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Fuzzy ideals and fuzzy congruences of distributive demi-p-algebras

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Abstract

In this article, we introduce *-fuzzy ideals of a distributive demi-p-algebra and its properties. We show that every *-fuzzy ideals of distributive demi-p-algebra L are precisely the kernels of the fuzzy congruences Θ on L if and only if its quotient is a Boolean algebra. Also we prove that the set of *-fuzzy ideals forms a complete Heyting algebra.

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1. Introduction

An algebra $(L, \wedge, \vee, {}^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a pseudocomplemented algebra (p-algebra) if $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, and for all $a \in L, x \leq a^*$ if and only if $x \wedge a = 0$.

Demi-pseudocomplemented algebra was introduced by, Sankappanavar [11] as a generalization pseudocomplemented algebra. Blyth, Fang, and Wang [5] studied the ideals and congruences of distributive demi-p-algebras.

The notion of fuzzy sets was introduced by Zadeh [14]. Next, Rosenfeld [10] studied fuzzy groups. These ideas have been applied to other algebraic structures like semigroups, rings, modules, topologies and

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so on. More recently Alaba and Alemayehu [3] introduced the concept of fuzzy ideals in demi-pseudocomplemented MS-algebras, Abbass and Atshan [1] studied η-smarandache pseudo fuzzy ideal of a pseudo BH-algebra, Baupradist, Chemat, Palanivel and Chinram [4] presented essential ideals and essential fuzzy ideals in semigroups and Rasuli [9] studied fuzzy congruence on product lattices under T-norms. In this paper, we introduce the notions of fuzzy ideals and fuzzy congruence of distributive demi-p-algebras. We prove that the set of $*$-fuzzy ideals forms Heyting algebras which is isomorphic to the interval $[\mathcal{G}_{\mathcal{F}}, \chi_{\iota}]$ of the congruence lattice of L, where $\mathcal{G}_{\mathcal{F}}$ is the Glivenko fuzzy congruence.

2. Preliminaries

In this topic, we recall basic definitions, theorems frequently used in this paper.

Definition 2.1: [11] An algebra $(L, \vee, \wedge, ^*, 0, 1)$ is a demi-p-lattice (and * is a demi-pseudocomplement) if the following conditions hold:

(1) $(L, \vee, \wedge, 0, 1)$ is bounded a distributive lattice,

(2) $(a \vee b)^* = a^* \wedge b^*$ for $a, b \in L$,

(3) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$ for $a, b \in L$,

(4) $0^* = 1$ and $1^* = 0$,

(5) $a^{***} = a^*$ for $a \in L$,

(6) $a^* \wedge a^{**} = 0$ for $a \in L$.

For any set X, a function $\nu : X \to [0,1]$ is called a fuzzy subset of X, $\gamma_1 \wedge \gamma_2 = \min\{\gamma_1, \gamma_2\}$ and $\gamma_1 \vee \gamma_2 = \max\{\gamma_1, \gamma_2\}$ for all $\gamma_1, \gamma_1 \in [0,1]$.

Let Ω be a nonempty indexing set and ν_i is a family of fuzzy subset of a nonempty set X. Then $(\vee \nu_i)(x) = \vee_{i \in \Omega} (\nu_i(x))$.

Definition 2.2 : [14, 10] Let ν be any fuzzy subset of a set S and let $t \in [0,1]$. The set $\nu_t = \{x \in L : \nu(x) \geq t\}$ is called a level subset of ν.

Definition 2.3 : [12] Let L be a bounded distributive lattice. A mapping $\nu : L \to [0,1]$ is a fuzzy ideal L if $\nu(0) = 1$, $\nu(a \vee b) \geq \nu(a) \wedge \nu(b)$ and $\nu(a \wedge b) \geq \nu(a) \vee \nu(b)$ for all $a, b \in L$.

Swamy and Raju [12] also show that a similar definition 2.3, i.e., a fuzzy subset ν is a fuzzy ideal of L if and only if $\nu(0) = 1$ and $\nu(a \vee b) = \nu(a) \wedge \nu(b)$ for all $a, b \in L$.

Definition 2.4 : [2, 8] A function $\Theta : X \times X \to [0,1]$ is called a fuzzy equivalence relation, if

(1) $\Theta(a,a) = 1$ for all $a \in X$ (fuzzy reflexive),

(2) $\Theta(a,b) = \Theta(b,a)$ for all $a,b \in L$ (fuzzy symmetric),

(3) $\Theta(a,c) \geq \Theta(a,b) \wedge \Theta(b,c)$ for all $a,b,c \in L$ (fuzzy transitive).

A fuzzy equivalence relation Θ on a lattice L is called a fuzzy congruence relation on L if it is compatible with $\wedge$ and $\vee$.

Throughout the next sections, L stands for demi-p- algebras.

3. Fuzzy ideals and fuzzy congruences of distributive demi-p-algebras

Definition 3.1 : A mapping $\Theta : L \times L \to [0,1]$ is called a fuzzy congruence relation on L if it is lattice fuzzy congruence and $\Theta(a,b) \leq \Theta(a^*,b^*)$ for all $a,b \in L$.

Theorem 3.2 : *A mapping $\Theta : L \times L \to [0,1]$ is a fuzzy congruence relation on L if and only if $\Theta(a,b) \leq \Theta(a \wedge c, b \wedge c) \wedge \Theta(a \vee c, b \vee c) \wedge \Theta(a^*,b^*)$, for all $a,b,c \in L$.*

Example 3.3 : A fuzzy relation $\mathcal{G}_{\mathcal{F}}$ in demi-p-algebra L is given by:

$$\mathcal{G}_{\mathcal{F}}(a,b) = \begin{cases} 1 & \text{if } a^* = b^* \\ 0 & \text{otherwise,} \end{cases}$$

for any $a,b \in L$. Easily show that $\mathcal{G}_{\mathcal{F}}$ is a fuzzy congruence of L,

We call $\mathcal{G}_{\mathcal{F}}$ is Glievenko fuzzy congruence on L.

Theorem 3.4 : *A mapping Θ is a fuzzy congruence relation on demi-p-algebra L if and only if level subsets Θ_α of L, $\alpha \in [0,1]$ is a congruence relation on L. (In particular χ_Θ is a fuzzy congruence of L iff Θ is a congruence relation of L).*

Lemma 3.5 : *The intersection of a class of fuzzy congruences of demi-p-algebra is fuzzy congruence.*

$\mathcal{FC}(L)$ denotes the set of all fuzzy congruences of L.

$$\chi_\omega(a,b) = \begin{cases} 1 & \text{if } (a,b) \in \omega \\ 0 & \text{otherwise,} \end{cases}$$

for all $a,b \in L$ and $\chi_\iota(a,b) = 1$ for all $a,b \in L$ are the smallest and the largest elements of $\mathcal{FC}(L)$ respectively, where $\omega = \{(a,b) \in L \times L : a = b\}$ and $\iota = L \times L$.

Theorem 3.6 : *The set of all fuzzy congruences* $\mathcal{FC}(L)$ *of a demi-p-algebra L is a complete lattice with set inclusion.*

***Proof* :** Clearly $(\mathcal{FC}(L),\subseteq)$ is a poset, with χ_{ω} is least elements and χ_{ι} is the greatest elements of $\mathcal{FC}(L)$. Also, its intersection is a lower bound of any family fuzzy congruence of L. Let ϕ be any lower bound of $\{\Theta_i : i \in I\}$. Then $\phi \subseteq \Theta_i$, for all $i \in I$. Thus $\phi \subseteq \bigcap_{i\in I}\Theta_i$. So $\bigcap_{i\in I}\Theta_i$ is a greatest lower bound of $\{\Theta_i : i \in \Theta\}$. Hence $(\mathcal{FC}(L),\subseteq)$ is a complete lattice.

Next, we define the fuzzy quotient demi-pseudocomplemented algebra induced by fuzzy congruence relation.

Definition 3.7 : Let L be a demi-p-algebra, $a \in L$ and Θ be a fuzzy congruence on L. The fuzzy congruence determined by a and Θ, denoted by a_{Θ}, is the fuzzy subset of L defined by $a_{\Theta}(b) = \Theta(a,b)$, for all $b \in L$.

Let L/Θ denoted the set of all fuzzy congruence classes, that is $L/\Theta = \{a_{\Theta} : a \in L\}$.

Remark 3.8 : $a_{\Theta} = b_{\Theta} \Leftrightarrow \Theta(a,b) = 1$, for any fuzzy congruence Θ of L and $a,b \in L$.

Theorem 3.9 : *Let L be a demi-p-algebra and* $\Theta \in \mathcal{FC}(L)$. *For any* $a_{\Theta}, b_{\Theta} \in L/\Theta$, *define*

$$a_{\Theta} \wedge b_{\Theta} = (a \wedge b)_{\Theta},\; a_{\Theta} \vee b_{\Theta} = (a \vee b)_{\Theta},\; \textit{and}\; (a_{\Theta})^* = a^*_{\Theta}.$$

Then $(L/\Theta, \wedge, \vee, ^*, 0_{\Theta}, 1_{\Theta})$ *is a distributive demi-p-algebra, where* 0_{Θ} *is least and* 1_{Θ} *is the greatest element of* L/Θ.

L/Θ *is called the fuzzy quotient a demi-p-algebra L induced by* Θ. *It easily proved that the mapping* $a \to L/\Theta$ *is a homomorphism from L onto* L/Θ.

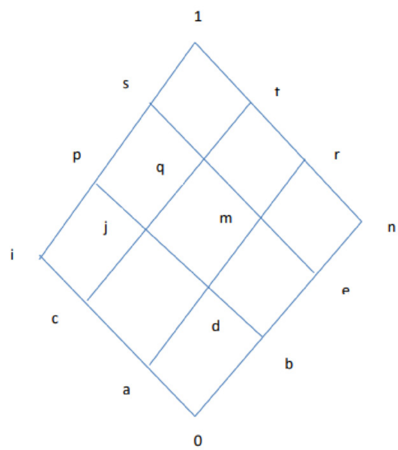
4. $*$-fuzzy ideals

In this section, we study $*$-fuzzy ideals of distributive demi-p-algebras and their properties.

Definition 4.1 : A fuzzy ideal ν is Kernel fuzzy ideal of a distributive demi-p-algebra L, there is a fuzzy congruence Θ on L such that $\nu = Ker\,\Theta$, where $Ker\,\Theta(a) = \Theta(a,0)$, for all $a \in L$.

Definition 4.2 : A fuzzy ideal ν of a demi-p-algebra L is said to be a $*$-fuzzy ideal if $\nu(a) = \nu(a^{**})$ for all $a \in L$.

Example 4.3 : Consider the following demi-p-algebra:



x	0	a	b	c	d	e	i	j	m	n	p	q	r	s	t	1
x^*	1	1	1	n	1	i	n	n	i	i	n	0	i	0	0	0

Define a fuzzy subset ν on L as follows: $\nu(0)=\nu(a)=1, \nu(b)=\nu(d)=\nu(c)=\nu(i)=\nu(j)=\mu(p)=0.7,\ \nu(e)=\nu(m)=\nu(q)=\nu(s)=0.5,\ \nu(n)=\nu(r)=\nu(t)=\nu(1)=0.3$. Clearly ν is a fuzzy ideal of L, but it is not $*$-fuzzy ideal of L, since $0.7=\nu(b)\neq\nu(b^{**})=\nu(0)=1$.

Example 4.4 : $Ker\mathcal{G}_{\mathcal{F}}$ is a $*$-fuzzy ideal of L.

Theorem 4.5 : *Let L be demi-p-algebra. Then* $Ker\mathcal{G}_{\mathcal{F}}$ *is the least $*$-fuzzy ideal of L.*

Corollary 4.6 : *If L be distributive demi-p-algebra, then every $*$-fuzzy ideal of L is a kernel fuzzy ideal of L.*

Proof : Let ν is $*$-fuzzy ideal of L. Consider the fuzzy relation $\nu^{\Diamond}$ on L defined by

$$\nu^{\Diamond}(a,b)=\vee\{\nu(i):(a\vee i)\wedge i^{*}=(b\vee i)\wedge i^{*}, i\in L\}.$$

We prove that $\nu^{\Diamond}$ is a fuzzy congruence of L. The proof of fuzzy reflexive and fuzzy symmetric are trivial. Next, we prove that $\nu^{\Diamond}$ is fuzzy transitive.

Suppose that $(a \vee i) \wedge i^* = (b \vee i) \wedge i^*$, $(b \vee j) \wedge j^* = (c \vee j) \wedge j^*$, for any $a, b, c, i, j \in L$. Put $k = i \vee j$.

After rootin verification, we have

$$(a \vee k) \wedge k^* = (c \vee k) \wedge k^*.$$

This implies
$$\begin{aligned}\nu^{\Diamond}(a,b) \wedge \nu^{\Diamond}(b,c) &= \vee\{\nu(i) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*\} \\ &\quad \wedge \vee\{\nu(j) : (b \vee j) \wedge j^* = (c \vee j) \wedge j^*\} \\ &= \vee\{\nu(i) \wedge \mu(j) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*, \\ &\quad (b \vee j) \wedge j^* = (c \vee j) \wedge j^*\} \\ &\leq \vee\{\nu(k) : (a \vee k) \wedge k^* = (c \vee k) \wedge k^*\} \\ &= \nu^{\Diamond}(a,c)\end{aligned}$$

Hence $\nu^{\Diamond}(a,b) \wedge \nu^{\Diamond}(b,c) \leq \nu^{\Diamond}(a,c)$. Thus we conclude that $\nu^{\Diamond}$ is transitive.

Put $k = i \vee j$. Then we have $k^* = i^* \wedge j^*$.

Easily, we proved that $(a \vee c) \vee k) \wedge k^* = (b \vee c) \vee k) \wedge k^*$.

For each $c \in L$,

$$\begin{aligned}\nu^{\Diamond}(a,b) &= \nu^{\Diamond}(a,b) \wedge \nu^{\Diamond}(c,c) \\ &= \vee\{\nu(i) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*\} \wedge \vee\{\nu(j) : (c \vee j) \wedge j^* = (c \vee j) \wedge j^*\} \\ &= \vee\{\nu(i) \wedge \nu(j) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*,\ (c \vee j) \wedge j^* = (c \vee j) \wedge j^*\} \\ &\leq \vee\{\nu(k) : (a \vee c) \vee k) \wedge k^* = (b \vee c) \vee k) \wedge k^*\} = \nu^{\Diamond}(a \vee c, b \vee c).\end{aligned}$$

This implies $\nu^{\Diamond}(a,b) \leq \nu^{\Diamond}(a \vee c, b \vee c)$. With a similarly way, $\nu^{\Diamond}(a,b) \leq \nu^{\Diamond}(a \wedge c, b \wedge c)$. So $\nu^{\Diamond}$ is a lattice fuzzy congruence of L.

Next, we show that $\nu^{\Diamond}(a,b) \leq \nu^{\Diamond}(a^*, b^*)$.

$$\begin{aligned} & (a \vee i) \wedge i^* = (b \vee i) \wedge i^* && (*) \\ \Rightarrow\ & (a^* \wedge i^*) \vee i^{**} = (b^* \wedge i^*) \vee i^{**} \\ \Rightarrow\ & (a^* \vee i^{**}) \wedge (i^* \vee i^{**}) = (b^* \vee i^{**}) \wedge (i^* \vee i^{**}) \\ \Rightarrow\ & (a^* \vee i^{**}) \wedge (i^* \vee i^{**}) \wedge i^* = (b^* \vee i^{**}) \wedge (i^* \vee i^{**}) \wedge i^* \\ \Rightarrow\ & (a^* \vee i^{**}) \wedge i^* = (b^* \vee i^{**}) \wedge i^* \\ \Rightarrow\ & (a^* \vee i^{**}) \wedge i^{***} = (b^* \vee i^{**}) \wedge i^{***} \ as\ i^{***} = x^*. \end{aligned}$$

$$\begin{aligned}\text{Now, } \nu^{\Diamond}(a,b) &= \vee\{\nu(i) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*\} \\ &\leq \vee\{\nu(i^{**}) : (a^* \vee i^{**}) \wedge i^{***} = (b^* \vee i^{**)} \wedge i^{***}\} \\ &= \nu^{\Diamond}(a^*, b^*)\end{aligned}$$

Thus $\nu^{\lozenge}$ is a fuzzy congruence of L.

Finally, we prove that $Ker\nu^{\lozenge} = \nu$.

$Ker\nu^{\lozenge}(a) = \nu^{\lozenge}(a,0) = \vee\{\nu(i) : (a \vee i) \wedge i^* = (0 \vee i) \wedge i^*\} \geq \nu(a)$. As $(a \vee a) \wedge a^* = (0 \vee a) \wedge a^*$ is true.

This implies $\nu \subseteq Ker\nu^{\lozenge}$.

Conversely, if $(a \vee i) \wedge i^* = i \wedge i^*$, then $(a \wedge i^*) \vee (i \wedge i^*) = i \wedge i^*$.

We observe that $a \wedge i^* \leq i^* \wedge i \leq i$.

This indicates that $(a \wedge i^*)^{**} \leq (i^* \wedge i)^{**} \leq i^{**}$. Now,

$$\begin{aligned}
Ker\nu^{\lozenge}(a) &= \nu^{\lozenge}(a,0) \\
&= \vee\{\nu(i) : (a \vee i) \wedge i^* = i \wedge i^*\} \\
&= \vee\{\nu(i^{**}) : (a \vee i) \wedge i^* = i \wedge i^*\} \\
&= \vee\{\nu(i^{**}) \wedge \nu(a^{**} \wedge i^*) : (a \vee i) \wedge i^* = i \wedge i^*\} \\
&\leq \nu(i^{**} \vee (a^{**} \wedge i^*)) \\
&= \nu((i^{**} \vee (a^{**} \wedge i^*))^{**}) \\
&= \nu(i^{**} \vee a^{**}) as (i^* \vee i^{**})^{**} = 1 \\
&\leq \nu(a^{**}) = \nu(a)
\end{aligned}$$

Hence $\nu = Ker\nu^{\lozenge}$.

Corollary 4.7 : *If ν is a *-fuzzy ideal of L, then $\nu^{\lozenge}$ is the smallest fuzzy congruence on L with fuzzy kernel ν.*

Proof : Let Θ be any fuzzy congruence on L with $Ker\Theta = \nu$. We prove that $\nu^{\lozenge} \subseteq \Theta$. Now

$$\begin{aligned}
\nu^{\lozenge}(a,b) &= \vee\{\nu(i) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*\} \\
&= \vee\{\Theta(i,0) : (a \vee i) \wedge i^* = (b \vee i) \wedge i^*\}.
\end{aligned}$$

Since Θ is fuzzy congruence on L, $\Theta(i,0) \leq \Theta(i^*,1)$. Then

$$\Theta(i,0) = \Theta(i,0) \wedge \Theta(i^*,1) \wedge \Theta(a,a) \leq \Theta(a,(a \vee i) \wedge i^*).$$

Similarly, $\Theta(i,0) \leq \Theta((b \vee i) \wedge i^*, b)$.

Thus $\Theta(i,0) \leq \Theta(a,(a \vee i) \wedge i^*) \wedge \Theta((b \vee i) \wedge i^*, b) \leq \Theta^{\lozenge}(a,b)$.

Thus $\nu^{\lozenge} \subseteq \Theta$. So $\nu^{\lozenge}$ is the smallest fuzzy congruence on L with fuzzy kernel ν.

Theorem 4.8 : *For every distributive demi-p-algebra L, the following propositions are equivalent:*

(1) *each fuzzy congruence kernel of L is ∗-fuzzy ideal,*

(2) $Ker\mathcal{G}_{\mathcal{F}} = \chi_{\{0\}}$

Proof : Suppose that (1) holds. Each fuzzy congruence kernel of L is a ∗-fuzzy ideal, by Theorem 4.5 , $Ker\mathcal{G}_{\mathcal{F}}(a) = 1 = Ker\chi_{\omega}(a)$. This implies $a = 0$. Thus $Ker\mathcal{G}_{\mathcal{F}} = \chi_{\{0\}}$.

Assume (2) holds. Since $(L,^{*})$ is a demi-p-algebra, for every $a \in L$, we have $(a \wedge a^{*})^{*} = a^{*} \vee a^{**} = 1 = 0^{*}$.

This implies $1 = \mathcal{G}_{\mathcal{F}}(a \wedge a^{*}, 0) = Ker\mathcal{G}_{\mathcal{F}}(a \wedge a^{*}) = \chi_{\{0\}}(a \wedge a^{*})$. This implies $a \wedge a^{*} = 0$. Suppose that ν is the kernel of fuzzy congruence Θ of L i.e., $\nu = Ker\Theta$.

Now $\nu(a) = Ker\Theta(a) = \Theta(a,0) \leq \Theta(a^{**},0) = \nu(a^{**})$.

Conversely, $\nu(a^{**}) = \Theta(a^{**},0) \leq \Theta(a^{*},1) = \Theta(a^{*},1) \wedge \Theta(a,a) \leq \Theta(a \wedge a^{*}, a) = \Theta(0,a) = \nu(a)$. Thus $\nu(a) = \nu(a^{**}), \forall\, a \in L$ and so ν is a ∗-fuzzy ideal of L.

Theorem 4.9 : *Let L be a distributive demi pseudo-p- algebra, for every* $\nu \in \mathcal{FI}_{*}(L)$, *define a fuzzy relation* $\nu^{\intercal}$ *on L by*

$$\nu^{\intercal}(a,b) = \nu((a^{**} \wedge b^{*}) \vee (a^{*} \wedge b^{**})).$$

Then $\nu^{\intercal}$ is the largest fuzzy congruence on L with kernel ν.

Theorem 4.10 : *Let* $\lambda, \nu \in \mathcal{FI}_{*}(L)$. *Then following propositions hold:*

(1) $\lambda \subseteq \nu \Leftrightarrow \lambda^{\intercal} \subseteq \nu^{\intercal}$,

(2) $(Ker\mathcal{G}_{\mathcal{F}})^{\intercal} = \mathcal{G}_{\mathcal{F}}$,

(3) $\mathcal{G}_{\mathcal{F}} \subseteq \lambda^{\intercal}$.

Theorem 4.11 : *∗-Fuzzy ideals of a distributive demi pseudo-complemented algebra L are precisely the kernels of the fuzzy congruences* Θ *on L if and only if* L/Θ *is a Boolean algebra.*

To each fuzzy ideal ν *of L, we consider the fuzzy subset* ν *given by*

$$\nu_{*}(a) = \vee\{\nu(i) : a^{**} \wedge i^{*} = 0\}.$$

Now, we have the following Theorems.

Theorem 4.12 : *Let* ν *be a fuzzy ideal of distributive demi-p-algebra L. Then the following propositions are equivalent:*

(1) ν *is a ∗-fuzzy ideal,*

(2) $\nu = \nu_{*}$.

Proof : $(1)\Rightarrow(2)$. Suppose that ν is a $*$-fuzzy ideal. Then

$$\begin{aligned}\nu_*(a) &= \bigvee\{\nu(a): i^{**}\wedge a^* = 0\}\\ &= \bigvee\{\nu(a^{**}): i^{**}\wedge a^* = 0\}\\ &\leq \nu(a^{**}) = \nu(a)\end{aligned}$$

Conversely $\nu_*(a) = \bigvee\{\nu(i): a^{**}\wedge i^* = 0\} \geq \mu(i)$ as $i^{**}\wedge i^* = 0$.

Hence $\nu = \nu_*$.

$(2)\Rightarrow(1)$. $\nu(a) = \nu_*(a) = \bigvee\{\nu(i): i^{**}\wedge a^* = 0\} = \bigvee\{\nu(i): i^{****}\wedge a^* = 0\} = \nu_*$ $(a^{**}) = \nu(a^{**})$ as $a^{****} = a^{**}$.

Theorem 4.13 : *Let μ and λ be fuzzy ideals distributive demi-p-algebra L. Then*

(1) μ_* *is a $*$-fuzzy ideal,*

(2) $\mu\subseteq\lambda\Rightarrow\mu_*\subseteq\lambda_*$,

(3) $(\mu\cap\lambda)_* = \mu_*\cap\lambda_*$.

Proof : (1) First, we prove that μ is a fuzzy ideal of L. Clearly $\mu_*(0)=1$.

$$\begin{aligned}\mu_*(a)\wedge\mu_*(b) &= \vee\{\mu(i): a^{**}\wedge j^* = 0\}\wedge\vee\{\mu(j): b^{**}\wedge j^* = 0\}\\ &= \vee\{\mu(i)\wedge\mu(j): a^{**}\wedge i^* = 0, b^{**}\wedge j^* = 0\}\\ &\leq \vee\{\mu(i\vee j): (a\vee b)^{**}\wedge(i\vee j)^* = 0\}\\ &= \mu_*(a\vee b)\end{aligned}$$

$$\begin{aligned}\text{Next, }\mu_*(a)\vee\mu_*(b) &= \vee\{\mu(i): a^{**}\wedge i^* = 0\}\vee\vee\{\mu(j): b^{**}\wedge j^* = 0\}\\ &= \vee\{\mu(i)\vee\mu(j): a^{**}\wedge i^* = 0, b^{**}\wedge j^* = 0\}\\ &\leq \vee\{\mu(i\wedge j): (a\wedge b)^{**}\wedge(i\wedge j)^* = 0\}\\ &= \mu_*(a\wedge b)\end{aligned}$$

Finally, we prove that $\mu_*(a) = \mu_*(a^{**})$.

$\mu_*(a) = \vee\{\mu(i): a^{**}\wedge i^* = 0\} = \vee\{\mu(i): a^{****}\wedge i^* = 0\} = \mu_*(a^{**})$. Thus μ_* is a $*$-fuzzy ideal.

(2) It is clear.

(3) By (2), $(\mu\cap\lambda)_*\subseteq\mu_*\cap\lambda_*$.

Conversely,

$$\begin{aligned}(\mu_* \cap \lambda_*)(a) &= \mu_*(a) \wedge \lambda_*(a)\\ &= \vee\{\mu(i) : a^{**} \wedge i^* = 0\} \wedge \vee\{\lambda(j) : a^{**} \wedge j^* = 0\}\\ &= \vee\{\mu(i) \wedge \lambda(j) : a^{**} \wedge j^* = 0, a^{**} \wedge j^* = 0\}\\ &\leq \vee\{\mu(i \wedge j) \wedge \lambda(i \wedge j) : a^{**} \wedge (i \wedge j)^* = 0\}\\ &\leq \vee\{(\mu \cap \lambda)(i \wedge j) : a^{**} \wedge (i \wedge j)^* = 0\}\\ &= (\mu \cap \lambda)_*(a)\end{aligned}$$

Hence $(\mu \cap \lambda)_* = \mu_* \cap \lambda_*$.

Corollary 4.14 : *The mapping* $\tau : \mathcal{FI}(L) \to \mathcal{FI}(L)$ *described by* $\tau(\mu) = \mu_*$ *is a closure.*

Proof : Clearly $\mu \subseteq \mu_* = \tau(\mu)$ for all $\mu \in \mathcal{FI}(L)$. By Theorem 4.12 $\tau^2(\mu) = \mu_{**} = \mu_* = \tau(\mu)$. Also by Theorem 4.13, τ is an isotone. Thus τ is a closure on $\mathcal{FI}(L)$.

Theorem 4.15 : *Let L be a distributive demi-p-algebra. Then the set of all *-fuzzy ideals* $\mathcal{FI}_*(L)$ *is a Heyting algebra.*

Proof : Easily proved that $\mathcal{FI}(L)$ is bounded distributive lattice and every subclass has the least element. $\tau : \mu \to \mu_*$ is a closure on $\mathcal{FI}(L)$, it follows that $\mathcal{FI}(L) = \mathrm{Im}\,\tau$ is a complete lattice in which $\wedge_{\alpha \in I} \lambda_\alpha = \bigcap_{\alpha \in I} \lambda_\alpha$, $\sqcup_{\alpha \in I} \lambda_\alpha = \tau(\vee_{\alpha \in I} \lambda_\alpha)$. To show that $\mathcal{FI}_*(L)$ is a Heyting algebra, it enough to show that for all $\mu, \lambda \in \mathcal{FI}_*(L)$ there exists the largest $\pi \in \mathcal{FI}_*(L)$ such that $\pi \cap \mu \subseteq \lambda$. For this purpose, given $\mu, \lambda \in \mathcal{FI}_*(L)$, consider the set $\pi = \vee\{\eta \in \mathcal{FI}_*(L) : \eta \cap \mu \subseteq \lambda\}$. Clearly, π is a *-fuzzy ideal of L. Now we show that π is the largest fuzzy ideal satisfying $\mu \cap \pi \subseteq \lambda$. Suppose that there exists a fuzzy ideal σ properly containing π such that $\mu \cap \sigma \subseteq \lambda$. This implies $\sigma \subseteq \pi$ by definition of π. Which is a contradiction. Hence $\mathcal{FI}_*(L)$ is a Heyting algebra.

Theorem 4.16 : *Let L be a distributive demi-p-algebra. Then* $\mathcal{FI}_*(L)$ *isomorphic to* $[\mathcal{G}_{\mathcal{F}}, \chi_\iota]$.

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