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Commutativity degree of chains of finite groups

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Abstract

The concepts of commutativity of two chains, and the commutativity degree of the chains of a finite group such as G which ends in G are introduced. Then, the relation between the commutativity degree of the chains of a finite group is examined, and eventually this measure is calculated for some finite groups and fuzzy subgroups.

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1. Introduction

In recent years, special attention has been given to the use of probability in the finite group theory. One of the important aspects examined is the probability that two elements of a finite group G are commuted. This is called the commutativity degree of group G, and explicit formulas have been proposed for some special finite groups [1, 2, 3, 4, 5]. On the other hand, results for some finite groups describe a

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relation regarding the commutativity degrees of these groups [4, 5]. A generalization of the concept of commutativity degree is examined in [6]. Moreover, the commutativity degree of the subgroups is also investigated, and this measure has been studied in some finite groups such as finite dihedral groups and finite p-groups [7]. Rasuli [8] characterized some basic properties of T-fuzzy subgroups and normal T-fuzzy subgroups, using a t-norm T. He also explained fuzzy and normal fuzzy subgroups induced by t-norm T in the direct product of groups.

In this paper, the concepts of product of two chains, the commutativity of two chains, and the commutativity degree of the chains of a finite group G, which ends in G are introduced, and the relation between commutativity degree of the chains of a finite group is examined. The rest of this paper is organized as follows. Section 2 presents the results related to the commutativity degree of the chains of groups. Section 3 includes the definition of the commutativity degree of two fuzzy subgroups and examples of commutativity degree of some finite fuzzy subgroups.

2. The commutativity degree of chains

In this section, we present some results on the commutativity degree of the chains of finite groups. We will show the set of all subgroups of the group G by $L(G)$. We call the n-tuple $(A_1, A_1, \ldots, A_n)$ of G's subgroups, a chain of length n, if we have $A_i \subset A_{i+1}$ for each $1 \leq i \leq n-1$. If the last term of the chain be G, we say that the chain ends in G. The set of all chains of subgroups that are ending in G, are important for us, and we show these series of chains with $Ch(G)$.

Definition 1 : Let $C_1(A_1, A_2, \ldots, A_n = G)$ and $C_2(B_1, B_2, \ldots, B_n = G)$ are two members of $Ch(G)$, the product of these two chains is defined as $C_1C_2(A_1B_1, A_2B_2, \ldots, A_rB_r = G)$ when $r = min\{i : A_iB_i = G\}$. Two chains C_1 and C_2 are commute $(C_1C_2 = C_2C_1)$, when the product of these two chains is a member of $Ch(G)$, and this is true only when $A_1B_1, A_2B_2, \ldots, A_{r-1}B_{r-1}$ are subgroups of G.

Definition 2 : The commutativity degree of chains of group G is defined as follows:

$$cd(G) = \frac{1}{|Ch(G)|^2} |\{(C_1, C_2) \in Ch(G)^2 : C_1C_2 = C_2C_1\}|$$

$$= \frac{1}{|Ch(G)|^2} |\{(C_1, C_2) \in Ch(G)^2 : C_1 C_2 \in Ch(G)\}|. \tag{1}$$

Now, we will have the following lemma.

Lemma 1 : *Consider the finite group G and the normal subgroup N. Both chains of subgroups G including N such as* $N \subset A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n = G$ *and* $N \subset B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_m = G$ *are commute, if and only if the chains* $1 \subset \frac{A_1}{N} \subset \frac{A_2}{N} \subset \cdots \subset \frac{A_{n-1}}{N} \subset \frac{G}{N}$ *and* $1 \subset \frac{B_1}{N} \subset \frac{B_2}{N} \subset \cdots \subset \frac{B_{m-1}}{N} \subset \frac{G}{N}$ *from the group* $\frac{G}{N}$ *can be commuted.*

Proof : Due to the one-to-one correspondence between subgroups G including N and subgroups $\frac{G}{N}$, and the relation $\frac{AB}{N} = \frac{A}{N} \cdot \frac{B}{N}$, it is easily proved. □

So, we have the following theorem.

Theorem 1 : *There is the following relation between the commutativity degree of a chain of finite group G and normal subgroup N:*

$$cd(G) \geqslant \frac{1}{|Ch(G)|^2}\left(\left(|Ch(N)| + \left|Ch\left(\frac{G}{N}\right)\right| - 1\right)^2 + (cd(N) - 1)|Ch(N)|^2 + \left(cd\left(\frac{G}{N}\right) - 1\right)\left|Ch\left(\frac{G}{N}\right)\right|^2\right).$$

Proof : We define the function f as follows:

$$f : Ch(G)^2 \to \{0, 1\}$$

$$f(C_1, C_2) = \begin{cases} 1 & C_1C_2 = C_2C_1 \\ 0 & C_1C_2 \neq C_2C_1 \end{cases}$$

It is obvious that

$$cd(G) = \frac{1}{|Ch(G)|^2} \sum_{C_1, C_2 \in Ch(G)} f(C_1, C_2).$$

Suppose N is the normal subgroup of G, we set:

$$\Gamma_1 = \{C \in Ch(G) : \text{ all components of } C \text{ containing } N\}.$$

$$\Gamma_2 = \{C \in Ch(G): \text{ all components of } C, \text{ which do not contain } N \text{ except } G\}.$$

Since $\Gamma_1 \cup \Gamma_2 \subset Ch(G)$, we have:

$$d(G) \geqslant \frac{1}{|Ch(G)|^2} \sum_{C_1, C_2 \in \Gamma_1 \cup \Gamma_2} f(C_1, C_2)$$

$$= \frac{1}{|Ch(G)|^2} \left(\sum_{C_1, C_2 \in \Gamma_1} f(C_1, C_2) + \sum_{C_1, C_2 \in \Gamma_2} f(C_1, C_2) + 2 \sum_{C_1 \in \Gamma_1} \sum_{C_2 \in \Gamma_2} f(C_1, C_2) \right).$$

Considering the one-to-one correspondence between subgroups G including N and subgroup $\frac{G}{N}$, we have:

$$cd\left(\frac{G}{N}\right) = \frac{\sum_{C_1, C_2 \in \Gamma_1} f(C_1, C_2)}{Ch\left|\left(\frac{G}{N}\right)\right|^2}.$$

Therefore,

$$\sum_{C_1, C_2 \in \Gamma_1} f(C_1, C_2) = cd\left(\frac{G}{N}\right) \left| Ch\left(\frac{G}{N}\right) \right|^2.$$

Moreover,

$$\sum_{C_1, C_2 \in \Gamma_2} f(C_1, C_2) = \sum_{C_1, C_2 \in \Gamma_2 \cup \{C^*\}} f(C_1, C_2) - 2 \sum_{C_1 \in \Gamma_2 \cup \{C^*\}} f(C_1, C^*)$$
$$= cd(N)\,|Ch(N)|^2 - 2\,|Ch(N)| + 1.$$

where $C^*(N, G)$ is a chain of length 2, and

$$2 \sum_{C_1 \in \Gamma_1} \sum_{C_2 \in \Gamma_2} f(C_1, C_2) = 2\,|\Gamma_1|\;|\Gamma_2| = 2\left| Ch\left(\frac{G}{N}\right) \right| (\,|Ch(N) - 1|\,).$$

By substitution, we have:

$$cd(G) \geqslant \frac{1}{|Ch(G)|^2} \left((\,|Ch(N)| + \left| Ch\left(\frac{G}{N}\right) \right| - 1)^2 + (cd(N) - 1)\,|Ch(N)|^2 + \left(cd\left(\frac{G}{N}\right) - 1 \right) \left| Ch\left(\frac{G}{N}\right) \right|^2 \right).$$

and this completes the proof. □

Based on the Theorem 1, we can obtain the following corollaries.

Corollary 1 : *If G be a finite group and N be a normal subgroup, such that N and $\frac{G}{N}$ be abelian, then:*

$$cd(G) \geqslant \left(\frac{|Ch(N)| + \left| Ch\left(\frac{G}{N}\right) \right| - 1}{|Ch(G)|} \right)^2.$$

Proof : It is obvious according to Theorem 1. □

Corollary 2 : *If G be a finite group and N be a normal subgroup of prime index, then*

$$cd(G) \geqslant \frac{1}{|Ch(G)|^2}(cd(N)\,|Ch(N)|^2 + 2\,|Ch(N)| + 1).$$

Proof : Since N is a normal subgroup of prime index, then $\frac{G}{N}$ has only two subgroups, and so $cd\left(\frac{G}{N}\right) = 1,$ and

$$\begin{aligned} cd(G) &\geqslant \frac{1}{|Ch(G)|^2}((|Ch(N)| + 2 - 1)^2 + (cd(N) - 1)\,|Ch(N)|^2 + 0) \\ &= \frac{1}{|Ch(G)|^2}(cd(N)\,|Ch(N)|^2 + 2\,|Ch(N)| + 1). \end{aligned}$$

and this completes the proof. □

Corollary 3 : *If G be a finite soluble group and $1 \subset G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_k = G$ is a series of subgroup of G, so that the factors $\frac{G_i}{G_{i-1}}$ for $i = 1, 2, \ldots, k$ are cyclic of prime order, then we have:*

$$cd(G) \geqslant \frac{1}{|Ch(G_{i-1})|^2}\left(2\sum_{i=1}^{k} |Ch(G_{i-1})| + k + 1 \right).$$

Proof : Given the Corollary 2, because $\frac{G_i}{G_{i-1}}$ for $i = 1, 2, \cdots, k$ is a cycle of a prime number, we have:

$$cd(G_i)\,|Ch(G_i)|^2 \geqslant cd(G_{i-1})\,|Ch(G_{i-1})|^2 + 2\,|Ch(G_{i-1})| + 1$$

for $i = 1, 2, \cdots, k$.

By adding to both sides of the inequality, we have:

$$cd(G_k)\,|\,Ch(G_k)\,|^2 \geqslant cd(G_0)\,|\,Ch(G_0)\,|^2 + 2\sum_{i=1}^{k} |\,Ch(G_{i-1})\,|$$

$$+k = 1 + 2\sum_{i=1}^{k} |\,Ch(G_{i-1})\,| + k.$$

Therefore

$$cd(G) \geqslant \frac{1}{|Ch(G)|^2}\left(2\sum_{i=1}^{k} |Ch(G_{i-1})| + k + 1\right).$$

□

Corollary 4 : *If G be a finite p-group of the order* p^k *which has a maximal cyclic subgroup, then*

$$cd(G) \geqslant \frac{2^{k+1} + k - 1}{|Ch(G)|^2}.$$

Proof: Since G is a finite group of order p^k, it has a maximal cyclic subgroup, then G is soluble and contains a series $1 \subset G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ in which G_{k-1} is cyclic, and so all subgroups $G_i\,(1 \geqslant i \geqslant k-2)$ are cyclic and $[G_i : G_{i-1}] = p$ for each $1 \leqslant i \leqslant k$. Using the Corollary 3 and considering that $|\,Ch(G_i)\,| = 2^i$, we have:

$$\begin{aligned} cd(G) &\geqslant \frac{1}{|Ch(G)|^2}\left(2\sum_{i=1}^{k} |Ch(G_{i-1})| + k + 1\right) \\ &= \frac{1}{|Ch(G)|^2}\left(2\sum_{i=1}^{k} 2^{i-1} + k + 1\right) \\ &= \frac{1}{|Ch(G)|^2}(2^{k+1} + k - 1). \end{aligned}$$

□

3. Commutativity degree of fuzzy subgroups

We know that the concept of the numbers of fuzzy subgroups is the same as the numbers of equivalence classes. On the other hand, the number of fuzzy subgroups of group G is equal to the number of chains of group G ending in G. For more details on fuzzy subgroups, the readers are referred to [9, 10]. We also have the following definition and theorem on fuzzy subgroups.

Definition 3 : Suppose μ and v are two fuzzy subgroups. Then, for each $x \in G$, the combination $\mu \circ v$ is defined as:

$$(\mu \circ v)(x) = sup\{\mu(y) \wedge \mu(z) : yz = x, \quad y, z \in G\}.$$

Theorem 2 : *(Theorem 1.2.10 [10]) Suppose μ and v are two fuzzy subgroups, then $\mu \circ v$ is fuzzy subgroup, if and only if $\mu \circ v = v \circ \mu$.*

In the following, we first define commutativity degree of fuzzy subgroups and then compute this measure for fuzzy subgroups.

Definition 4 : Suppose G is a group. The commutativity degree of fuzzy subgroup of G is defined as:

$$cd_F(G) = \frac{|\{(\mu, v) : \mu, v \in F_G, \quad \mu \circ v = v \circ \mu\}|}{|F_G|^2} \tag{2}$$

where F_G is the number of all fuzzy subgroups of the group G.

The number of fuzzy subgroups F_G for some specific groups can be found in [12]. Now, we can get the following results for the fuzzy subgroups.

Theorem 3 : *For Abelian group G, the commutativity degree of fuzzy subgroup is equal to one.*

Proof : In Abelian groups, if μ and v are two arbitrary fuzzy subgroups, then $\mu \circ v$ are fuzzy subgroups and so $\mu \circ v = v \circ \mu$. Therefore

$$cd_F(G) = \frac{|\{(\mu, v) : \mu, v \in F_G, \quad \mu \circ v = v \circ \mu\}|}{|F_G|^2} = \frac{|\{(\mu, v) : \mu, v \in F_G\}|}{|F_G|^2} = \frac{|F_G|^2}{|F_G|^2} = 1.$$

□

Dedekind groups are groups whose subgroups are normal. The non-Abliean Dedekind groups are called Hamiltonian groups. According to [11], group G is Dedekind, if and only if, it is Abelian or it is the direct product of Q_8 (Quaternion group of order 8) and an elementary Abelian 2-group and an Abelian group whose all elements are of odd order. The commutativity degree of fuzzy subgroups of Dedekind groups is equal to one. Therefore, the Hamiltonian groups are examples of non-Abelian groups in which commutativity degree of their fuzzy subgroups is equal to one.

Lemma 2 : *Suppose* μ *and* v *are two fuzzy subgroups of group G, and* $x \in G$, *then we have:*

$$(1)\quad (\mu \circ v)(x) \geq \mu(x)$$
$$(2)\quad (\mu \circ v)(x) \geq v(x)$$
$$(3)\quad (\mu \circ v)(x) \geq \mu(x) \vee v(x)$$
$$(4)\quad (\mu \circ v)(x) \geq \mu(x) \wedge v(x)$$

Proof : By definition, they are obvious. □

Theorem 4 : *If* μ *be a fuzzy subgroup, then* $\mu \circ \mu = \mu$.

Proof : By using Lemma 2, we have:

$$(\mu \circ \mu)(x) \geq \mu(x). \tag{3}$$

Now, we prove $(\mu \circ \mu)(x) \leq \mu(x)$. Suppose that is not the case, then there exists y and z, where $x = yz$ and $\mu(x) \wedge \mu(z) > \mu(x)$. According to the definition of fuzzy subgroup, we have: $\mu(yz) \geq \mu(y) \wedge \mu(z)$. So, $\mu(x) = \mu(yz) \geq \mu(y) \wedge \mu(z) > \mu(x)$ that is a contradiction. So, the assumption is invalid and

$$(\mu \circ \mu)(x) \leq \mu(x). \tag{4}$$

Therefore, from equations (3) and (4) the theorem is proved. □

Theorem 5 : *In Abelian groups, if* μ *and* v *are fuzzy subgroups, then* $\mu \circ v$ *is a fuzzy subgroup. In other words, the composition of fuzzy subgroups has the commutativity property.*

Proof :

$$\begin{aligned}(\mu \circ v)(x) &= sup\{\mu(y) \wedge v(z) : yz = x\}\\ &= sup\{\mu(y) \wedge v(z) : zy = x\}\\ &= sup\{v(y) \wedge \mu(y) : zy = x\} = (v \circ \mu)(x).\end{aligned}$$

and this completes the proof. □

Theorem 6 : *Suppose* μ *and* v *are two fuzzy subgroups and* $\mu \subset v$, *in this case* $\mu \circ v = v$.

Proof : By using Lemma 2, we have:

$$(\mu \circ v)(x) \geq v(x). \tag{5}$$

On the other hand, suppose $x = yz$, because v is a fuzzy subgroup and $\mu \subset v$, we have:

$$v(x) = v(yz) \geq v(y) \wedge v(z) \geq \mu(y) \wedge v(z).$$

Therefore,

$$v(x) \geq \{\mu(y) \wedge v(z) : yz = x\} = (\mu \circ v)(x) \tag{6}$$

From equations (5) and (6), the theorem is proved. □

From basic algebra we know that if *H* and *K* be two subgroups of *G*, then *HK* is a subgroup of *G*, if and only if, we have $HK = KH$.

Lemma 3 : *Suppose G is a group, and H and K are two subgroups of G. The fuzzy subgroups corresponding to chains* $H \subset G$ *and* $K \subset G$ *are commute, if and only if* $HK = KH$.

Proof : Suppose μ and v are fuzzy subgroups corresponding to chains $H \subset G$ and $K \subset G$, respectively. If so, μ and v are as follows:

$$v(t) = \begin{cases} 1 & t \in K \\ 0.5 & t \in G - K \end{cases}$$

and

$$\mu(t) = \begin{cases} 1 & t \in H \\ 0.5 & t \in G - H \end{cases}$$

and so

$$(\mu \circ v)(t) = \begin{cases} 1 & t \in HK \\ 0.5 & t \in G - HK \end{cases}$$

The necessary and sufficient condition for the commutativity of μ and v is that

$$(\mu \circ v)_a = \{x \in G : (\mu \circ v)(x) \geq a\}, \quad \forall a \in Im(\mu \circ v)$$

be a subgroup of *G*, and this is equivalent to that *HK* is a subgroup and is equivalent to $HK = KH$ with respect to the above lemma. □

Remark 1 : The following example shows that for two fuzzy subgroups such as μ and v, we may have $\mu \circ v = v \circ \mu$, $\mu_1 \subset \mu$ and $v_1 \subset v$, but $\mu_1 \circ v_1 \neq v_1 \circ \mu_1$.

Example 1 : For the group $G = D_8$, the fuzzy subgroups corresponding to the chains $\langle a^2, b\rangle \subset D_8$ and $\langle a^2, ab\rangle \subset D_8$ are commute, but the fuzzy subgroups corresponding to the chains $\langle b\rangle \subset D_8$ and $\langle ab\rangle \subset D_8$ are not commute.

Lemma 4 : *Suppose G be a group, and H, K and L are subgroups of G. Then, we have:*

(1) $HK \cup HL = H(K \cup L)$.

(2) $HL \cup KL = (H \cup K)L$.

Proof : By definition they are obvious. □

Theorem 7 : *Suppose* μ *is a fuzzy subgroup corresponding to chain* $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n = G$ *and* v *is a fuzzy subgroup corresponding to chain* $B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_m = G$ *and* $n \leq m$, *and let* $r = min\{i : A_iB_i = G\}$. *In this case,* $\mu \circ v = v \circ \mu$ *is a fuzzy subgroup, if and only if* $A_1B_1, A_2B_2, \cdots, A_{r-1}B_{r-1}$ *are subgroups of G.*

Proof : Since μ is a fuzzy subgroup corresponding to chain $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n = G$ and v is a fuzzy subgroup corresponding to chain $B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_m = G$. There are numbers $0 \leq t_{m-1} < \cdots < t_n < t_{n-1} < \cdots < t_2 < t_1 < 1$, such that

$$\mu(x) = \begin{cases} 1 & A_1 \\ t_1 & A_2 - A_1 \\ t_2 & A_3 - A_2 \\ & . \\ & . \\ & . \\ t_{n-1} & G - A_{n-1} \end{cases}$$

and

$$v(x)=\begin{cases} 1 & B_1 \\ t_1 & B_2-B_1 \\ & . \\ & . \\ & . \\ t_{n-1} & B_n-B_{n-1} \\ t_n & B_{n+1}-B_n \\ & . \\ & . \\ & . \\ t_{m-1} & G-B_{m-1} \end{cases}$$

The fuzzy subgroup $(\mu \circ v)(x)$ equals 1, if and only if $x \in A_1B_1$. Now, if $A_1B_1 = G$, for each $x \in G$, we have $(\mu \circ v)(x) = 1$.

According to the definition of fuzzy subgroup composition, $(\mu \circ v)(x)$ is equal to t_1, if and only if $x \in A_1(B_2 - B_1) \cup (A_2 - A_1)B_1 \cup (A_2 - A_1)(B_2 - B_1)$. But, according to Lemma 4, we have:

$$A_1(B_2-B_1)\cup(A_2-A_1)B_1\cup(A_2-A_1)(B_2-B_1)=A_1(B_2-B_1)\cup(A_2-A_1)(B_1\cup(B_2-B_1))$$

$$A_1(B_2-B_1)\cup(A_2-A_1)B_2,$$

and

$$A_1(B_2-B_1)\cup(A_2-A_1)B_2\cup A_1B_1 = A_1((B_2-B_1)\cup B_1)\cup(A_2-A_1)B_2$$

$$= A_1B_2\cup(A_2-A_1)B_2 = (A_1\cup(A_2-A_1))B_2 = A_2B_2,$$

and so

$$A_1(B_2-B_1)\cup(A_2-A_1)B_2\cup(A_2-A_1)(B_2-B_1) = A_2B_2 - A_1B_1.$$

Now, if $A_2B_2 = G$, the equivalent chain for $\mu \circ v$ is $A_1B_1 \subset A_2B_2 = G$, and if we continue this process, we have:

$$\mu(x) = \begin{cases} 1 & A_1B_1 \\ t_1 & A_2B_2 - A_1B_1 \\ t_2 & A_3B_3 - A_2B_2 \\ \quad . & \\ \quad . & \\ \quad . & \\ t_{r-1} & G - A_{r-1}B_{r-1} \end{cases}$$

Then, $\mu \circ v$ is a fuzzy subgroups, if and only if $A_1B_1, A_2B_2, \cdots A_{r-1}B_{r-1}$ are subgroups of G. In this case, the equivalent chain to $\mu \circ v$ is $A_1B_1 \subset A_2B_2 \subset \cdots \subset A_{r-1}B_{r-1} \subset G$, and this completes the proof. □

3.1 *Commutativity degree of fuzzy subgroups of* D_{2p}

In this subsection, we calculate the commutativity degrees of fuzzy subgroups of dihedral group D_{2p} where *p* is a prime number.

Theorem 8 : *The commutativity degree of two fuzzy subgroups in the group* D_{2p} *is:*

$$cd_F(D_{2p}) = \frac{p^2 + 9p + 8}{2p^2 + 8p + 8}. \tag{7}$$

where p is a prime number.

Proof : We have

$$G = D_{2p} = \langle a, b : a^p = b^2 = 1, a^b = a^{-1} \rangle.$$

The number of fuzzy subgroups is equal to $F_{D_{2p}} = 2p + 4.$

Equivalent chains to fuzzy subgroups are:

(1) D_{2p}

(2) $\langle a \rangle \subset D_{2p}$

(3) $1 \subset \langle a \rangle \subset D_{2p}$

(4) $1 \subset D_{2p}$

(5) $\langle a^i b \rangle \subset D_{2p} \quad (0 \le i \le p-1),$

(6) $1 \subset \langle a^i b \rangle \subset D_{2p} \quad (0 \le i \le p-1).$

Fuzzy subgroups equivalent to chains (1), (2), (3) and (4) are commutate with all fuzzy subgroups. No two different chains of group (5) can be commutated together. In other words, the combination of fuzzy subgroups $\langle a^i b\rangle \subset D_{2p}$ and $\langle a^j b\rangle \subset D_{2p}$, $i \neq j$ is not a fuzzy subgroup. Moreover, no two different chains of group (6) can be commutated together. In other words, the combination of two fuzzy subgroups $1 \subset \langle a^i b\rangle \subset D_{2p}$ and $1 \subset \langle a^j b\rangle \subset D_{2p}$ for $i \neq j$, is not a fuzzy subgroup. Each fuzzy subgroup equivalent to a chain of (5) and each fuzzy subgroup equivalent to a chain of (6), can be commutated together. That is, the combination of fuzzy subgroups equivalent to chains $\langle a^i b\rangle \subset D_{2p}$ and $1 \subset \langle a^j b\rangle \subset D_{2p}$ for each i, j, is a fuzzy subgroup.

Therefore, the number of ordered pairs (μ, v) of fuzzy subgroups of group D_{2p} when $\mu \circ v = v \circ \mu$, is:

$$4(2p+4) + p(2p+4-(p-1)) + p(2p+4-(p-1)) = 2p^2 + 18p + 16.$$

Therefore, the commutativity degree of two fuzzy subgroups is:

$$cd_F(G) = \frac{|\{(\mu, v): \quad \mu \circ v = v \circ \mu\}|}{|F_{D_{2p}}|^2} = \frac{2(p^2+9p+8)}{(2p+4)^2} = \frac{p^2+9p+8}{2p^2+8p+8}.$$

□

Finally, we have the following corollary.

Corollary 5 : *The commutativity degree of two fuzzy subgroups in group* D_{2p} *is greater than* $\frac{1}{2}$ *and less than* 0.88, *and also* $\lim_{p\to+\infty} cd_F(D_{2p}) = \frac{1}{2}$.

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