 5 & 3 & 4 & 5 \\ 6 & 7 & 8 & 6 & 7 & 8 \end{bmatrix}.$$

It is easy to check that f is an $8d - L(1, d)$ -labeling of $C_3 \times C_6$. Then, $\lambda_{1,d}(C_3 \times C_6) \leq 8d$. Hence we have the theorem. □

Theorem 2.7 : For $d \geq 2$, $\lambda_{1,d}(C_3 \times C_9) = 6d$.

Proof : Given a labeling function f for graph $C_3 \times C_9$ and its corresponding labeling matrix as follows.

$$M_f(C_3 \times C_9) = \begin{bmatrix} 0 & 3d & 3d & 6d & 6d & 2d & 2d & 5d+1 & 0 \\ d & d & 4d & 4d & 0 & 0 & 3d & 3d & 5d \\ 2d & 2d & 5d & 5d & d & d & 4d & 4d & 6d \end{bmatrix}.$$

It is obviously that f is a $6d - L(1, d)$ -labeling. Then $\lambda_{1,d}(C_3 \times C_9) \leq 6d$.

On the other hand, from Lemma 2.4, we have $\lambda_{1,d}(C_3 \times C_9) \geq 6d$ for $d \geq 1$. Hence, we have the theorem. $\square$

Let f be a $\lambda - L(1, d)$ labeling of $C_3 \times C_m$, where $\lambda < 7d$ and $m \geq 5$. Similar to the proof of Lemma 2.4, we let *sub-interval* $I_i = [id, (i+1)d)$, where $0 \leq i \leq 6$. Clearly $[0, \lambda] \subseteq \bigcup_{i=0}^6 I_i$. Let M be the labeling matrix of f , where $i \in \mathbb{Z}_3$ and $j \in \mathbb{Z}_m$. Let M_j be the set of entries of the j -th column of M . Note that, each M_j is d -separated. Under these notation and conditions we have

Claim 1 : For each $j \in \mathbb{Z}_m$, there are two sub-intervals such that each sub-interval contains an element from M_j and an element from M_{j+4} .

Proof : Since the statement is cyclically under modulo m , it suffices to consider the case $j = 0$. Since 6 vertices $v_{0,2}, v_{1,2}, v_{2,2}, v_{0,4}, v_{1,4}$ and $v_{2,4}$ are at distance two from each other, the set $M_2 \cup M_4$ is d -separated. It means that members in $M_2 \cup M_4$ lie in 6 different sub-intervals, respectively. Similarly, the set $M_0 \cup M_2$ has the same property as $M_2 \cup M_4$. Since elements of M_2 are occupied 3 sub-intervals, elements of $M_0 \cup M_4$ must lie into the remaining 4 sub-intervals. Note that, each of such sub-interval contains at most two elements of $M_0 \cup M_4$. By pigeonhole principle, at least two pigeonholes (sub-intervals) contain exactly 2 elements of $M_0 \cup M_4$. Moreover, these two elements lying in the same sub-interval must come from difference M_j , $j = 0, 4$. Hence, the claim is proved. $\square$

Theorem 2.8 : $\lambda_{1,d}(C_3 \times C_5) = 7d$ for $d \geq 2$ and $\lambda_{1,d}(C_3 \times C_{10}) = 7d$ for $d \geq 1$.

Proof : Suppose $d \geq 2$. We define a labeling f by an augmented matrix $A_f(C_3 \times C_5)$ as follows.

$$\left[\begin{array}{cc|cccc|cc} 6d+1 & 0 & 0 & 3d & 3d & 6d+1 & 0 & 0 & 3d \\ 4d & 7d & d & d & 4d & 4d & 7d & d & d \\ 5d & 6d & 2d & 2d & 5d & 5d & 6d & 2d & 2d \\ \hline 6d+1 & 0 & 0 & 3d & 3d & 6d+1 & 0 & 0 & 3d \end{array} \right].$$

It is directly to check that f is a $7d - L(1, d)$ -labelings for $C_3 \times C_5$. Then $\lambda_{1,d}(C_3 \times C_5) \leq 7d$.

On the other hand, suppose $\lambda_{1,d}(C_3 \times C_5) = \lambda < 7d$. Let f be a $\lambda - L(1, d)$ -labeling of $C_3 \times C_5$ with labeling matrix M , where $i \in \mathbb{Z}_3$ and $j \in \mathbb{Z}_5$.

Keep the notation defined before Claim 1. Since $M_0 = M_{1+4}$ and $M_1 = M_{2+4}$, by Claim 1, at least one sub-interval contains 3 labels which come from 3 different set M_0 , M_1 and M_2 . It yields a contradiction since the length of each sub-interval is less than d .

Hence, $\lambda_{1,d}(C_3 \times C_5) = 7d$.

Suppose $d \geq 1$. We define a labeling g by an augmented matrix $A_g(C_3 \times C_{10})$ as follows.

$$d \left[\begin{array}{cc|cccccccc|cc} 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 & 1 & 1 & 4 & 4 & 0 & 0 \\ 5 & 5 & 1 & 1 & 4 & 4 & 7 & 7 & 2 & 2 & 5 & 5 & 1 & 1 \\ \hline 6 & 6 & 2 & 2 & 5 & 5 & 0 & 0 & 3 & 3 & 6 & 6 & 2 & 2 \\ 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 & 1 & 1 & 4 & 4 & 0 & 0 \end{array} \right].$$

It is directly to check that g is a $7d - L(1, d)$ -labeling for $C_3 \times C_{10}$. Then $\lambda_{1,d}(C_3 \times C_{10}) \leq 7d$.

Similarly, suppose $\lambda_{1,d}(C_3 \times C_{10}) = \lambda < 7d$. Let g be a $\lambda - L(1, d)$ -labeling of $C_3 \times C_{10}$ with labeling matrix M , where $i \in \mathbb{Z}_3$ and $j \in \mathbb{Z}_{10}$. By considering the sets M_0 , M_4 and M_8 and applying Claim 1, similar to the $C_3 \times C_5$ case, at least one sub-interval contains 3 labels which come from 3 different set M_0 , M_4 and M_8 , which yields a contradiction. Hence, $\lambda_{1,d}(C_3 \times C_5) = 7d$. $\square$

Theorem 2.9 : For $1 \leq d \leq 2$, $\lambda_{1,d}(C_3 \times C_3) = 8$.

Proof : Note that, $v_{s,t}$ is adjacent to $v_{s\pm 1, t\pm 1}$ and is of distance 2 with $v_{s, t\pm 1}$ and $v_{s\pm 1, t}$, where indices are taken modulo 3. Let f be a labeling of $C_3 \times C_3$ so that its labeling matrix $M_f(C_3 \times C_3)$ is defined below:

$$M_f(C_3 \times C_3) = \begin{bmatrix} 0 & 3 & 6 \\ 7 & 1 & 4 \\ 5 & 8 & 2 \end{bmatrix}.$$

It is easy to check directly that f is an $8 - L(1, d)$ -labeling of $C_3 \times C_3$. Then $\lambda_{1,d}(C_3 \times C_3) \leq 8$.

On the other hand, since the diameter of $C_3 \times C_3$ is 2, the difference of each pair of vertex-labels is at least 1. Here $\lambda_{1,d}(C_3 \times C_3) \geq 8$. Hence $\lambda_{1,d}(C_3 \times C_3) = 8$. $\square$

Theorem 2.10 : For $d \geq 2$, $\lambda_{1,d}(C_3 \times C_3) = 2d + 4$.

Proof : Let

$$M = \begin{bmatrix} 0 & d+1 & 2d+2 \\ 2d+3 & 1 & d+2 \\ d+3 & 2d+4 & 2 \end{bmatrix}.$$

Clearly, M corresponds a $(2d+4)$ - $L(1, d)$ -labeling of $C_3 \times C_3$.

Suppose f is a λ - $L(1, d)$ -labeling of $C_3 \times C_3$. Without loss of generality, let $f_{0,0} = 0$. Note that, since the diameter of $C_3 \times C_3$ is 2, the difference between any pair of labels is at least 1. So, $f_{s,t} \geq 1$ for $(s, t) \neq (0, 0)$.

By symmetry we may assume that $f_{0,1} < f_{1,0}$ and $f_{1,0}$ is less than both $f_{0,2}$ and $f_{2,0}$. Hence, $f_{0,1} \in [d, \lambda - d - 1]$, $f_{1,0} \in [1 + d, \lambda - d]$ and $f_{0,2}, f_{2,0} \in [2d, \lambda]$. Since $f_{2,0}$ is d -separated from $f_{1,0}$, $f_{2,0} \in [1 + 2d, \lambda]$.

Consider $f_{1,1}$. There are three cases: $f_{0,1} < f_{1,1} < f_{1,0}$, $f_{1,1} < f_{0,1} < f_{1,0}$ or $f_{0,1} < f_{1,0} < f_{1,1}$.

1. Suppose $f_{0,1} < f_{1,1} < f_{1,0}$. Since $f_{1,1}$ is d -separated from $f_{1,0}$ and $f_{0,1}$, we have $(\lambda - d) - d \geq 2d$. Hence $\lambda \geq 4d$.
2. Suppose $f_{1,1} < f_{0,1} < f_{1,0}$. Thus, we have $f_{0,1} \in [1 + d, \lambda - d - 1]$, $f_{1,0} \in [2 + d, \lambda - d]$. Furthermore, we have $f_{1,1} \in [1, \lambda - 2d - 1]$, $f_{0,2} \in [1 + 2d, \lambda]$ and $f_{2,0} \in [2 + 2d, \lambda]$.

Since $\{f_{2,1}, f_{1,1}, f_{0,1}\}$ is d -separated, then $f_{1,1} < f_{2,1} < f_{0,1}$ or $f_{1,1} < f_{0,1} < f_{2,1}$. The former case implies $(\lambda - d - 1) - 1 \geq 2d$. Hence, $\lambda \geq 3d + 2$.

Now we assume $f_{1,1} < f_{0,1} < f_{2,1}$. We have $f_{2,1} \in [1 + 2d, \lambda]$.

If $f_{2,1} > f_{2,0} \geq 2 + 2d$, then we have $\lambda \geq 3d + 2$. So the remaining case is $f_{1,1} < f_{0,1} < f_{2,1} < f_{2,0}$. Hence $f_{2,1} \in [1 + 2d, \lambda - d]$, $f_{0,1} \in [1 + d, \lambda - 2d]$ and $f_{2,0} \in [1 + 3d, \lambda]$.

Consider $f_{1,2}$. Note that $\{f_{1,2}, f_{1,1}, f_{1,0}\}$ is d -separated. If $f_{1,2} \in [1, \lambda - 2d - 1]$, then we have $\lambda \geq 3d + 2$. So we only need to consider that $f_{1,2} > f_{1,1}$. Since $f_{0,1} < f_{1,0}$, there are two cases: $f_{1,1} < f_{1,0} < f_{1,2}$ or $f_{1,1} < f_{1,2} < f_{1,0}$.

- 2a: Suppose $f_{1,1} < f_{1,0} < f_{1,2}$. We have $f_{1,2} \in [2 + 2d, \lambda]$. Since vertices $v_{1,2}, v_{2,1}, v_{0,2}, v_{2,0}$ induce a C_4 , by the result in [6], we have $\lambda_{1,d}(C_4) = 1 + d$ for $d \geq 2$. Hence $\lambda - (1 + 2d) \geq 1 + d$, i.e., $\lambda \geq 3d + 2$.

- 2b: Suppose $f_{1,1} < f_{1,2} < f_{1,0}$. We have $f_{1,2} \in [1 + d, \lambda - 2d]$. Since $f_{0,1} \in [1 + d, \lambda - 2d]$, we have $\lambda \geq 3d + 2$.

3. Suppose $f_{0,1} < f_{1,0} < f_{1,1}$. Then $f_{1,1} \in [1+2d, \lambda]$ and $f_{1,0} < \min\{f_{1,1}, f_{2,0}\}$. Since $f_{1,0}$ is d -separated from $f_{1,1}$ and $f_{2,0}$, $f_{1,0} \leq \min\{f_{1,1}, f_{2,0}\} - d \leq \lambda - 1 - d$. That is, $f_{1,0} \in [1+d, \lambda-d-1]$. Thus $f_{0,1} \in [d, \lambda-d-2]$.
- If $f_{2,1} > \max\{f_{1,1}, f_{2,0}\}$, then $\lambda \geq f_{2,1} \geq (2+2d)+d = 3d+2$. So we assume $f_{2,1} < \min\{f_{1,1}, f_{2,0}\}$. Hence, $f_{2,1} \in [1, \lambda-d-1]$.
- 3a: Suppose $f_{0,1} < f_{2,1}$, then we obtain that $d \leq f_{0,1} \leq \lambda-2d-2$. That is, $\lambda \geq 3d+2$.
- 3b: Suppose $f_{2,1} < f_{0,1}$. We have $f_{2,1} \in [1, \lambda-2d-2]$. This implies that $f_{0,1} \in [1+d, \lambda-d-2]$. If $f_{1,0} \in [1+d, \lambda-d-2]$, then $(\lambda-d-2)-(1+d) \geq 1$ implies that $\lambda \geq 2d+4$. So, it suffices to deal with $f_{1,0} \in (\lambda-d-2, \lambda-d-1]$. Since $f_{1,0} < \min\{f_{0,2}, f_{1,1}, f_{2,0}\}$, $f_{0,2} \in (\lambda-d-1, \lambda]$ and $f_{1,1}, f_{2,0} \in (\lambda-2, \lambda]$. Since $\ell((\lambda-2, \lambda]) < 2$, $f_{0,2} \in (\lambda-d-1, \lambda-2]$. Consider $f_{1,0}, f_{0,2}$ and $f_{2,2}$. Since $\ell((\lambda-d-2, \lambda-2]) < d$, $f_{2,2} \in [1, \lambda-d-2] \cup (\lambda-2, \lambda]$. Since $\ell((\lambda-2, \lambda]) < 2 \leq d$ and $\ell([1, d]) < d$, $f_{2,2} \in [1+d, \lambda-d-2]$. Combining with $f_{0,1} \in [1+d, \lambda-d-2]$, we get $(\lambda-d-2)-(1+d) \geq 1$. Hence $\lambda \geq 2d+4$.

Thus, for each case, we have $\lambda \geq 2d+4$. □

3. $L(1, d)$ -number of $C_6 \times C_m$

In this section, we shall determine $L(1, d)$ -numbers for all $C_6 \times C_m$, where $m \geq 4$ and $d \geq 1$. Note that, we have determined the case when $m = 3$ at the last section.

Theorem 3.1 : For $d \geq 1$, $\lambda_{1,d}(C_6 \times C_{4p}) = 5d$.

Proof : Define a labeling f for $C_6 \times C_{4p}$ and its corresponding labeling matrix as follows.

$$\text{Let } A = d \begin{bmatrix} 0 & 0 & 3 & 3 \\ 2 & 2 & 5 & 5 \\ 1 & 1 & 4 & 4 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 5 & 5 \\ 4 & 4 & 1 & 1 \end{bmatrix} \text{ and let } M_f(C_6 \times C_{4p}) \text{ be a } 1 \times p \text{ block matrix}$$

whose blocks are A .

It is not difficult to verify that f is a $5d$ - $L(1, d)$ -labeling of $C_6 \times C_{4p}$ for $d \geq 1$. That is, $\lambda_{1,d}(C_6 \times C_{4p}) \leq 5d$ for $d \geq 1$.

On the other hand, six vertices $v_{0,0}$, $v_{0,2}$, $v_{2,0}$, $v_{2,2}$, $v_{4,0}$, and $v_{4,2}$ are at distance two from each other. Hence their labels are d -separated pairwise. This implies that $\lambda_{1,d}(C_6 \times C_{4p}) \geq 5d$.

Hence, $\lambda_{1,d}(C_6 \times C_{4p}) = 5d$. $\square$

Lemma 3.2 : For $d \geq 1$ and $m \geq 4$, $\lambda_{1,d}(C_6 \times C_m) \geq 6d$ if $m \not\equiv 0 \pmod{4}$. In addition, $\lambda_{1,d}(C_6 \times C_m) \geq 7d$ if $m = 5, 10$.

Proof: Suppose $\lambda_{1,d}(C_6 \times C_m) = \lambda < 6d$, where $m \not\equiv 0 \pmod{4}$. Let $M = (M_{i,j})$ be the labeling matrix corresponding to a $\lambda - L(1, d)$ -labeling of $C_6 \times C_m$. Let $[0, \lambda] = \bigcup_{i=0}^{i=5} I_i$, where $I_i = [id, (i+1)d)$ for $0 \leq i \leq 4$ and $I_5 = [5d, \lambda]$. For each $t \in \mathbb{Z}_m$, since $\{M_{0,t}, M_{2,t}, M_{4,t}, M_{0,t+2}, M_{2,t+2}, M_{4,t+2}\}$ form a d -separated set, without loss of generality, let $M_{0,0}, M_{2,0}, M_{4,0} \in I_a \cup I_b \cup I_c$, then $M_{0,2}, M_{2,2}, M_{4,2} \in I_x \cup I_y \cup I_z$, where $\{a, b, c, x, y, z\} = \{0, 1, 2, 3, 4, 5\}$. By a similar proof as that of Lemma 2.4 we obtain a contradiction.

Thus, we have $\lambda_{1,d}(C_6 \times C_m) \geq 6d$ when $m \not\equiv 0 \pmod{4}$.

When $m = 5, 10$. Suppose $\lambda_{1,d}(C_6 \times C_m) = \lambda < 7d$. Let f be $\lambda - L(1, d)$ -labeling of $C_6 \times C_m$. Let sub-interval $J_i = [id, (i+1)d)$ for $0 \leq i \leq 6$. Hence $[0, \lambda] \subset \bigcup_{i=0}^6 J_i$.

By a similar argument as above, without loss of generality, let $M_{0,0}, M_{2,0}, M_{4,0} \in J_a \cup J_b \cup J_c$ and $M_{0,2}, M_{2,2}, M_{4,2} \in J_x \cup J_y \cup J_z$, where $\{a, b, c, x, y, z\} = \{0, 1, 2, 3, 4, 5, 6\}$. Similarly, since six vertices $v_{0,0}, v_{2,0}, v_{4,0}, v_{0,m-2}, v_{2,m-2},$ and $v_{4,m-2}$ are of distance 2 pairwise and $M_{0,0}, M_{2,0}, M_{4,0} \in J_a \cup J_b \cup J_c$, $M_{0,m-2}, M_{2,m-2}, M_{4,m-2} \in J_w \cup J_x \cup J_y \cup J_z$. Similar to the above discussion, we have $M_{0,2}, M_{2,2}, M_{4,2}, M_{0,4}, M_{2,4}, M_{4,4}$ are d -separated pairwise. Since $M_{0,2}, M_{2,2}, M_{4,2} \in J_x \cup J_y \cup J_z$, $M_{0,4}, M_{2,4}, M_{4,4} \in J_a \cup J_b \cup J_c \cup J_w$. Moreover, since three d -separated numbers $M_{0,6}, M_{2,6}$ and $M_{4,6}$ are d -separated from the numbers $M_{0,4}, M_{2,4}, M_{4,4}, M_{0,m-2}, M_{2,m-2},$ and $M_{4,m-2}$, they lie in three different sub-intervals from $M_{0,4}, M_{2,4}, M_{4,4}, M_{0,m-2}, M_{2,m-2},$ and $M_{4,m-2}$ do. Since $M_{0,4}, M_{2,4}, M_{4,4}, M_{0,m-2}, M_{2,m-2},$ and $M_{4,m-2}$ have lied in at least five different sub-intervals, at most two sub-intervals are left for $M_{0,6}, M_{2,6}, M_{4,6}$, here 6 is taken under modulo m , which is impossible. Thus $\lambda_{1,d}(C_6 \times C_m) \geq 7d$, where $m = 5, 10$.

Lemma 3.3 : For $d \geq 1$ and $p \geq 0$, $\lambda_{1,d}(C_6 \times C_{4p+14}) \leq 6d$.

Proof : Define a labeling f for graph $C_6 \times C_{4p+2}$ and its corresponding labeling matrix as follows.

$$\text{Let } A = d \begin{bmatrix} 0 & 3 & 3 & 6 & 6 & 2 & 2 & 5 & 5 & 1 & 1 & 4 & 4 & 0 \\ 2 & 2 & 5 & 5 & 1 & 1 & 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 \\ 1 & 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 & 2 & 2 & 5 & 5 & 1 \\ 3 & 3 & 6 & 6 & 2 & 2 & 5 & 5 & 1 & 1 & 4 & 4 & 0 & 0 \\ 2 & 5 & 5 & 1 & 1 & 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 & 2 \\ 4 & 4 & 0 & 0 & 3 & 3 & 6 & 6 & 2 & 2 & 5 & 5 & 1 & 1 \end{bmatrix} \text{ and } B = d \begin{bmatrix} 0 & 3 & 3 & 0 \\ 2 & 2 & 5 & 5 \\ 1 & 4 & 4 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 5 & 5 & 2 \\ 4 & 4 & 1 & 1 \end{bmatrix}.$$

Let $M_f(C_6 \times C_{4p+14}) = [A \ B \ \cdots \ B]_{1 \times (p+1)}$ be a $1 \times (p+1)$ block matrix, where p is the number of entry B . It is easy to check that f is a $6d$ - $L(1, d)$ -labeling of $C_6 \times C_{4p+14}$.

Lemma 3.4 : For $d \geq 1$ and $l \geq 0$, $\lambda_{1,d}(C_6 \times C_{2l+7}) \leq 6d$.

Proof : Define a labeling f for graph $C_6 \times C_{2l+7}$ as follows.

$$\text{Let } A = d \begin{bmatrix} 0 & 3 & 3 & 1 \\ 5 & 2 & 2 & 5 \\ 1 & 4 & 4 & 2 \\ 6 & 3 & 3 & 6 \\ 2 & 5 & 5 & 3 \\ 0 & 4 & 4 & 1 \end{bmatrix}, B = d \begin{bmatrix} 0 & 5 & 3 & 1 & 6 & 4 & 2 \\ 4 & 2 & 0 & 5 & 3 & 1 & 6 \\ 1 & 6 & 4 & 2 & 0 & 5 & 3 \\ 5 & 3 & 1 & 6 & 4 & 2 & 0 \\ 2 & 0 & 5 & 3 & 1 & 6 & 4 \\ 6 & 4 & 2 & 0 & 5 & 3 & 1 \end{bmatrix},$$

$$C = d \begin{bmatrix} 0 & 3 & 2 & 5 & 4 & 0 & 6 & 2 & 1 \\ 5 & 4 & 0 & 6 & 2 & 1 & 4 & 3 & 6 \\ 1 & 4 & 3 & 6 & 5 & 1 & 0 & 3 & 2 \\ 6 & 5 & 1 & 0 & 3 & 2 & 5 & 4 & 0 \\ 2 & 5 & 4 & 0 & 6 & 2 & 1 & 4 & 3 \\ 0 & 6 & 2 & 1 & 4 & 3 & 6 & 5 & 1 \end{bmatrix}.$$

1. Suppose $l = 2p$. Let $M_f(C_6 \times C_{4p+7}) = [A \ \cdots \ A \ B]_{1 \times (p+1)}$.
2. Suppose $l = 2p+1$. Let $M_f(C_6 \times C_{4p+9}) = [A \ \cdots \ A \ C]_{1 \times (p+1)}$.

It is easy to verify that each labeling above is a $6d$ - $L(1, d)$ -labeling of $C_6 \times C_{2l+7}$ for $l \geq 0$. Hence, we have the lemma. $\square$

According to Lemmas 3.2, 3.3 and 3.4, we have the following result.

Theorem 3.5: For $d \geq 1$, $\lambda_{1,d}(C_6 \times C_m) = 6d$ if $m \not\equiv 0 \pmod{4}$ and $m \neq 5, 6, 10$.

Theorem 3.6: For $d \geq 1$, $\lambda_{1,d}(C_6 \times C_6) = 8d$.

Proof : Define the labeling matrix of a labeling f for graph $C_6 \times C_6$ by

$$M_f(C_6 \times C_6) = d \begin{bmatrix} 0 & 0 & 3 & 3 & 6 & 6 \\ 2 & 2 & 5 & 5 & 8 & 8 \\ 1 & 1 & 4 & 4 & 7 & 7 \\ 3 & 3 & 6 & 6 & 0 & 0 \\ 2 & 2 & 5 & 5 & 8 & 8 \\ 4 & 4 & 7 & 7 & 1 & 1 \end{bmatrix}.$$

It is easy to verify that f be an $8d$ - $L(1, d)$ -labeling of $C_6 \times C_6$ for $d \geq 1$. That is, $\lambda_{1,d}(C_6 \times C_6) \leq 8d$.

On the other hand, since nine vertices in set $\{v_{s,t} \mid s, t = 0, 2, 4\}$ are at distance two from each other, the set of their labels is d -separated. This implies that $\lambda_{1,d}(C_6 \times C_6) \geq 8d$. Hence, $\lambda_{1,d}(C_6 \times C_6) = 8d$. $\square$

Theorem 3.7 : For $d \geq 1$, $\lambda_{1,d}(C_6 \times C_m) = 7d$ if $m = 5, 10$.

Proof : Define the labeling matrices of labelings f and g for $C_6 \times C_5$ and $C_6 \times C_{10}$ by

$$M_f(C_6 \times C_5) = d \begin{bmatrix} 0 & \frac{3}{2} & 3 & \frac{9}{2} & 6 \\ \frac{1}{2} & 2 & \frac{7}{2} & 5 & \frac{13}{2} \\ 1 & \frac{5}{2} & 4 & \frac{11}{2} & 7 \\ \frac{3}{2} & 3 & \frac{9}{2} & 6 & 0 \\ 2 & \frac{7}{2} & 5 & \frac{13}{2} & \frac{1}{2} \\ \frac{5}{2} & 4 & \frac{11}{2} & 7 & 1 \end{bmatrix}, \text{ and}$$

$$M_g(C_6 \times C_{10}) = d \begin{bmatrix} 0 & 0 & 3 & 3 & 6 & 6 & 1 & 1 & 4 & 4 \\ 6 & 2 & 2 & 5 & 5 & 0 & 0 & 3 & 3 & 6 \\ 1 & 1 & 4 & 4 & 7 & 7 & 2 & 2 & 5 & 5 \\ 0 & 3 & 3 & 6 & 6 & 1 & 1 & 4 & 4 & 0 \\ 2 & 2 & 5 & 5 & 0 & 0 & 3 & 3 & 6 & 6 \\ 1 & 4 & 4 & 7 & 7 & 2 & 2 & 5 & 5 & 1 \end{bmatrix}, \text{ respectively.}$$

It is easy to verify that f and g are $7d$ - $L(1, d)$ -labelings of $C_6 \times C_m$ for $m = 5, 10$, respectively. Combining with Lemma 3.2, we have the theorem. $\square$

4. $L(1, d)$ -numbers of $C_5 \times C_m$ and $C_{10} \times C_m$

Lemma 4.1 : Let $M = (M_{i,j})$ be a labeling matrix corresponding to a λ - $L(1, d)$ -labeling of $C_5 \times C_m$, where $d \geq 1$, $m \geq 3$. Let J be a subinterval of $[0, \lambda]$ such that $\ell(J) < d$. For a fixed $t \in \mathbb{Z}_m$, J contains at most 2 entries come from the t -th and $(t+2)$ -nd columns of M , where $t+2$ is taken under modulo m .

Proof : By symmetric of the graph, without loss of generality, we may assume that $t = 0$ and $M_{0,2} \in J$. Consider the 0-th and 2-nd columns of M . Since $\ell(J) < d$, only $M_{1,0}$, $M_{4,0}$, $M_{1,2}$ and $M_{4,2}$ may lie in J . It is easy to see that when one of the above entries lie in J , there is no other entry may lie in J .

Now, suppose M is a labeling matrix of a λ - $L(1, d)$ -labeling of $C_5 \times C_m$ with $m \geq 3$. Let $I_i = [id, (i+1)d)$, where $i \geq 0$. In order to cover all entries of M , by Lemma 4.1, we need at least 5 I_i 's. So, in the following we always assume that:

$$\left\{ \begin{array}{l} M \text{ is a labeling matrix of a } \lambda\text{-}L(1, d)\text{-labeling of } C_5 \times C_m \text{ and } [0, \lambda] \subset \bigcup_{i=0}^4 I_i. \\ \text{The first and the second subscripts of } M_{s,t} \text{ are taken under modulo 5 and } m, \text{ respectively.} \end{array} \right. \quad (4.1)$$

Lemma 4.2 : Under the condition (4.1), the following statements hold.

- (1) For fixed $t \in \mathbb{Z}_m$ and $s \in \mathbb{Z}_5$, there is exactly one of $M_{s,t}$ and $M_{s,t+2}$ lies in the same subinterval as one of $M_{s+1,t}$ and $M_{s+1,t+2}$.
- (2) For a fixed $t \in \mathbb{Z}_m$, each subinterval contains exactly two entries from the t -th and $(t+2)$ -nd columns of M .

Proof : By symmetric we may assume that $s = 0$. Since $v_{0,t}$, $v_{0,t+2}$, $v_{3,t}$ and $v_{3,t+2}$ are of distance two pairwise, we may assume that $M_{0,t} \in I_a$, $M_{0,t+2} \in I_b$, $M_{3,t} \in I_c$ and $M_{3,t+2} \in I_d$, where a, b, c, d are distinct and $\{a, b, c, d\} \subset \{0, 1, 2, 3, 4\}$. Let $e \in \{0, 1, 2, 3, 4\} \setminus \{a, b, c, d\}$. Since $v_{0,t}$, $v_{0,t+2}$, $v_{2,t}$ and $v_{2,t+2}$ are of distance two pairwise, $M_{2,t}, M_{2,t+2} \in I_c \cup I_d \cup I_e$. Similarly, $M_{1,t}, M_{1,t+2} \in I_a \cup I_b \cup I_e$.

Suppose $M_{1,t}, M_{1,t+2} \in I_a \cup I_b$. By the same argument above, $M_{4,t}, M_{4,t+2} \in I_c \cup I_d \cup I_e$. This is a contradiction since $M_{2,t}, M_{2,t+2}, M_{4,t}, M_{4,t+2}$ are i -separated pairwise and the length of each I_i is less than d . Thus, $M_{1,t}, M_{1,t+2} \in I_a \cup I_e$ or $M_{1,t}, M_{1,t+2} \in I_b \cup I_e$. Hence we have (1).

Since there are only 5 subintervals and the total considered entries is 20, by Lemma 4.1 we have (2). $\square$

Lemma 4.3 : *Under the condition (4.1), if there are two adjacent vertices lie in I_i , for some $i \in \{0, 1, 2, 3, 4\}$, then $\lambda \geq 4d + 1$.*

Proof : Suppose $M_{s,t}, M_{s+1,t+1} \in I_i$, where $0 \leq i \leq 4$. Without loss of generality, let $M_{s,t} > M_{s+1,t+1}$. Hence $M_{s,t} \geq id + 1$. Applying Lemma 4.2 repeatedly if necessary, we have $\lambda \geq id + 1 + (4 - i)d = 4d + 1$. $\square$

Lemma 4.4 : *Under the condition (4.1) and $\lambda < 4d + 1$, all 5 entries of each column of M lie in 5 difference I_i 's.*

Proof : Suppose two entries of the t -th column of M lie in the same subinterval, say I_a . These two entries must be $M_{s,t}$ and $M_{s+1,t}$ for some $s \in \mathbb{Z}_5$. Without loss of generality, we assume that $s = 0$. Look at the $(t - 1)$ -st and $(t + 1)$ -st columns of M , by the contrapositive of Lemma 4.3, only $M_{4,t-1}$ and $M_{4,t+1}$ may lie in I_a . Since $v_{4,t-1}$ and $v_{4,t+1}$ are of distance 2, at most one of $M_{4,t-1}$ and $M_{4,t+1}$ may lie in I_a . This contradicts (2) of Lemma 4.2. $\square$

Under the condition (4.1), we introduce a new $5 \times m$ matrix N defined by $N_{h,k} = x$ if $M_{h,k} \in I_x$.

Lemma 4.5 : *Under the condition (4.1) and $\lambda < 4d + 1$, the matrix N defined above is column periodic matrix of period 5, i.e., the k -th column of N is the same as the $(k + 5)$ -th column of N for all k . Here $k + 5$ is taken under modulo m .*

Proof : Consider the 0-th and the 2-nd column of M . Without loss of generality, Lemma 4.4, we may assume that $M_{0,0} \in I_a$, $M_{1,0} \in I_b$, $M_{2,0} \in I_c$, $M_{3,0} \in I_d$ and $M_{4,0} \in I_e$, where $\{a, b, c, d, e\} = \{0, 1, 2, 3, 4\}$. Then only $M_{1,2} \in I_a$ or $M_{4,2} \in I_a$. By symmetric, we may assume $M_{1,2} \in I_a$. By the contrapositive of Lemma 4.3 $M_{3,1} \in I_a$. Now let introduce a new $5 \times m$ matrix N defined by $N_{h,k} = x$ if $M_{h,k} \in I_x$. Applying (1) of Lemma 4.2, the contrapositive of Lemma 4.3 and Lemma 4.4, we get that the first 3 columns of N is

$$\begin{pmatrix} a & c & e \\ b & d & a \\ c & e & b \\ d & a & c \\ e & b & d \end{pmatrix}$$

By the contrapositive of Lemma 4.3 repeatedly, we get that the first 5 columns of N is

$$\begin{pmatrix} a & c & e & b & d \\ b & d & a & c & e \\ c & e & b & d & a \\ d & a & c & e & b \\ e & b & d & a & c \end{pmatrix}$$

When $m \geq 6$, by the contrapositive of Lemma 4.3, we shall see that the 5-th column is the same as the 0-th of N , the 6-th column is the same as the 1-st of N , and so on. $\square$

Lemma 4.6 : Let $d \geq 1$, $m \geq 3$ and $m \not\equiv 0 \pmod{5}$, $\lambda_{1,d}(C_5 \times C_m) \geq 4d + 1$.

Proof : Suppose $\lambda = \lambda_{1,d}(C_5 \times C_m) < 4d + 1$. Since the m -th and the 0-th columns of N is the same, by Lemma 4.5, only when $m \equiv 0 \pmod{5}$ may agree. $\square$

Theorem 4.7 : For $d \geq 2$, $\lambda_{1,d}(C_5 \times C_m) = 4d + 1$ if $m \geq 4$, $m \neq 6$ and $m \not\equiv 0 \pmod{5}$.

Proof : By Lemma 4.6, we know that $\lambda_{1,d}(C_5 \times C_m) \geq 4d + 1$.

On the other hand, we shall define a $(4d + 1)$ - $L(1, d)$ -labeling f for each following case, where $m \geq 4$, $m \neq 6$ and $m \not\equiv 0 \pmod{5}$.

Case 1 : Suppose $m = 4p$, where $p \geq 1$.

$$\text{Let } A = \begin{bmatrix} 0 & 1 & d & 1+d \\ d & 1+d & 2d & 1+2d \\ 2d & 1+2d & 3d & 1+3d \\ 3d & 1+3d & 4d & 1+4d \\ 4d & 1+4d & 0 & 1 \end{bmatrix}. \text{ We define } M_f(C_5 \times C_{4p}) = [A \cdots A]_{1 \times p} = M_1.$$

Case 2 : Suppose $m = 5 + 4p$, where $p \geq 1$.

$$\text{Let } B = \begin{bmatrix} 0 & d & 2d & 3d+1 \\ 3d & 4d & 0 & 1+d \\ d & 2d & 3d & 1+4d \\ 4d & 0 & d & 1+2d \\ 2d & 3d & 4d & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & d & 2d & 3d & 4d \\ 3d & 4d & 0 & d & 2d \\ d & 2d & 3d & 4d & 0 \\ 4d & 0 & d & 2d & 3d \\ 2d & 3d & 4d & 0 & d \end{bmatrix}.$$

$$\text{Let } M_f(C_5 \times C_{4p+5}) = [C B \cdots B]_{1 \times (1+p)}.$$

Case 3 : Suppose $m = 7 + 4p$, where $p \geq 0$.

$$\text{Let } D = \begin{bmatrix} 0 & 3d+1 & 2d & 1 & 4d & 2d+1 & d \\ 3d & d+1 & 0 & 3d+1 & 2d & 1 & 4d \\ d & 4d+1 & 3d & d+1 & 0 & 3d+1 & 2d \\ 4d & 2d+1 & d & 4d+1 & 3d & d+1 & 0 \\ 2d & 1 & 4d & 2d+1 & d & 4d+1 & 3d \end{bmatrix} \text{ and } E = \begin{bmatrix} 0 & 3d+1 & 2d & 1 \\ 3d & d+1 & 0 & 3d+1 \\ d & 4d+1 & 3d & d+1 \\ 4d & 2d+1 & d & 4d+1 \\ 2d & 1 & 4d & 2d+1 \end{bmatrix}.$$

Let $M_f(C_5 \times C_{7+4p}) = [D E \cdots E]_{1 \times (1+p)}$.

Case 4 : Suppose $m = 14 + 4p$, where $p \geq 0$.

Let $M_f(C_5 \times C_{14+4p}) = [D D E \cdots E]_{1 \times (2+p)} = M_2$, where D and E are defined above.

It is not difficult to check that each f defined above is a $(4d+1) - L(1, d)$ -labeling. Hence, we have the theorem. $\square$

Since $C_{10} \times C_6 \cong C_6 \times C_{10}$, from Theorem 3.7, we have $\lambda_{1,d}(C_{10} \times C_6) = 7d$ for $d \geq 1$.

Theorem 4.8 : For $d \geq 2$, $\lambda_{1,d}(C_{10} \times C_4) = \lambda_{1,d}(C_{10} \times C_m) = 4d+1$ if $m \not\equiv 0(\text{mod } 5)$ and $m \geq 7$.

Proof : For odd m , one may check that $C_{10} \times C_m \cong C_5 \times C_{2m} \cong C_{2m} \times C_5$. By Theorems 4.7 and 3.7, we have $\lambda_{1,d}(C_{10} \times C_m) = 4d+1$ for $m \geq 7$.

For even $m \neq 6$, let $S = \{v_{s,t} \mid t \in_m \text{ and } s+t \equiv 0(\text{mod } 2)\}$ be a set of vertices in $C_{10} \times C_m$. Let $H(5, m)$ be the subgraph induced by S . One may check that $H(5, m) \cong C_5 \times C_m$. By Theorem 4.7 and Lemma 1.1, we have $\lambda_{1,d}(C_{10} \times C_m) \geq \lambda_{1,d}(H(5, m)) = 4d+1$.

On the other hand, we define a $(4d+1) - L(1, d)$ -labeling f for each case, where $m \not\equiv 0(\text{mod } 5)$.

Case 1 : Suppose $m = 4p$, where $p \geq 1$.

Let $M_f(C_{10} \times C_{4p}) = \begin{bmatrix} M_1 \\ M_1 \end{bmatrix}$, where M_1 is defined in Case 1 of Theorem 4.7.

Case 2 : Suppose $m = 14 + 4p$, where $p \geq 0$.

Let $M_f(C_{10} \times C_{14+4p}) = \begin{bmatrix} M_2 \\ M_2 \end{bmatrix}$, where M_2 is defined in Case 4 of Theorem 4.7.

It is not difficult to check that each f is a $(4d+1) - L(1, d)$ -labeling. This completes the proof. $\square$

When $m \equiv 0(\text{mod } 5)$, we have the following upper bound for $d \geq 1$.

Lemma 4.9 : For $d \geq 1$ and $p, q \geq 1$, $\lambda_{1,d}(C_{5p} \times C_{5q}) \leq 4d$.

Proof: Let $A = d \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}$. Let M be a $p \times q$ block matrix whose blocks

are A . Clearly M defines a $4d$ - $L(1, d)$ -labeling for $C_{5p} \times C_{5q}$. Hence, we have the lemma. $\square$

5. $L(1, d)$ -number of $C_{4p} \times C_m$

Before studying the $L(1, d)$ -number of $C_{4p} \times C_m$ for $m \geq 4$, we consider the following three graphs first.

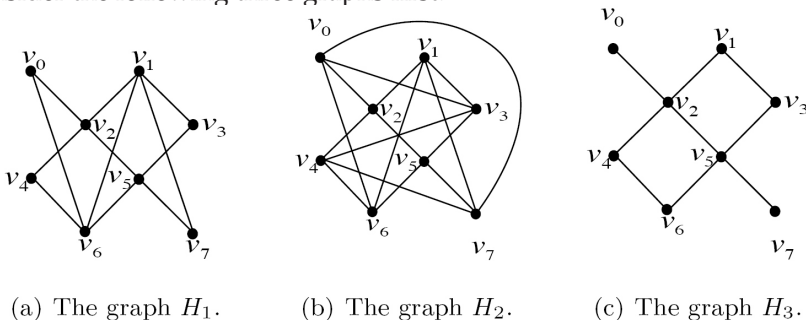


Figure 1

Lemma 5.1 : For $d \geq 1$, $\lambda_{1,d}(H_i) \geq 3d + 1$, where $i = 1, 2, 3$.

Proof : In each graph, there are two sets of vertices, namely $A_5 = \{v_0, v_1, v_4, v_5\}$ and $A_2 = \{v_7, v_6, v_3, v_2\}$. Any two vertices in the same set are of distance 2. Moreover, v_2 is adjacent to each vertex of A_5 and v_5 is adjacent to each vertex of A_2 . Let f be a λ - $L(1, d)$ -labeling of H_i , $1 \leq i \leq 3$. $\{f(v_7), f(v_6), f(v_3), f(v_2)\}$ is d -separated. Hence $\lambda \geq 3$.

Since the proofs for H_1 , H_2 and H_3 are same as each other, here we only show the proof for H_1 .

Note that, either a vertex in A_5 or a vertex in A_2 is labeled by 0. By the symmetry property stated above, we may assume that $0 \in \{f(v_0), f(v_1), f(v_4), f(v_5)\}$. Then, each of the sets $\{0\}$, $[d, d+1)$, $[2d, 2d+1)$ and $[3d, \lambda]$ contains exactly one label $f(v_j)$, $j \in \{0, 1, 4, 5\}$.

Let $I_i = [(i-1)d+1, id)$, where $i = 1, 2, 3$. Since v_2 is adjacent to each vertex of A_5 and the lengths of intervals $[d, d+1)$ are $[2d, 2d+1)$ are less than 1, $f(v_2) \in I_i$ for some $i \in \{1, 2, 3\}$ or $f(v_2) \in [3d, \lambda]$.

- Suppose $f(v_2) \in [3d, \lambda]$. Since $f(v_j) \in [3d, \lambda]$ for some $v_j \in A_5$, $\lambda - 3d \geq 1$, i.e., $\lambda \geq 3d + 1$.
- If $f(v_2) \in I_1 = [1, d]$, then $f(v_3), f(v_6), f(v_7) > f(v_2)$. This implies that $\lambda \geq \max\{f(v_3), f(v_6), f(v_7)\} \geq 3d + 1$.
- If $f(v_2) \in I_2 = [1 + d, 2d]$, then at most one label of $\{f(v_3), f(v_6), f(v_7)\}$ is less than $f(v_2)$ since the length of $[0, d]$ is less than d . That is, at least two labels of $\{f(v_3), f(v_6), f(v_7)\}$ are larger than $f(v_2)$. This implies that $\lambda \geq \max\{f(v_3), f(v_6), f(v_7)\} \geq 3d + 1$.
- If $f(v_2) \in I_3 = [1 + 2d, 3d]$, then at most two labels of $\{f(v_3), f(v_6), f(v_7)\}$ are less than $f(v_2)$ since the length of $[0, 2d]$ is less than $2d$. That is, at least one of $\{f(v_3), f(v_6), f(v_7)\}$ is larger than $f(v_2)$. This implies that $\lambda \geq \max\{f(v_3), f(v_6), f(v_7)\} \geq 3d + 1$.

Hence, the lemma is proved. $\square$

Theorem 5.2 : For $d \geq 2$, $\lambda_{1,d}(C_{4p} \times C_{4q}) = 3d + 1$.

Proof: Define a labeling f for $C_{4p} \times C_{4q}$ and its corresponding labeling matrix as follows.

$$\text{Let } A = \begin{bmatrix} 0 & 0 & 2d & 2d \\ 1 & 1 & 2d+1 & 2d+1 \\ d & d & 3d & 3d \\ d+1 & d+1 & 3d+1 & 3d+1 \end{bmatrix} \text{ and let } M_f(C_{4p} \times C_{4q}) \text{ be a } p \times q \text{ block}$$

matrix, whose blocks are A .

It is easy to check that f is a $(3d + 1) - L(1, d)$ -labeling of $C_{4p} \times C_{4q}$ for $d \geq 2$. That is, $\lambda_{1,d}(C_{4p} \times C_{4q}) \leq 3d + 1$.

On the other hand, we have following three cases.

- Suppose $p = q = 1$. The graph H_2 (See Figure 1(b)) is an induced subgraph of $C_4 \times C_4$. According to Lemmas 1.1 and 5.1, we have $\lambda_{1,d}(C_4 \times C_4) \geq \lambda_{1,d}(H_2) \geq 3d + 1$ for $d \geq 1$.
- Suppose $p = 1$ and $q \geq 2$. The graph H_1 (See Figure 1(a)) is an induced subgraph of $C_4 \times C_{4q}$. By Lemmas 1.1 and 5.1, we obtain that $\lambda_{1,d}(C_4 \times C_{4q}) \geq \lambda_{1,d}(H_1) \geq 3d + 1$ for $d \geq 1$.
- Suppose $p, q \geq 2$. The graph H_3 (See Figure 1(c)) is an induced subgraph of $C_{4p} \times C_{4q}$. By Lemmas 1.1 and 5.1, we obtain that $\lambda_{1,d}(C_{4p} \times C_{4q}) \geq \lambda_{1,d}(H_3) \geq 3d + 1$ for $d \geq 1$.

Hence $\lambda_{1,d}(C_{4p} \times C_{4q}) = 3d + 1$ for $d \geq 2$. $\square$

Based on the definition of $L(j, k)$ -labeling [7] and $L(1, d)$ -labeling, it is easy to obtain that $d = \frac{k}{j}$. That is, $\lambda_{1,d}(G) = \lambda_{1, \frac{k}{j}}(G) = \frac{1}{j} \lambda_{j,k}(G)$. According to the proof of the lower bound on $\lambda_{j,k}(P_3 \times C_m)$ in [7, Theorem 2.4], we have the following result directly.

Lemma 5.3 : ([7]) For $d \geq 1$, $\lambda_{1,d}(P_3 \times C_m) \geq 4d$ if $m \not\equiv 0 \pmod{4}$.

Since graph $P_3 \times C_m$ is an induced subgraph of $C_n \times C_m$ if $m, n \geq 4$, by Lemmas 1.1 and 5.3, we have the following result.

Lemma 5.4 : For $d \geq 1$ and $m, n \geq 4$, $\lambda_{1,d}(C_n \times C_m) = \lambda_{1,d}(C_m \times C_n) \geq 4d$ if $m \not\equiv 0 \pmod{4}$.

Combining Lemmas 5.4 and 4.9 we have

Corollary 5.5 : For $d \geq 1$, $\lambda_{1,d}(C_{5p} \times C_{5q}) = 4d$ if $p \not\equiv 0 \pmod{4}$ or $q \not\equiv 0 \pmod{4}$.

Now we have known the $L(1, d)$ -labeling numbers of $C_{4p} \times C_{4q}$. From the previous sections we have also known the $L(1, d)$ -labeling numbers of $C_3 \times C_{4p}$, $C_5 \times C_{4p}$, $C_6 \times C_{4p}$ and $C_{10} \times C_{4p}$, when $d \geq 2$. Following we consider the graph $C_{4p} \times C_m$ when $m \geq 7$, $m \neq 10$ and $m \not\equiv 0 \pmod{4}$.

Lemma 5.6 : Suppose $d \geq 2$, $m \not\equiv 0 \pmod{4}$, $m \geq 7$ and $m \neq 10$. Then $\lambda_{1,d}(C_{4p} \times C_m) \leq 4d$.

Proof : Following we will define a labeling f for each graph $C_n \times C_m$ by providing its corresponding labeling matrix $M_f(C_{4p} \times C_m)$.

1 Suppose $m = 4q + 5 \geq 9$.

$$\text{Let } A = d \begin{bmatrix} 0 & \frac{1}{2} & 1 & \frac{3}{2} \\ 0 & \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & \frac{5}{2} & 3 & \frac{7}{2} \\ 2 & \frac{5}{2} & 3 & \frac{7}{2} \end{bmatrix} \text{ and } B = d \begin{bmatrix} 2 & \frac{5}{2} & 3 & \frac{7}{2} & 4 \\ 2 & \frac{5}{2} & 3 & \frac{7}{2} & 4 \\ 4 & 0 & \frac{1}{2} & 1 & \frac{3}{2} \\ 4 & 0 & \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix}.$$

$$\text{Define } M_f(C_n \times C_m) = \begin{bmatrix} A & \cdots & A & B \\ \vdots & \vdots & \vdots & \vdots \\ A & \cdots & A & B \end{bmatrix}_{p \times (q+1)}.$$

It is directly to verify that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p} \times C_{4q+5}$ when $q \geq 1$.

2. Suppose $m = 4q + 7 \geq 7$.

$$\text{Let } C = d \begin{bmatrix} 0 & 4 & 1 & \frac{5}{2} \\ 0 & 4 & 1 & \frac{5}{2} \\ 2 & \frac{3}{2} & 3 & \frac{1}{2} \\ 2 & \frac{3}{2} & 3 & \frac{1}{2} \end{bmatrix} \text{ and } D = d \begin{bmatrix} 0 & 4 & 1 & 0 & 2 & 1 & 3 \\ 0 & 4 & 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 \\ 2 & \frac{3}{2} & 3 & \frac{5}{2} & 4 & \frac{7}{2} & \frac{1}{2} \\ 2 & \frac{3}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} \end{bmatrix}.$$

$$\text{Define } M_f(C_n \times C_m) = \begin{bmatrix} C & \cdots & C & D \\ \vdots & \vdots & \vdots & \vdots \\ C & \cdots & C & D \end{bmatrix}_{p \times (q+1)}.$$

It is directly to verify that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p} \times C_{4q+7}$ when $q \geq 0$. Note that, when $q = 0$, only the last column of the block matrix $M_f(C_{4p} \times C_{4q+7})$ appears. This convention also applies other cases in this paper.

3. Suppose $m = 14$.

$$\text{Let } E = d \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 0 & 0 & 1 & 1 \\ 3 & 4 & 4 & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & 0 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & 4 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & \frac{1}{2} \end{bmatrix}. \text{ Define}$$

$$M_f(C_{4p} \times C_{14}) = \begin{bmatrix} E \\ \vdots \\ E \end{bmatrix}_{p \times 1}.$$

It is obviously that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p} \times C_{14}$.

4. Suppose $m = 4q + 14 \geq 18$.

$$\text{Let } G = d \begin{bmatrix} 2 & 2 & 3 & 3 & 4 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} \\ \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 4 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

$$\text{and } F = d \begin{bmatrix} 0 & 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 2 & 2 & 3 & 3 \\ \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{7}{2} \end{bmatrix}.$$

$$\text{Let } M_f(C_n \times C_m) = \begin{bmatrix} F & \cdots & F & G \\ \vdots & \vdots & \vdots & \vdots \\ F & \cdots & F & G \end{bmatrix}_{p \times (q+1)}.$$

It is directly to verify that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p} \times C_{4q+14}$ when $q \geq 1$.

Hence, we have the lemma. $\square$

Combining Corollary 5.5, Lemmas 5.4 and 5.6 we have

Theorem 5.7 : Suppose $m \not\equiv 0 \pmod{4}$, $m \geq 7$, $m \neq 10$. For $d \geq 2$, $\lambda_{1,d}(C_{4p} \times C_m) = 4d$.

Lemma 5.8 : Let M be a labeling matrix corresponding to a λ - $L(1, d)$ -labeling of $C_4 \times C_5$, where $d \geq 1$. Let I be a subinterval of $[0, \lambda]$ such that $\ell(I) < d$. I contains at most 4 entries of M . Moreover, if $\ell(I) < 1$, then I contains at most 3 entries of M .

Proof : The first part follows from Lemma 4.1. For the last case, it is easy to check that I contains at most 3 entries of M . $\square$

Corollary 5.9 : $\lambda_{1,1}(C_4 \times C_5) = 6$.

Proof : Let $I_j = [jd, (j+1)d)$, $j \geq 0$. By Lemma 5.8, $\bigcup_{j=0}^6 I_j$ cannot cover all entries of M . Thus $\lambda_{1,1}(C_4 \times C_5) \geq 6$.

On the other hand, let $M = \begin{bmatrix} 2 & 4 & 0 & 6 & 5 \\ 0 & 1 & 5 & 4 & 3 \\ 5 & 2 & 3 & 0 & 1 \\ 3 & 6 & 1 & 2 & 4 \end{bmatrix}$. M defines a

6- $L(1, 1)$ -labeling for $C_4 \times C_5$. This completes the proof. $\square$

Theorem 5.10 : For $1 \leq d \leq 2$, $\lambda_{1,d}(C_4 \times C_4) = 7$.

Proof: Let f be any λ - $L(1, d)$ -labeling of $C_4 \times C_4$ with labeling matrix M . Let $I_i = [i, i+1)$, $0 \leq i \leq \lfloor \lambda \rfloor$.

By a similar argument of the proof of Lemma 4.1, we will get that there are at most 2 entries of M lying in the same subinterval I_i . Since there are 16 entries, $\lambda \geq 7$.

On the other hand, $M = \begin{bmatrix} 0 & 0 & 4 & 4 \\ 1 & 1 & 5 & 5 \\ 2 & 2 & 6 & 6 \\ 3 & 3 & 7 & 7 \end{bmatrix}$ defines a 7- $L(1, d)$ -labeling of

$C_4 \times C_4$. This completes the proof. $\square$

6. $L(1, d)$ -number of $C_n \times C_m$ for other cases

Let us summarize what we have in the previous sections.

For $n = 4p$, $m \geq 3$ and $d \geq 2$, the $\lambda_{1,d}(C_{4p} \times C_m)$ are:

m	$4q$	3	5	6	10	others
$\lambda_{1,d}$	$3d+1$	$5d$	$4d+1$	$5d$	$\begin{cases} 4d+1, & \text{if } p \not\equiv 0 \pmod{5} \\ 4d, & \text{if } p \equiv 0 \pmod{5} \end{cases}$	$4d$

For $n = 4p$, $m \geq 3$ and $1 \leq d \leq 2$, the $\lambda_{1,d}(C_n \times C_m)$ are:

n	$4p$	4
m	3	6
$\lambda_{1,d}$	$5d$	7

and $\lambda_{1,1}(C_4 \times C_5) = 6$.

Following we assume that $n, m \geq 3$, $n \not\equiv 0 \pmod{4}$ and $m \not\equiv 0 \pmod{4}$.

For $d \geq 2$, the $\lambda_{1,d}(C_n \times C_m)$ are:

n	3						6				5p	5, 10
m	3	5	6	9	10	others	5	6	10	others	$5q$	$\not\equiv 0 \pmod{5}$ and ≥ 7
$\lambda_{1,d}$	$2d+4$	$7d$	$8d$	$6d$	$7d$	$6d$	$7d$	$8d$	$7d$	$6d$	$4d$	$4d+1$

For $1 \leq d \leq 2$, the $\lambda_{1,d}(C_n \times C_m)$ are:

n	3					6				5p
m	3	6	7	10	≥ 11	5	6	10	others	$5q$
$\lambda_{1,d}$	8	$8d$	$6d$	$7d$	$6d$	$7d$	$8d$	$7d$	$6d$	$4d$

Lemma 6.1 : Suppose $n, m \not\equiv 0 \pmod{4}$, $n \geq m \geq 9$, $m, n \notin \{10, 14\}$ and $n \notin \{11, 15, 19, 23\}$ if $m \equiv 3 \pmod{4}$. For $d \geq 2$, $\lambda_{1,d}(C_n \times C_m) \leq 4d$ and $\lambda_{1,d}(C_{23} \times C_{23}) \leq 4d$.

Proof: We consider the cases when $n = 4p+9$, $4p+11$ and $4p+18$ separately. We shall define a labeling f for the graph $C_n \times C_m$ by providing its corresponding labeling matrix $M_f(C_n \times C_m)$ for each case. The detail labelings please see the appendix. $\square$

According to Lemmas 5.4 and 6.1., we have the following results.

Theorem 6.2 : Suppose $n, m \not\equiv 0(\text{mod } 4)$, $n \geq m \geq 9$, $m, n \notin \{10, 14\}$ and $n \notin \{11, 15, 19, 23\}$ if $m \equiv 3(\text{mod } 4)$. For $d \geq 2$, $\lambda_{1,d}(C_n \times C_m) = 4d$ and $\lambda_{1,d}(C_{23} \times C_{23}) = 4d$.

7. Conclusion

In this paper, we have discussed $L(1, d)$ -labeling number of direct product of two cycles for $d \geq 1$. There are some unsolved problems and conjectures as follows.

Conjecture 7.1 : $\lambda_{1,d}(C_{4p} \times C_{4q}) = \min\{7, 3d + 3\}$ when $1 \leq d < 2$.

Conjecture 7.2 : For $d \geq 2$, $\lambda_{1,d}(C_7 \times C_m) = \lambda_{1,d}(C_{14} \times C_m) = 4d + 1$ when $m \not\equiv 0(\text{mod } 4)$ and $m \geq 7$.

Conjecture 7.3 : Suppose $n, m \not\equiv 0(\text{mod } 4)$, $n \geq m$. For $d \geq 2$, $\lambda_{1,d}(C_n \times C_m) = 4d$ if $n \in \{11, 15, 19, 23\}$ and $m \equiv 3(\text{mod } 4)$.

Problem: What is the $L(1, d)$ -labeling number of graph $C_n \times C_m$, for $1 \leq d < 2$, $n, m \not\equiv 0(\text{mod } 4)$, and $n, m \geq 9$?

8. Appendix: The proof of Lemma 6.1

Firstly, suppose $n = 4p + 9 \geq 9$.

1(a) Suppose $m = 4q + 14 \geq 18$.

$$\text{Let } T_1 = d \begin{bmatrix} 0 & 0 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} \\ 1 & 1 & 3 & 3 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} \end{bmatrix}, \quad S_1 = d \begin{bmatrix} 0 & 0 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} \\ 1 & 1 & 3 & 3 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} \end{bmatrix},$$

$$P_1 = d \begin{bmatrix} 0 & 0 & 2 & 2 & 4 & 4 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 0 \\ 1 & 1 & 3 & 3 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \text{ and}$$

$$Q_1 = d \begin{bmatrix} 4 & 4 & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} & 1 & 1 & 3 & 3 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 2 & 2 & 4 & 4 & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} & 1 & 1 & 3 & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 0 & 2 & 2 & 4 & 4 & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} \\ 1 & 1 & 3 & 3 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 0 & 2 & 2 & 4 & 4 \\ \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} & 1 & 1 & 3 & 3 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 0 \\ 2 & 2 & 4 & 4 & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} & 1 & 1 & 3 & 3 & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{2} & \frac{5}{2} & 0 & 0 & 2 & 2 & 4 & 4 & \frac{3}{2} & \frac{3}{2} & \frac{7}{2} & \frac{7}{2} & 1 & 1 \\ 3 & 3 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 0 & 2 & 2 & 4 & 4 & \frac{3}{2} & \frac{3}{2} \\ \frac{7}{2} & \frac{7}{2} & 1 & 1 & 3 & 3 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} & 0 & 0 & 2 & 2 \end{bmatrix}.$$

$$\text{Let } M_f(C_{4p+9} \times C_{4q+14}) = \begin{bmatrix} T_1 & \cdots & T_1 & P_1 \\ \vdots & \vdots & \vdots & \vdots \\ T_1 & \cdots & T_1 & P_1 \\ S_1 & \cdots & S_1 & Q_1 \end{bmatrix}_{(p+1) \times (q+1)}.$$

It is directly to verify that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p+9} \times C_{4q+14}$ when $p \geq 0$ and $q \geq 1$.

1(b) Suppose $m = 4q + 5 \geq 9$.

$$\text{Let } T_2 = d \begin{bmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 1 & 2 & 3 & 4 \\ \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \end{bmatrix}, \quad P_2 = d \begin{bmatrix} 4 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 0 & 1 & 2 & 3 & 4 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 \\ 1 & 2 & 3 & 4 & \frac{1}{2} \end{bmatrix}, \quad S_2 = d \begin{bmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 1 & 2 & 3 & 4 \\ \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 2 & 3 & 4 & \frac{1}{2} \\ \frac{5}{2} & \frac{7}{2} & 0 & 1 \\ \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 4 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \end{bmatrix}$$

$$\text{and } Q_2 = d \begin{bmatrix} 4 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 0 & 1 & 2 & 3 & 4 \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 \\ 1 & 2 & 3 & 4 & \frac{1}{2} \\ \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & 0 & 1 \\ 2 & 3 & 4 & \frac{1}{2} & \frac{3}{2} \\ \frac{5}{2} & \frac{7}{2} & 0 & 1 & 2 \\ 3 & 4 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{7}{2} & 0 & 1 & 2 & 3 \end{bmatrix}. \text{ Let } M_f(C_{4p+9} \times C_{4q+5}) = \begin{bmatrix} T_2 & \cdots & T_2 & P_2 \\ \vdots & \vdots & \vdots & \vdots \\ T_2 & \cdots & T_2 & P_2 \\ S_2 & \cdots & S_2 & Q_2 \end{bmatrix}_{(p+1) \times (q+1)}.$$

It is directly to verify that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p+9} \times C_{4q+5}$ when $p \geq 0$ and $q \geq 1$.

1(c) Suppose $m = 4q + 11 \geq 11$.

$$\text{Let } T_3 = d \begin{bmatrix} 0 & \frac{3}{2} & 1 & \frac{5}{2} \\ 1 & \frac{5}{2} & 2 & \frac{7}{2} \\ 2 & \frac{7}{2} & 3 & 0 \\ 3 & \frac{1}{2} & 4 & \frac{3}{2} \end{bmatrix}, \quad P_3 = d \begin{bmatrix} 0 & \frac{3}{2} & 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} \\ 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} \\ 2 & \frac{7}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} & 3 & \frac{5}{2} \\ 3 & 0 & 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} & 3 & \frac{5}{2} & 4 & \frac{7}{2} \end{bmatrix},$$

$$S_3 = d \begin{bmatrix} 0 & \frac{3}{2} & 1 & \frac{5}{2} \\ 1 & \frac{5}{2} & 2 & \frac{7}{2} \\ 2 & \frac{7}{2} & 3 & 0 \\ 3 & 0 & 4 & 1 \\ 4 & 1 & \frac{1}{2} & 2 \\ \frac{1}{2} & 2 & \frac{3}{2} & 3 \\ \frac{3}{2} & 3 & \frac{5}{2} & 4 \\ \frac{5}{2} & 4 & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}, \quad Q_3 = d \begin{bmatrix} 0 & \frac{3}{2} & 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} \\ 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} \\ 2 & \frac{7}{2} & 3 & 0 & 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} & 3 & \frac{5}{2} \\ 3 & 0 & 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} & 3 & \frac{5}{2} & 4 & \frac{7}{2} \\ 4 & 1 & \frac{1}{2} & 2 & \frac{3}{2} & 3 & \frac{5}{2} & 4 & \frac{7}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 2 & \frac{3}{2} & 3 & \frac{5}{2} & 4 & \frac{7}{2} & \frac{1}{2} & 0 & \frac{3}{2} & 1 \\ \frac{3}{2} & 3 & \frac{5}{2} & 4 & \frac{7}{2} & \frac{1}{2} & 0 & \frac{3}{2} & 1 & \frac{5}{2} & 2 \\ \frac{5}{2} & 4 & \frac{7}{2} & \frac{1}{2} & 0 & \frac{3}{2} & 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 \\ \frac{7}{2} & \frac{1}{2} & 0 & \frac{3}{2} & 1 & \frac{5}{2} & 2 & \frac{7}{2} & 3 & 0 & 4 \end{bmatrix}.$$

$$\text{Let } M_f(C_{4p+9} \times C_{4q+11}) = \begin{bmatrix} T_3 & \cdots & T_3 & P_3 \\ \vdots & \vdots & \vdots & \vdots \\ T_3 & \cdots & T_3 & P_3 \\ S_3 & \cdots & S_3 & Q_3 \end{bmatrix}_{(p+1) \times (q+1)}.$$

It is directly to verify that f is a $4d-L(1, d)$ -labeling of $C_{4p+9} \times C_{4q+11}$ when $p, q \geq 0$.

Secondly, suppose $n = 4p + 11 \geq 11$.

2(a) Suppose $m = 4q + 18 \geq 18$. By using the matrices defined in Case 1(c),

$$\text{let } M_f(C_{4p+11} \times C_{4q+18}) = \begin{bmatrix} T_3^t & \cdots & T_3^t & S_3^t & S_3^t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ T_3^t & \cdots & T_3^t & S_3^t & S_3^t \\ P_3^t & \cdots & P_3^t & Q_3^t & Q_3^t \end{bmatrix}_{(p+1) \times (q+2)},$$

where X^t is the transpose of the matrix X . It is directly to verify that f is a $4d-L(1, d)$ -labeling of $C_{4p+11} \times C_{4q+18}$ when $p, q \geq 0$.

2(b) Suppose $m = 4q + 9 \geq 9$. Since $C_n \times C_m \cong C_m \times C_n$, this case is same as Case 1(c).

2(c) Suppose $m = 4q + 27 \geq 27$. Let

$$M_f(C_{4p+11} \times C_{4q+27}) = \left[\begin{array}{ccc|ccc} T_2 & \cdots & T_2 & T_2 & P_2 & T_2 & P_2 & T_2 & P_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_2 & \cdots & T_2 & T_2 & P_2 & T_2 & P_2 & T_2 & P_2 \\ \hline S_2 & \cdots & S_2 & S_2 & Q_2 & S_2 & Q_2 & S_2 & Q_2 \\ S_2 & \cdots & S_2 & S_2 & Q_2 & S_2 & Q_2 & S_2 & Q_2 \\ S_2 & \cdots & S_2 & S_2 & Q_2 & S_2 & Q_2 & S_2 & Q_2 \end{array} \right]_{(p+3) \times (q+6)},$$

where S_2, T_2, P_2, Q_2 are defined in Case 1(b) and $p \geq 4$ and $q \geq 0$.

It is directly to verify that f is a $4d-L(1, d)$ -labeling of $C_{4p+11} \times C_{4q+27}$ when $p \geq 4$ and $q \geq 0$.

For the remaining case $C_{23} \times C_{23}$, let $M_f(C_{23} \times C_{23})$ be

$$d$$

Finally, suppose $n = 4p + 18 \geq 18$.

3(a) Suppose $m = 4q + 14 \geq 18$. Let

$$M_f(C_{4p+18} \times C_{4q+18}) = \begin{bmatrix} T_1 & \cdots & T_1 & P_1 \\ \vdots & \vdots & \vdots & \vdots \\ T_1 & \cdots & T_1 & P_1 \\ S_1 & \cdots & S_1 & Q_1 \\ S_1 & \cdots & S_1 & Q_1 \end{bmatrix}_{(p+2) \times (q+1)},$$

where P_1, Q_1, S_1 and T_1 are defined in Case 1(a).

It is not difficult to verify that f is a $4d$ - $L(1, d)$ -labeling of $C_{4p+18} \times C_{4q+18}$ when $p, q \geq 0$.

3(b) Suppose $m = 4q + 9 \geq 9$. This case is same as Case 1(a).

3(c) Suppose $m = 4q + 11 \geq 11$. This case is same as Case 2(a).

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