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Generalized (α, β)-derivations in 3-prime near-rings

Abdelkarim Boua *
Polydisciplinary Faculty
Sidi Mohammed Ben Abdellah University
LSI
Taza
Morocco

Ahmed Y. Abdelwanis †
Department of Mathematics
Faculty of Science
Cairo University
Giza
Egypt

Abstract

Let $\mathcal{N}$ be a 3-prime near-ring and $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ are endomorphisms. In this paper, we generalize a few results concerning generalized derivations and two sided α -generalized derivations of 3-prime near rings to generalized (α, β) -derivations. Cases demonstrating the need of the 3-primeness speculation is given. When $\beta = id_{\mathcal{N}}$ (resp. $\alpha = \beta = id_{\mathcal{N}}$), one can easily obtain the main results of [1] (resp. [4]).

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1. Introduction

In the present paper, $\mathcal{N}$ is a zero symmetric right near-ring and $Z(\mathcal{N})$ is the multiplication center of $\mathcal{N}$, that is, $Z(\mathcal{N}) = \{x \in N \mid xy = yx \text{ for all } y \in \mathcal{N}\}$. It is clear that $Z(\mathcal{N}) \neq \varnothing$ because

* *E-mail:* **abdelkarimbouaboua@yahoo.fr (Corresponding Author)**
† *E-mail:* **ayunis@sci.cu.edu.eg**

$0 \in Z(\mathcal{N})$. We recall $\mathcal{N}$ will be 3-prime near-ring, if for $x, y \in \mathcal{N}$, we have the following implication $x\mathcal{N}y = \{0\} \Rightarrow x = 0$ or $y = 0$.

A nonempty set U of $\mathcal{N}$ is called a semigroup right ideal (resp. a semigroup left ideal) if $U\mathcal{N} \subseteq U$ (resp. $\mathcal{N}U \subseteq U$). U is said to be a semigroup ideal if it is both a semigroup right ideal and a semigroup left ideal. An additive subgroup J of $\mathcal{N}$ is called a right Jordan ideal of $\mathcal{N}$ (resp. left Jordan ideal of $\mathcal{N}$), if $i \circ n \in J$ for all $i \in J$, $n \in \mathcal{N}$ (resp. $n \circ i \in J$ for all $i \in J$, $n \in \mathcal{N}$). J is called a Jordan ideal of $\mathcal{N}$ if it is both right and left Jordan ideal of $\mathcal{N}$. Recalling that $\mathcal{N}$ is 2-torsion free if for $x \in \mathcal{N}$, we have $2x = 0 \Rightarrow x = 0$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$. From [9], we also have the equivalence with the following formula: $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. As in [9], an additive mapping $F : \mathcal{N} \to \mathcal{N}$ is a right or left generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ or $F(xy) = d(x)y + xF(y)$ holds for all $x, y \in \mathcal{N}$ respectively.

Let $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ be endomorphisms, an additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called (α, β)-derivation, if $d(xy) = \alpha(x)d(y) + d(x)\beta(y)$ for all $x, y \in \mathcal{N}$, and or equivalently from [2] that $d(xy) = d(x)\beta(y) + \alpha(x)d(y)$, for all $x, y \in \mathcal{N}$.

Let $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ be endomorphisms, an additive mapping $F : \mathcal{N} \to \mathcal{N}$ is called a right generalized (α, β)-derivation (resp. left generalized (α, β)-derivation) if there exists a (α, β)-derivation d such that $F(xy) = F(x)\beta(y) + \alpha(x)d(y)$ (resp. $F(xy) = d(x)\beta(y) + \alpha(x)F(y)$) holds for all $x, y \in \mathcal{N}$. An additive mapping $F : \mathcal{N} \to \mathcal{N}$ is called a generalized (α, β)-derivation if F is both right generalized (α, β)-derivation and left generalized (α, β)-derivation.

The structure of rings is directly linked to additive mappings. In [13] Posner proved the initial pioneer result. He discovered a relationship between the commutativity of the ring and derivation on the same ring. The structure of prime near-rings, as well as other near-rings admitting derivations, generalized derivations, automorphisms, orthogonal generalized higher derivations, and many other maps satisfying certain algebraic conditions on appropriate subsets of the rings, has been studied in the literature. For such results, we refer the reader to Bell [3], Ö. Gölbasi [9], S. Huang et al. [11], E. M. Khazil [12], where further references can be found.

Example 1 : Let S be a zero-symmetric near-ring. Let us define $\mathcal{N}$ and $d, F, \alpha, \beta : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in S \right\}, d\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

$$F\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \alpha\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \text{ and } \beta\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

Clearly, $\mathcal{N}$ is a zero symmetric near-ring, d is a (α, β)-derivation of $\mathcal{N}$, which not is a derivation and F is a generalized (α, β)-derivation associated with d, but F is not a (α, β)-derivation of $\mathcal{N}$.

We will write, for all $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie and Jordan products, respectively. Usually we denote for all $x, y \in \mathcal{N}$, $[x, y]_{\alpha,\beta} = \alpha(x)y - y\beta(x)$, and $(x \circ y)_{\alpha,\beta} = \alpha(x)y + y\beta(x)$. In particular $[x, y]_{id_{\mathcal{N}}, id_{\mathcal{N}}} = [x, y]$ and $(x \circ y)_{id_{\mathcal{N}}, id_{\mathcal{N}}} = x \circ y$, for all $x, y \in \mathcal{N}$, where $id_{\mathcal{N}}$ be the identity map on $\mathcal{N}$.

In this paper, we generalized Theorems 3.1 and 3.5 of [1] and some Theorems which are existing in [4].

2. Preliminaries

We'll start with the following lemmas, which will be valuable in the next sections.

Lemma 1 : [3, Lemma 1.3 (i)]. *Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup left ideal (or semigroup right ideal) of $\mathcal{N}$. If $x \in \mathcal{N}$ and $xI = \{0\}$, (or $Ix = \{0\}$,) then $x = 0$.*

Lemma 2 : [3, Lemma 1.5]. *Let $\mathcal{N}$ be a 3-prime near-ring, and U is a non-zero semigroup right ideal or a semigroup left ideal of $\mathcal{N}$. If $U \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 3 : [3, Lemmas 1.2 (i), 1.2 (iii) & 1.3 (iii)]. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) If $z \in Z(\mathcal{N}) \setminus \{0\}$, then z is not a zero divisor.

(ii) If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $uz \in Z(\mathcal{N})$, then $u \in Z(\mathcal{N})$.

(iii) If z centralizes a non zero semigroup left ideal, then $z \in Z(\mathcal{N})$.

Lemma 4 : [10, Theorem 2]. *Let $\mathcal{N}$ be a 3-prime near-ring, U is a nonzero semigroup left ideal of $\mathcal{N}$. If d is a non-trivial (α, β)-derivation on $\mathcal{N}$ such that $d(U) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 5 : *Let $\mathcal{N}$ be a 3-prime near-ring and α, β maps of $\mathcal{N}$ such as α is additive. If $\mathcal{N}$ admits an additive mapping F, then the following assertions are equivalent:*

(i) $F(xy)=F(x)\beta(y)+\alpha(x)d(y)$ *for all* $x,y\in\mathcal{N}$,

(ii) $F(xy)=\alpha(x)d(y)+F(x)\beta(y)$ *for all* $x,y\in\mathcal{N}$.

Proof : $(i)\Rightarrow(ii)$. Assume that $F(xy)=F(x)\beta(y)+\alpha(x)d(y)$, for all $x,y\in\mathcal{N}$, so

$$\begin{aligned}F((x+x)y)&=F(x+x)\beta(y)+\alpha(x+x)d(y)\\&=F(x)\beta(y)+F(x)\beta(y)+\alpha(x)d(y)+\alpha(x)d(y)\text{ for all }x,y\in\mathcal{N},\end{aligned}$$

and

$$\begin{aligned}F((x+x)y)&=F(xy)+F(xy)\\&=F(x)\beta(y)+\alpha(x)d(y)+F(x)\beta(y)+\alpha(x)d(y)\text{ for all }x,y\in\mathcal{N}.\end{aligned}$$

Comparing the two equations, then we get

$$F(x)\beta(y)+\alpha(x)d(y)=\alpha(x)d(y)+F(x)\beta(y)\text{ for all }x,y\in\mathcal{N}.$$

$(ii)\Rightarrow(i)$. Similarly, we can prove the other implication. □

Lemma 6 : [11, Lemma 2.2]. *Let F be a generalized (α,β)-derivation of near-ring $\mathcal{N}$ associated with d. Then*

$$z(F(x)\beta(y)+\alpha(x)d(y))=zF(x)\beta(y)+z\alpha(x)d(y)\text{ for all }x,y,z\in\mathcal{N}.$$

Lemma 7 : *Let $\mathcal{N}$ be a 3-prime near-ring and I is a nonzero semigroup ideal of $\mathcal{N}$. If F is a generalized (α,β)-derivation of $\mathcal{N}$ associated with a nonzero (α,β)-derivation d such that $F(I)\subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Proof : We have $zF(xy)=F(xy)z$ for all $x\in I$, $y,z\in\mathcal{N}$, using Lemma 6 we obtain

$$zF(x)\beta(y)+z\alpha(x)d(y)=F(x)\beta(y)z+\alpha(x)d(y)z\text{ for all }x\in I,y,z\in\mathcal{N}.$$

This expression can be simplified under the following form

$$z\alpha(x)d(y)=\alpha(x)d(y)z\text{ for all }x\in I,y,z\in\mathcal{N}.$$

Replacing x by $nmxt$, where $m,n,t\in\mathcal{N}$ in last expression, we arrive at

$$[z,\alpha(n)]\mathcal{N}\alpha(x)\mathcal{N}d(y)=\{0\}\text{ for all }x\in I,n,y,z\in\mathcal{N}.$$

Since $\mathcal{N}$ is 3-prime, $\alpha \neq 0$ and $d \neq 0$, we get $\alpha(\mathcal{N}) \subseteq Z(\mathcal{N})$. Using the fact that α is an automorphism of $\mathcal{N}$, we conclude that $\mathcal{N}$ is a commutative ring. □

The following lemmas are very interesting in the next sections:

Lemma 8 : *Let $\mathcal{N}$ be a 3-prime near-ring, I is a nonzero semigroup left ideal and α, β automorphisms on $\mathcal{N}$. If $a \in \mathcal{N}$ and $[a, x]_{\alpha,\beta} = 0$ for all $x \in I$, then $a \in Z(\mathcal{N})$.*

Proof : Let $a \in \mathcal{N}$ such that $[a, x]_{\alpha,\beta} = 0$ for all $x \in I$, then $\alpha(a)x = x\beta(a)$ for all $x \in I$. Replace *x* by *tx*, where $t \in \mathcal{N}$, we get

$$\begin{aligned}\alpha(a)tx &= tx\beta(a)\\ &= t\alpha(a)x \text{ for all } x \in I, t \in \mathcal{N}.\end{aligned}$$

Then $[\alpha(a), t]x = 0$ for all $x \in I$, $t \in \mathcal{N}$. By Lemma 1, we obtain $\alpha(a) \in Z(\mathcal{N})$, but α is an automorphism, so $a \in Z(\mathcal{N})$. □

Lemma 9 : *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ are automorphisms and J a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a (α, β)-derivation d such that $d(J) = \{0\}$, then $d = 0$ or J is commutative.*

Proof : Suppose that *d* is a nonzero (*a, b*) derivation such that $d(J) = \{0\}$ and *J* a nonzero Jordan right ideal of $\mathcal{N}$. Then $d(j \circ n) = 0$ for all $j \in J$, $n \in \mathcal{N}$ which implies that $d(n)\alpha(j) = -\beta(j)d(n)$ for all $j \in J$, $n \in \mathcal{N}$. Replacing *n* by kn, where $k \in J$ in the last expression and using it again, we obtain $(\beta(j)\beta(k) - \beta(k)\beta(j))d(n) = 0$ for all $j, k \in J$, $n \in \mathcal{N}$. Taking *mn* in place of *n*, where $m \in \mathcal{N}$ in the last expression and using it, we arrive at $(\beta(j)\beta(k) - \beta(k)\beta(j))\beta(m)d(n) = 0$ for all $j, k \in J$, $m, n \in \mathcal{N}$. Since β is an automorphism of $\mathcal{N}$, we conclude that $(\beta(j)\beta(k) - \beta(k)\beta(j))\mathcal{N}d(n) = \{0\}$ for all $j, k \in J$, $n \in \mathcal{N}$. Using 3-primeness of $\mathcal{N}$, we obtain $d = 0$ or $\beta(jk) = \beta(kj)$ for all $j, k \in J$. Since β is an automorphism of $\mathcal{N}$, we get $jk = kj$ for all $j, k \in J$. Using a method similar to that employed in the proofs of [8, Lemma 2.4, and Lemma 2.7], we obtain the needed result. □

3. Commutativity conditions and generalized (α, β)-derivations

In this section, $\mathcal{N}$ is assumed to be a zero symmetric near-ring and $\alpha, \beta : \mathcal{N} \to \mathcal{N}$ are automorphisms.

Our next theorem is a generalization of Theorem 3.1 in [1] and Theorem 2.9 in [4].

Theorem 1 : *Let $\mathcal{N}$ be a 3-prime near-ring, I is a nonzero semigroup ideal of $\mathcal{N}$ and F is a generalized (α,β)-derivation of $\mathcal{N}$ associated with a nonzero (α,β)-derivation d. Then the following expressions are equivalent:*

(i) $[F(x),y]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $x,y \in I$;

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof : $(ii) \Rightarrow (i)$ is obvious.

$(i) \Rightarrow (ii)$ Assume that

$$[F(x),y]_{\alpha,\beta} \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{1}$$

Replacing y by $y\beta(F(x))$ in (1) and using the fact that $[F(x),y\beta(F(x))]_{\alpha,\beta} = [F(x),y]_{\alpha,\beta}\beta(F(x)),$ we get

$$[F(x),y]_{\alpha,\beta}\beta(F(x)) \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{2}$$

By Lemma 3 (ii), we conclude that for each $x \in I,$ we have

$$[F(x),y]_{\alpha,\beta} = 0 \text{ or } \beta(F(x)) \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{3}$$

But β is an automorphism, so (3) implies that

$$[F(x),y]_{\alpha,\beta} = 0 \text{ or } F(x) \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{4}$$

By Lemma 8, we get $F(x) \in Z(\mathcal{N})$ for all $x \in I,$ i.e $F(I) \subseteq Z(\mathcal{N}).$ Hence $\mathcal{N}$ is a commutative ring by Lemma 7. ☐

Corollary 1 : *Let $\mathcal{N}$ be a 3-prime near-ring, I is a nonzero semigroup ideal of $\mathcal{N}$ and d is a nonzero (α,β)-derivation of $\mathcal{N}$. Then the following expressions are equivalent:*

(i) $[d(u),v]_{\alpha,\beta} \in Z(\mathcal{N})$ *for all* $u,v \in I$;

(ii) $\mathcal{N}$ *is a commutative ring.*

It is worthy to note here that the results of Theorem 1 is a generalizations of [1, Theorem 3.1] if we put $\beta = id_{\mathcal{N}}$ and [4, Theorem 2.9] if we put $F = d$ and $\alpha)\ \beta = idN$

If $\mathcal{N}$ is 2-torsion free, Theorem 1 stays legitimate if we replace $[x,y]_{\alpha,\beta}$ by $(x \circ y)_{\alpha,\beta}$. In fact, we obtain the following result:

The next theorem is a generalization of Theorem 3.5 in [1] and Theorem 2.10 in [4].

Theorem 2 : *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, I is a nonzero semigroup ideal of $\mathcal{N}$ and F is a generalized (α,α)-derivation associated with a nonzero (α,α)-derivation d on $\mathcal{N}$, then the following statements are equivalent:*

(i) $(F(x)\circ y)_{\alpha,\alpha} \in Z(\mathcal{N})$ *for all* $x,y \in I$;

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof : $(ii) \Rightarrow (i)$ is obvious.

$(i) \Rightarrow (ii)$ Assume that

$$(F(x)\circ y)_{\alpha,\alpha} \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{5}$$

As above replacing y by $y\alpha(F(x))$ in (5), we get

$$(F(x)\circ y)_{\alpha,\alpha}\alpha(F(x)) \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{6}$$

By Lemma 3(ii), we conclude that

$$(F(x)\circ y)_{\alpha,\alpha} = 0 \text{ or } \alpha(F(x)) \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{7}$$

Again (7) implies that

$$(F(x)\circ y)_{\alpha,\alpha} = 0 \text{ or } F(x) \in Z(\mathcal{N}) \quad \text{for all } x,y \in I. \tag{8}$$

Assume there exists $x_0 \in I$ such that $F(x_0) \in Z(\mathcal{N})$. Since α is an automorphism of $\mathcal{N}$, $\alpha(F(x_0)) \in Z(\mathcal{N})$. Then (5) implies $(y+y)\alpha(F(x_0)) \in Z(\mathcal{N})$ for all $y \in I$. By Lemma 3 (ii), we obtain $\alpha(F(x_0)) = 0$ or $y+y \in Z(\mathcal{N})$ for all $y \in I$ which implies that $F(x_0) = 0$ or $(y+y)y = y^2+y^2 \in Z(\mathcal{N})$ for all $y \in I$. Using again Lemma 3 (ii) with 2-torsion freeness of $\mathcal{N}$, we get $F(x_0)=0$ or $y \in Z(\mathcal{N})$ for all $y \in I$ which means that $F(x_0)=0$ or $I \subseteq Z(\mathcal{N})$. By Lemma 2, we conclude that $F(x_0)=0$ or $\mathcal{N}$ is a commutative ring. In this case (8) becomes

$$(F(x)\circ y)_{\alpha,\alpha} = 0 \quad \text{for all } x,y \in I \text{ or } \mathcal{N} \text{ is a commutative ring.}$$

If $(F(x)\circ y)_{\alpha,\alpha} = 0$ for all $x,y \in I$. We get $\alpha(F(x))y = -y\alpha(F(x))$ for all $x,y \in I$. Putting yt in place of y, we obtain

$$\begin{aligned}\alpha(F(-x))yt &= yt\alpha(F(x))\\ &= y(t\alpha(F(x))\\ &= y\alpha(F(-x))t \quad \text{for all } x,y,t \in I,\end{aligned}$$

which implies that $\alpha(F(-x))y - y\alpha(F(-x))I = \{0\}$ for all $x,y \in I$. So, $\alpha(F(-x))y = y\alpha(F(-x))$ for all $x,y \in I$. Replacing y by ny, where $n \in \mathcal{N}$ in the last expression and using it again, we arrive at $\alpha(F(-x)) \in Z(\mathcal{N})$ for all $x \in I$. Since α is an automorphism of $\mathcal{N}$, we obtain $F(-x) \in Z(\mathcal{N})$ for all $x \in I$. i.e. $F(-I) \subseteq Z(\mathcal{N})$ and $\mathcal{N}$ is a commutative ring by Lemma 7. □

Corollary 2 : *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, I is a nonzero semigroup ideal of $\mathcal{N}$ and d is a nonzero (α,α)-derivation d on $\mathcal{N}$. Then, the following expressions are equivalent:*

(i) $(d(u) \circ v)_{\alpha,\alpha} \in Z(\mathcal{N})$ *for all* $u, v \in I$;

(ii) $\mathcal{N}$ *is a commutative ring.*

Note that we can obtained Theorem 3.5 in [1] and Theorem 2.10 in [4] from Theorem 2 by taking $F = d$ and $\alpha = id_{\mathcal{N}}$.

The following example shows that the 3-primeness hypothesis in Theorems 1 and 2 can't be discarded.

Example 2 : Let S be a 2-torsion free zero-symmetric near-ring, which is not abelian. Let us defined $\mathcal{N}, I$ and $F, d, \alpha, \beta : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}, \quad I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid y \in S \right\},$$

$$F\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \alpha\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ y & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$F = d, \beta = id_{\mathcal{N}}$.

It is clear that $\mathcal{N}$ is a 2-torsion free non 3-prime near-ring and I is a nonzero semigroup ideal of $\mathcal{N}$. Moreover, F is a generalized (α,β)-derivation associated with a nonzero (α,β)-derivation d satisfying the conditions:

(i) $[F(A), B]_{\alpha,\beta} \in Z(\mathcal{N})$,

(ii) $(F(A) \circ B)_{\alpha,\beta} \in Z(\mathcal{N})$

for all $A, B \in I$, but $\mathcal{N}$ is not a commutative ring.

4. (α,β)-derivations acts as homomorphisms and anti-homomorphisms

Spurred by the results in [7] our primary goal in this section is to establish similar results in the setting of Jordan right ideals or Jordan left ideals on right 3-prime near-rings admitting a nonzero (α,β)-derivation d, where $\alpha,\beta:\mathcal{N}\to\mathcal{N}$ are automorphisms, which satisfy the conditions: $d(ny)=d(y)d(n)$ and $d(yn)=d(n)d(y)$ for all $y\in J, n\in\mathcal{N}$.

The next theorem is a generalization of Theorem 2 in [7].

Theorem 3 : *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, $\alpha,\beta:\mathcal{N}\to\mathcal{N}$ automorphisms and J a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero (α,β)-derivation d that satisfies any one of the following conditions:*

(i) $d(ny)=d(y)d(n)$ *for all* $y\in J, n\in\mathcal{N}$,

(ii) $d(yn)=d(n)d(y)$ *for all* $y\in J, n\in\mathcal{N}$,

Then J is commutative.

Proof : (i) Suppose that

$$d(ny)=d(y)d(n) \quad \text{for all } y\in J, n\in\mathcal{N}. \tag{9}$$

Substituting n by ny in (9) and using the definition of d, we get

$$d(ny)\alpha(y)+\beta(ny)d(y)=d(y)d(ny) \quad \text{for all } y\in J, n\in\mathcal{N}. \tag{10}$$

From definition of d and Lemma 4 it follows that

$$d(y)d(n)\alpha(y)+\beta(ny)d(y)=d(y)d(n)\alpha(y)+d(y)\beta(n)d(y). \tag{11}$$

So,

$$\beta(ny)d(y)=d(y)\beta(n)d(y) \quad \text{for all } y\in J, n\in\mathcal{N}. \tag{12}$$

Putting mn in place of n in (12) and use it again with the fact that β is an automorphism of $\mathcal{N}$, we get

$$[d(y),\beta(m)]\mathcal{N}d(y)=\{0\} \quad \text{for all } y\in J, m\in\mathcal{N}. \tag{13}$$

3-primeness of $\mathcal{N}$ implies that $d(J)\subseteq Z(\mathcal{N})$, and hence (12) becomes $d(y)\mathcal{N}(\beta(y)-d(y))=\{0\}$ for all $y\in J$ and 3-primeness of $\mathcal{N}$ again gives

$$d(y)=0 \text{ or } d(y)=\beta(y) \quad \text{for all } y\in J. \tag{14}$$

Suppose there $y_0\in J$ such that $d(y_0)=\beta(y_0)$, using our hypothesis we have

$$\begin{aligned}\beta(y_0)d(n) &= d(ny_0)\\ &= d(n)d(y_0)+\beta(n)d(y_0)\\ &= d(n)\alpha(y_0)+\beta(n)\beta(y_0) \text{ for all } n\in\mathcal{N},\end{aligned}$$

which can be rewritten in the following form

$$d(j)(\beta(y_0)-\alpha(y_0)) = \beta(j)\beta(y_0) \quad \text{for all} \quad j\in J. \tag{15}$$

Putting $2j^2$ in place of j in the last expression and using 2-torsion freeness of $\mathcal{N}$, we obtain $d(j^2)(\beta(y_0)-\alpha(y_0)) = \beta(j)\beta(y_0)$ for all $j\in J$. From the last expression, we can easily arrive at $\beta(y_0)\mathcal{N}(d(j)\beta(j)-\beta(j^2)) = \{0\}$ for $j\in J$. Since $\mathcal{N}$ is 3-prime, we get $\beta(y_0)=0$ or $d(j)\beta(j)=\beta(j^2)$ for all $j\in J$. Since β is an automorphism of $\mathcal{N}$, we obtain either $y_0=0$ or $d(j)\beta(j)=\beta(j^2)$ for all $j\in J$. In this case (14) becomes $d(J)=\{0\}$ or $d(j)\beta(j)=\beta(j^2)$ for all $j\in J$.

Now suppose $d(j)\beta(j)=\beta(j^2)$ for all $j\in J$. By our hypothesis, we have

$$\begin{aligned}d(j)d(j^2) &= d(j^3)\\ &= d(j^2)\alpha(j)+\beta(j^2)d(j)\\ &= d(j^2)\alpha(j)+\beta(j^2\beta(j) \text{ for all } j\in J.\end{aligned}$$

This can be further written as $d(j)\mathcal{N}d(j)\mathcal{N}(d(j)-\alpha(j)-\beta(j))=\{0\}$ for all $j\in J$. By 3-primeness of $\mathcal{N}$, we obtain $d(j)=0$ or $d(j)=\alpha(j)+\beta(j)$ for all $j\in J$.

Suppose there exists $j_0\in J$ such that $d(j_0)=\alpha(j_0)+\beta(j_0)$ and right-multiplying it by $\beta(j_0)$, we can easily arrive at $\alpha(j_0)\beta(j_0)=0$. By hypothesis, we have $d(n)\alpha(j_0)+\beta(n)d(j_0)=d(j_0)d(n)$ for all $n\in\mathcal{N}$ and right-multiplying it again by $\beta(j_0)$, we obtain $\beta(n)d(j_0)\beta(j_0)=d(j_0)d(n)\beta(j_0)$ for all $n\in\mathcal{N}$ which implies that $\beta(n)\beta(j_0^2)=d(n)\beta(j_0^2)$ for all $n\in\mathcal{N}$. Replacing n by mn in the last equation and developing it, we find that $d(m)\mathcal{N}\beta(j_0^2)=\{0\}$ for all $m\in\mathcal{N}$. Since $d\neq 0$, by 3-primeness of $\mathcal{N}$, we get $\beta(j_0^2)=0$ which implies that $d(j_0)\mathcal{N}\beta(j_0)=\{0\}$, 3-primeness of $\mathcal{N}$ with β is an automorphism force that $d(j_0)=0$. In all cases, we obtain $d(J)=\{0\}$. By Lemma 9, we conclude that J is commutative.

(ii) We consider

$$d(yn)=d(n)d(y) \quad \text{for all} \quad y\in J, n\in\mathcal{N}. \tag{16}$$

Replacing n with nm in (16) and expanding this, we get

$$d(yn)\alpha(m)+\beta(yn)d(m)=d(nm)d(y) \quad \text{for all } y\in J, m,n\in\mathcal{N}.$$

This, by our assumption, implies that

$$d(n)d(y)\alpha(m)+\beta(yn)d(m)=d(nm)d(y) \quad \text{for all } y\in J, m,n\in\mathcal{N}. \tag{17}$$

Equivalently,

$$\begin{aligned}\beta(yn)d(m)=&-d(n)d(y)\alpha(m)+d(n)\alpha(m)d(y)\\ &+\beta(n)d(m)d(y) \quad \text{forall } y\in J, m,n\in\mathcal{N}.\end{aligned}$$

Taking $\alpha^{-1}(d(y))$ instead of m, we can reformulate the above expression in the following form

$$\beta(yn)d(\alpha^{-1}(d(y)))=\beta(n)d(\alpha^{-1}(d(y)))d(y) \quad \text{for all } y\in J, n\in\mathcal{N}. \tag{18}$$

Now putting mn in place of n where $m\in\mathcal{N}$ in Eq. (18) and use it to obtain

$$[\beta(y),\beta(m)]\beta(n)d(\alpha^{-1}(d(y)))=0 \quad \text{for all } y\in J, n\in\mathcal{N}$$

i.e.

$$[\beta(y),\beta(m)]\mathcal{N}d(\alpha^{-1}(d(y)))=\{0\} \quad \text{for all } y\in J, m\in\mathcal{N}. \tag{19}$$

By 3-primeness of $\mathcal{N}$ and since β is an automorphism of $\mathcal{N}$, we can write the last equation in the following way

$$\beta(y)\in Z(\mathcal{N}) \text{ or } d(\alpha^{-1}(d(y)))=0 \quad \text{for all } y\in J. \tag{20}$$

If there is an element $y_0\in J$ such that $\beta(y_0)\in Z(\mathcal{N})$, then (16) leads to $d(y_0n)=d(n)d(y_0)$ for all $n\in\mathcal{N}$. Which means that

$$d(y_0)\alpha(n)=(d(y_0)-\beta(y_0))d(n) \quad \text{for all } n\in\mathcal{N}. \tag{21}$$

Putting mn instead of n in the pervious equation and use it we get

$$(d(y_0)-\beta(y_0))\mathcal{N}d(n)=\{0\} \quad \text{for all } n\in\mathcal{N}. \tag{22}$$

By 3-primeness of $\mathcal{N}$ and $d\neq 0$, we have $d(y_0)=\beta(y_0)$. Returning to (21) we get $d(y_0)\alpha(n)=0$ for all $n\in\mathcal{N}$ so that $d(y_0)\mathcal{N}\alpha(n)=\{0\}$. Since α is an automorphism of $\mathcal{N}$ and $\mathcal{N}$ is 3-prime we get $d(y_0)=0$. So, (20) leads to $d(\alpha^{-1}(d(y)))=0$ for all $y\in J$. Replacing n with $\alpha^{-1}(d(x))$ in (16) where $x\in J$ we get $d(y)d(x)=0$ for all $x,y\in J$, i.e.

$$d(xy)=d(x)\alpha(y)+\beta(x)d(y)=0 \quad \text{for all } x,y\in J. \tag{23}$$

Substituting x with $2x^2$ in (23), we get $2\beta(x^2)d(y)=0$ for all $x,y\in J$. Then $\beta(x^2)d(y)=0$ for all $x,y\in J$. Now, taking $x\circ nx$ in place of x where $n \in \mathcal{N}$ in (23), we get $\beta(x \circ nx)d(y) = 0$ for all $x, y \in J, n \in \mathcal{N}$ which becomes $\beta(xnx)d(y) = 0$ for all $x, y \in J$, $n \in \mathcal{N}$ and 3-primeness of $\mathcal{N}$ forces that $\beta(x)d(y) = 0$ for all $x, y \in J$. Since β is an automorphism and J is a nonzero right Jordan ideal, we arrive at $d(J) = \{0\}$, by another applications of Lemma 9, we conclude that J is commutative.

For a nonzero Jordan left ideal J of $\mathcal{N}$, utilizing the equivalent previous demonstration with minor changes, we can undoubtedly find the required outcome. □

As an immediate consequence of Theorem 3 we have the following:

Corollary 3 : [7, Theorem 2] *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and J a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If d is a nonzero derivation of $\mathcal{N}$ which any of the following expressions are satisfied:*

(i) $d(nu) = d(u)d(n)$ *for all* $u \in J, n \in \mathcal{N}$,

(ii) $d(un) = d(n)d(u)$ *for all* $u \in J, n \in \mathcal{N}$,

then J is commutative.

References

[1] A. Y. Abdelwanis, Applications of two sided α-generalized derivations to 3-prime near-rings, Comm. Alg., 45 (2017), 4631-4645.

[2] M. Ashraf, A. Ali & S. Ali, (σ, τ)-derivations on prime near-rings, Arch. Math., 40 (3) (2004), 281-286.

[3] H. E. Bell, On derivations in near-rings. Vol. II. In: Nearrings, Nearfields and K-loops (Hamburg, 1995) Math. Appl., Vol. 426. Dordrecht: Kluwer Acad. Publ., (1997), 191-197.

[4] H. E. Bell, A. Boua & L. Oukhtite, Semigroup ideals and commutativity in 3-prime near-rings, Comm. Alg., 43 (2015), 1757-1770.

[5] A. Boua, Lahcen Oukhtite & Abderrahmane Raji, Jordan ideals and derivations in prime near-rings Comment. Math. Univ. Carolin., 55 (2) (2014), 131-139.

[6] A. Boua, Commutativity of near-rings with certain constrains on Jordan ideals. Bol. Soc. Paran. Mat. 36 (4) (2018), 159-170.

[7] A. Boua, Commutativity of Jordan ideals in 3-prime near-rings with derivations, Commun. *Korean Math. Soc.* 33 (2018), No. 1, pp. 37-44, https://doi.org/10.4134/CKMS.c170015.

[8] En-guady, A., Boua, A. & Farhan, E, Some functional identities on near-rings involving Jordan ideals and left generalized derivations. *Iraqi journal of science*, 62 (6) (2021), 1961-1967.

[9] A. Boua & E. Farhan, Generalized derivations on near-rings with identities, *Journal of Discrete Mathematical Sciences and Cryptography*, (In press).

[10] Ö. Gölbasi, Notes on prime near-rings with generalized derivation, South. Asian Bull. Math., 30 (2006), 49-54.

[11] Ö. Gölbasi & N. Aydin, On near-ring ideals with (σ, τ)-derivation, Arch. Math., 43 (2007), 87-92.

[12] S. Huang & Ö. Gölbasi, Notes on Prime Near-Rings with Generalized (σ, τ)-Derivations, South. Asian Bull. Math., 36 (2012), 353-361.

[13] E. M. Khazil, Orthogonal generalized higher derivations in semi prime gamma near-rings, *Journal of Discrete Mathematical Sciences and Cryptography*, 23 (5) (2020), 1009-1020, DOI: 10.1080/09720529.2020.1723933

[14] E. C. Posner, Derivations in prime rings, Proc. Amer. Math. Soc., 8 (1957), 1093-1100. https://doi.org/10.1090/s0002-9939-1957-0095863-0

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