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Study of some properties of *P*-field

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Abstract

This work studies some properties of $P-field$, which is a generalization of σ–algebra, $\delta-field$ and $\lambda-$ algebra. The relationship between each of an $\alpha-field$, $\alpha-\sigma-field$, $\beta-field$, $\beta-\sigma-field$, $\pi-$system and monotone class with $P-field$ have been studied where it is shown that $P-field$ is a stronger form of these concepts.

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1. Introduction

Algebra is a significant topic in algebraic and real analysis [1]. Wang [2] and Ahmed et al. [3-5] have studied many notions of classes of sets that are generalizations of σ–field like σ–ring, ring, $field$, $\alpha-field$, $\alpha-\sigma-field$,

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$\beta - field$, $\beta - \sigma - field$ and explained some significant findings of these notions, where a non-empty class $\mathcal{M}$ is named a ring if and only if M,N $\epsilon\mathcal{M}$, then we have M\N and M$\bigcup$N$\epsilon\mathcal{M}$. Endou [6] has studied the σ–ring concept as a family $\mathcal{M}$ such that it is closed under countable union and the difference between any two sets. In 2021 Ahmed et al. [7] have been studied the $\sigma - field$ concept to define an outer measure. The idea of $\sigma - field$ and field are studied by [8] and as a family of sets $\mathcal{M}$ which satisfied the following statements: The universal set belong to $\mathcal{M}$. If $D\epsilon\mathcal{M}$, then $D^c\epsilon\mathcal{M}$. If $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. So $\bigcup_{i=1}^{\infty} \mathrm{M}_i \epsilon \mathcal{M}$.

If statement 3 holds only finite sets in $\mathcal{M}$, then we say that $\mathcal{M}$ field.

In this work, the universal set is denoted by $\mathcal{U}$ and the power set of $\mathcal{U}$ by P($\mathcal{U}$). $\mathcal{J} \subseteq$ P($\mathcal{U}$) means that $\mathcal{J}$ is a subcollection of P($\mathcal{U}$).

2. The main results

We will investigate the definition of $\mathcal{P}-$field and prove many propositions and give important examples.

Definition 2.1: Let $\mathcal{M}$ be a non−empty collection for subsets of $\mathcal{U}$. Then $\mathcal{M}$ is called $\mathcal{P} - field$ of a set $\mathcal{U}$ if the following conditions hold:

Φ ,$\mathcal{U} \epsilon \mathcal{M}$ and if $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. Thus $\bigcup_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$ and $\bigcap_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$.

Definition 2.2: A universal set $\mathcal{U}$ and let $\mathcal{M}$ be a non−empty collection for subsets of the universal set $\mathcal{U}$. So $\mathcal{M}$ is named $\mathcal{P}^* - field$ of $\mathcal{U}$ iff $\Phi \epsilon \mathcal{M}$. If N, M$\epsilon\mathcal{M}$, Hence N$\bigcap$M $\epsilon \mathcal{M}$. If $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. Thus $\bigcup_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$.

Proposition 2.3: If $\mathcal{M}$ is a $\sigma - field$, then $\mathcal{M}$ is a $\mathcal{P} - field$.

Proof: Clearly $\mathcal{U} \epsilon \mathcal{M}$, but $\mathcal{U}^c$=Φ, implies that $\Phi\epsilon \mathcal{M}$. Let $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. So $\bigcup_{i=1}^{\infty} \mathrm{M}_i \epsilon \mathcal{M}$ since $\mathcal{M}$ is a $\sigma - field$.

Let $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. Then $\mathrm{M}_1^c, \mathrm{M}_2^c, \ldots \epsilon \mathcal{M}$ & $\bigcup_{k=1}^{\infty} \mathrm{M}_k^c \epsilon \mathcal{M}$ because $\mathcal{M}$ is a $\sigma - field$, hence $(\bigcup_{k=1}^{\infty} \mathrm{M}_k^c)^c \epsilon \mathcal{M}$. By Law of De-Morgan, we will have $\bigcap_{k=1}^{\infty} \mathrm{M}_k = (\bigcup_{k=1}^{\infty} \mathrm{M}_k^c)^c$ and thus $\bigcap_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$.

Thus, $\mathcal{M}$ is $\mathcal{P} - field$ of a set $\mathcal{U}$.

Example 2.4: Let $\mathcal{U}$ =$\{m_1,m_2,m_3\}$ and $\mathcal{M}$ =$\{ \Phi, \{ m_1\}, \{m_3\}, \{m_1, m_3\}, \mathcal{U} \}$. Then $\mathcal{M}$ is $\mathcal{P} - field$ but not $\sigma - field$, since $\{m_1, m_3\} \in \mathcal{M}$ but $\{m_1, m_3\}^c = \{m_2\} \notin \mathcal{M}$.

Proposition 2.5: Let $\mathcal{P}(\mathcal{J})$ be the smallest $\mathcal{P} - field$ that includes $\mathcal{J}$ and $\sigma(\mathcal{J})$ be the smallest $\sigma - field$ that includes $\mathcal{J}$ and let $\mathcal{J} \subseteq$ P($\mathcal{U}$). Then $\mathcal{P}(\mathcal{J}) \subseteq \sigma(\mathcal{J})$.

Proof: Let $\mathcal{J} \subseteq$ P($\mathcal{U}$). So $\sigma(\mathcal{J})$ is a $\sigma - field$ of a set the universal set $\mathcal{U}$ that contains $\mathcal{J}$. Since every σ- $field$ is $\mathcal{P}$- $field$, then $\sigma(\mathcal{J})$ is a $\mathcal{P}$- $field$ that contains $\mathcal{J}$. But, $\mathcal{P}(\mathcal{J})$ is the smallest $\mathcal{P}-$ *field* of a set $\mathcal{U}$ which holds $\mathcal{J}$ this leads to $\mathcal{P}(\mathcal{J}) \subseteq \sigma(\mathcal{J})$.

Definition 2.6 [3]: Assume $\mathcal{U}$ is a nonempty set. A nonempty class $\mathcal{M}$ is called $\delta-field$ on $\mathcal{U}$, iff $\Phi\epsilon\mathcal{M}$. If M is a nonempty set in $\mathcal{M}$ and $\mathrm{M} \subset \mathrm{N} \subseteq \mathcal{U}$, then $\mathrm{N}\epsilon\mathcal{M}$. If $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon\mathcal{M}$, then $\bigcap_{k=1}^{\infty} \mathrm{M}_k \epsilon\mathcal{M}$.

Proposition 2.7: If $\mathcal{M}$ *is a* $\delta -$ *field*, So $\mathcal{M}$ is a $\mathcal{P} -$ *field*.

Proof: Assume that $\mathcal{M}$ is a $\delta -$ *field* of universal set $\mathcal{U}$. So $\Phi \epsilon \mathcal{M}$, but $\Phi \subset \mathcal{U}$, implies that $\mathcal{U}\epsilon \mathcal{M}$. Suppose that $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. Then $\bigcap_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$. Now, $\bigcap_{k=1}^{\infty} \mathrm{M}_k \subseteq \bigcup_{k=1}^{\infty} \mathrm{M}_k$ and $\bigcap_{i=1}^{\infty} \mathrm{M}_i \epsilon \mathcal{M}$, then we have, $\bigcup_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$. Therefore $\mathcal{M}$ is $\mathcal{P} -$ *field* of $\mathcal{U}$.

Example 2.8: Let $\mathcal{U}=\{m_1,m_2,m_3\}$ and $\mathcal{M} =\{ \Phi,\{ m_1\},\{m_2\},\{m_1,m_2\},\mathcal{U} \}$. So $\mathcal{M}$ be a $\mathcal{P}-field$ of a universal set $\mathcal{U}$, but it is not $\delta -$ *field*, since $\{m_2\}\in \mathcal{M}$ and $\{m_2\}\subset\{m_2,m_3\}$, but $\{m_2,m_3\}\notin \mathcal{M}$.

Proposition 2.9: Let $\delta(\mathcal{J})$ be the smallest $\delta -$ *field* that includes $\mathcal{J}$ and let $\mathcal{J} \subseteq \mathrm{P}(\mathcal{U})$. Then $\mathcal{P}(\mathcal{J}) \subseteq \delta(\mathcal{J})$.

Proof: Let $\mathcal{J} \subseteq \mathrm{P}(\mathcal{U})$. So $\delta(\mathcal{J})$ is a $\delta - field$ of $\mathcal{U}$ that contain $\mathcal{J}$. Since every $\delta -$ *field* is $\mathcal{P} -$ *field*, then $\delta(\mathcal{J})$ is a $\mathcal{P} -$ *field* that contains $\mathcal{J}$. But, $\mathcal{P}(\mathcal{J})$ be the smallest $\mathcal{P} -$ *field* of a universal set $\mathcal{U}$ which holds $\mathcal{J}$ this leads to $\mathcal{P}(\mathcal{J}) \subseteq \delta(\mathcal{J})$.

Definition 2.10 [9]: A class $\mathcal{M} \neq \{\mathcal{U}\}$ is a $\lambda-$ algebra of a nonempty set $\mathcal{U}$ iff $\mathcal{U}\epsilon\mathcal{M}$ and If $\mathrm{B}\ \epsilon\mathcal{M}$ and $\mathrm{A} \subset \mathrm{B} \subset \mathcal{U}$, then $\mathrm{A}\epsilon\mathcal{M}$. If $\mathrm{B}_1, \mathrm{B}_2, \ldots \epsilon\mathcal{M}$, then $\bigcup_{n=1}^{\infty} \mathrm{B}_n \epsilon\mathcal{M}$.

Proposition 2.11: Every $\lambda -$ algebra is a $\mathcal{P} -$ *field*.

Proof: Suppose that $\mathcal{M}$ is a $\lambda -$ algebra on $\mathcal{U}$, then $\mathcal{U} \epsilon \mathcal{M}$, but $\Phi \subset \mathcal{U}$, implies that $\Phi\epsilon \mathcal{M}$. Assume that $\mathrm{M}_1, \mathrm{M}_2, \ldots \epsilon \mathcal{M}$. Then $\bigcup_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$. Now, $\bigcup_{k=1}^{\infty} \mathrm{M}_k \subseteq \bigcap_{k=1}^{\infty} \mathrm{M}_k$ and $\bigcup_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$, then we have $\bigcap_{k=1}^{\infty} \mathrm{M}_k \epsilon \mathcal{M}$. Hence $\mathcal{M}$ is $\mathcal{P} -$ *field* on $\mathcal{U}$.

Example 2.12: Let $\mathcal{U}=\{m_1,m_2,m_3, m_4\}$ and $\mathcal{M} =\{ \Phi,\{ m_1\},\{m_3\},\{m_1,m_3\}, \{m_1, m_3,m_4\},\mathcal{U} \}$. Then $\mathcal{M}$ is a $\mathcal{P}-$ field but not $\lambda-$ algebra, since $\{m_1,m_3, m_4\}\in \mathcal{M}$, $\{m_4\}\subset\{m_1,m_3,m_4\}$, but $\{m_4\}\notin \mathcal{M}$.

Proposition 2.13: Let $\lambda(\mathcal{J})$ be the smallest $\lambda-$ algebra that holds $\mathcal{J}$ and let $\mathcal{J} \subseteq \mathrm{P}(\mathcal{U})$. Then $\mathcal{P}(\mathcal{J}) \subseteq \lambda(\mathcal{J})$.

Proof: Let $\mathcal{J} \subseteq \mathrm{P}(\mathcal{U})$. So $\lambda(\mathcal{J})$ is a λ- algebra of a universal set $\mathcal{U}$ that contains $\mathcal{J}$. Since every λ- algebra is $\mathcal{P}-$ $field$, then $\lambda(\mathcal{J})$ is a $\mathcal{P}-$ $field$ that contains $\mathcal{J}$. But, $\mathcal{P}(\mathcal{J})$ is the smallest $\mathcal{P}-$ $field$ of a set $\mathcal{U}$ which holds $\mathcal{J}$ this leads to $\mathcal{P}(\mathcal{J}) \subseteq \lambda(\mathcal{J})$.

Definition 2.14 [4]: A class $\mathcal{M}$ is called $\alpha - field$ on a nonempty set $\mathcal{U}$ iff $\mathcal{U} \epsilon \mathcal{M}$ and if $\mathrm{M}_1, \mathrm{M}_2, \dots, \mathrm{M}_\mathrm{k} \, \epsilon \mathcal{M}$, then $\bigcup_{\mathrm{n}=1}^{\mathrm{k}} \mathrm{M}_\mathrm{n} \, \epsilon \mathcal{M}$.

Remark 2.15: If $\mathcal{M}$ is a $\mathcal{P} -$ $field, then$ $\mathcal{M}$ is an $\alpha -$ $field$.

Example 2.16: Let $\mathcal{U}$ =$\{m_1, m_2, m_3, m_4\}$ and $\mathcal{M}$ =$\{ \Phi, \{ m_1, m_3\}, \{m_3, m_4\}, \{m_1, m_3, m_4\}, \mathcal{U} \}$, then $\mathcal{M}$ is α–field, but it is not $\mathcal{P}$–field, since $\{ m_1, m_3\}, \{m_3, m_4\} \in \mathcal{M}$ but $\{ m_1, m_3\} \cap \{m_3, m_4\}$=$\{m_3\} \notin \mathcal{M}$.

Definition 2.17 [3]: A class $\mathcal{M}$ is called α- σ- field on a non−empty universal set $\mathcal{U}$ iff $\Phi, \mathcal{U} \epsilon \mathcal{M}$ and if $\mathrm{M}_1, \mathrm{M}_2, \dots \, \epsilon \mathcal{M}$, then $\bigcup_{\mathrm{n}=1}^{\infty} \mathrm{M}_\mathrm{n} \, \epsilon \mathcal{M}$.

Proposition 2.18: If $\mathcal{M}$ is a $\mathcal{P} -$ $field, then$ $\mathcal{M}$ is a α- σ- $field$.

Proof: By the definition of $\mathcal{P} -$ $field$ directly, we prove this proposition.

Example 2.19: Let $\mathcal{U}$ =$\{m_1, m_2, m_3, m_4, m_5\}$ and

$$\mathcal{M} = \{ \Phi, \{ m_1, m_2, m_3\}, \{m_1, m_2, m_4\}, \{m_1, m_2, m_3, m_4\}, \mathcal{U} \}.$$

Then $\mathcal{M}$ $\alpha - \sigma - field$, but not $\mathcal{P} -$ $field$, since $\{ m_1, m_2, m_3\} \cap \{m_1, m_2, m_4\} = \{m_1, m_2\} \notin \mathcal{M}$.

Definition 2.20 [4]: A class $\mathcal{M}$ is called $\beta - field$ of a nonempty set $\mathcal{U}$, iff $\Phi \epsilon \mathcal{M}$ and $\bigcap_{\mathrm{n}=1}^{\mathrm{k}} \mathrm{M}_\mathrm{n} \, \epsilon \mathcal{M}$ whenever $\mathrm{M}_1, \mathrm{M}_2, \dots, \mathrm{M}_\mathrm{k} \, \epsilon \mathcal{M}$

Proposition 2.21: If $\mathcal{M}$ is a $\mathcal{P} -$ $field, then$ $\mathcal{M}$ is a $\beta - field$.

Proof: Clearly that $\Phi, \mathcal{U} \in \mathcal{M}$. Let $\mathrm{M}_1, \mathrm{M}_2, \dots, \mathrm{M}_\mathrm{n} \in \mathcal{M}$. Put, $\mathrm{M}_\mathrm{k} = \mathcal{U} \; \forall k > n$, So $\bigcap_{\mathrm{i}=1}^{\infty} \mathrm{M}_\mathrm{i} \in \mathcal{M}$, but $\bigcap_{\mathrm{i}=1}^{\infty} \mathrm{M}_\mathrm{i} = \mathrm{M}_1 \cap \mathrm{M}_2 \cap \dots \cap \mathrm{M}_\mathrm{n} \cap \mathcal{U} \cap \mathcal{U} \cap \dots = \bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{M}_\mathrm{i}$ and $\bigcap_{\mathrm{i}=1}^{\infty} \mathrm{M}_\mathrm{i} \in \mathcal{M}$ thus $\bigcap_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{M}_\mathrm{i} \in \mathcal{M}$. Therefore, $\mathcal{M}$ is $\beta - field$ on $\mathcal{U}$.

Example 2.22: Let $\mathcal{U}$ =$\{m_1, m_2, m_3, m_4\}$ and $\mathcal{M}$ =$\{\Phi, \{ m_2\}, \{ m_1, m_2\}, \{m_2, m_3\}, \mathcal{U} \}$, then $\mathcal{M}$ $\beta - field$, but not $\mathcal{P} - field$,

Definition 2.23 [3]: A class $\mathcal{M}$ is called $\beta - \sigma - field$ of a nonempty set $\mathcal{U}$, iff $\Phi, \mathcal{U} \, \epsilon \mathcal{M}$ and $\bigcap_{\mathrm{n}=1}^{\infty} \mathrm{M}_\mathrm{n} \, \epsilon \mathcal{M}$ for $\mathrm{M}_1, \mathrm{M}_2, \dots \, \epsilon \mathcal{M}$.

Proposition 2.24: If $\mathcal{M}$ is a $\mathcal{P} -$ $field, then$ $\mathcal{M}$ is a $\beta - \sigma - field$.

Proof: The result immediately by definition of $\mathcal{P}-$ $field$.

While in general, the converse need not be true, as shown in the example below.

Example 2.25: Let $\mathcal{U}=\{m_1,m_2,m_3, m_4, m_5\}$ and

$$\mathcal{M} =\{ \Phi,\{ m_1, m_2,m_3\},\{m_1, m_2,m_4\},\{m_1, m_2\},\mathcal{U} \}.$$

Then $\mathcal{M}$ is $\beta-\sigma-field$, but it is not a $\mathcal{P}-\ field$,

Definition 2.26 [10]: A class $\mathcal{M}$ is $\pi-$system, if $M_1 \cap M_2 \in \mathcal{M}$ for $M_1, M_2 \in \mathcal{M}$.

Proposition 2.27 [10]: Every $\mathcal{P}-\ field$ is a $\pi-$system.

Proof: Assume that $\mathcal{M}$ is a $\mathcal{P}-\ field$, then $\mathcal{U} \in \mathcal{M}$. Let $M_1, M_2 \in \mathcal{M}$. put, $M_k = \mathcal{U}\ \forall k > 2$, then $\bigcap_{i=1}^{\infty} M_i \in \mathcal{M}$, but $\bigcap_{i=1}^{\infty} M_i = M_1 \cap M_2 \cap \mathcal{U} \cap \ldots = M_1 \cap M_2$, thus $M_1 \cap M_2 \in \mathcal{M}$. So, $\mathcal{M}$ is $\pi-$system.

Example 2.28: Let $\mathcal{U}=\{m_1,m_2,m_3,m_4\}$ and $\mathcal{M} =\{\Phi,\{ m_3\},\{ m_1,m_3\},\{m_2, m_3\},\mathcal{U} \}$. Then $\mathcal{M}$ is a $\pi-$system, but it is not $\mathcal{P}-$field, since $\{ m_1,m_3\} \cup \{m_2,m_3\}=\{m_1,m_2,m_3\} \notin \mathcal{M}$.

Definition 2.29 [2]: A nonempty collection $\mathcal{M}$ of s of a nonempty that subset of $\mathcal{U}$ is named monotone class iff whenever $M_1, M_2, \ldots \in \mathcal{M}$ like $M_i \uparrow M$, then $M \in \mathcal{M}$ Also if $M_i \downarrow M$, then $M \in \mathcal{M}$.

Proposition 2.30: If $\mathcal{M}$ is a $\mathcal{P}-\ field$, then it is a monotone class.

Proof: Assume $\mathcal{M}$ is a $\mathcal{P}-\ field$ on universal set $\mathcal{U}$ and $M_1, M_2, \ldots \in \mathcal{M}$ such that $M_i \uparrow M$. Then $\bigcup_{i=1}^{\infty} M_i = M$ Since $\mathcal{M}$ is a $-field$, So by using the definition of $\mathcal{M}$, we will have $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$ which leads to $M \in \mathcal{M}$. Let $M_1, M_2, \ldots \in \mathcal{M}$ such that $M_i \downarrow M$. Then $\bigcap_{i=1}^{\infty} M_i = M$, but $\mathcal{M}$ is a $\mathcal{P}-\ field$, hence $\bigcap_{i=1}^{\infty} M_i \in \mathcal{M}$ and $M \in \mathcal{M}$. Thus $\mathcal{M}$ is a monotone class.

The next example shows that the converse of proposition(2.30) generally is untrue.

Example 2.31: If $\mathcal{U}=\{m_1,m_2,m_3\}$ and $\mathcal{K} =\{ \Phi,\{ m_1\},\{m_1, m_2\} \}$, then $\mathcal{M}$ is a monotone class, but not $\mathcal{P}-\ field$, because$\mathcal{U} \notin \mathcal{M}$.

Definition 2.32 [2]: Let $\mathcal{J} \subseteq P(\mathcal{U})$. Then the intersection of all monotone classes of $\mathcal{U}$ that contain $\mathcal{J}$ is called the monotone class created by $\mathcal{J}$ and it is denoted by $\mathbb{M}(\mathcal{J})$.

Lemma 2.33 [2]: *Let $\{\mathcal{M}_i\}_{i\in I}$be a collection of monotone classes on $\mathcal{U}$. Then $\bigcap_{i\in I} \mathcal{M}_i$ is a monotone class on $\mathcal{U}$.*

Proposition 2.34 [2]: Let $\mathcal{J} \subseteq P(\mathcal{U})$. Then $\mathbb{M}(\mathcal{J})$ is the smallest monotone class of $\mathcal{U}$ that contains $\mathcal{J}$.

Proposition 2.35: Let $\mathcal{J} \subseteq P(\mathcal{U})$. Then $\mathbb{M}(\mathcal{J}) \subseteq \mathcal{P}(\mathcal{J})$.

Proof: Let $\mathcal{J} \subseteq P(\mathcal{U})$. Then $\mathcal{P}(\mathcal{J})$ is a $\mathcal{P}$-field of $\mathcal{U}$ which hold $\mathcal{J}$. Since every $\mathcal{P}$-field is a monotone class; it leads to $\mathcal{P}(\mathcal{J})$ is a monotone class which contains $\mathcal{J}$. But $\mathbb{M}(\mathcal{J})$ is the smallest monotone class which includes $\mathcal{J}$, then $\mathbb{M}(\mathcal{J}) \subseteq \mathcal{P}(\mathcal{J})$.

Proposition 2.36: If $\mathcal{M}$ is a $\mathcal{P}-$ $field$, then it is a $\mathcal{P}^*-$ $field$.

Proof: Suppose a $\mathcal{M}$ is a $\mathcal{P}$-field on universal set $\mathcal{U}$, So $\Phi, \mathcal{U} \in \mathcal{M}$. Let $M_1, M_2 \in \mathcal{M}$. put, $M_k = \mathcal{U}\ \forall k > 2$, then $M_1, M_2, \dots, M_k, \dots \in \mathcal{M}$, hence $\bigcap_{i=1}^{\infty} M_i \in \mathcal{M}$, but $\bigcap_{i=1}^{\infty} M_i = M_1 \cap M_2 \cap \mathcal{U} \cap \mathcal{U} \cap \dots = M_1 \cap M_2$, thus $M_1 \cap M_2 \in \mathcal{M}$. Assume $M_1, M_2, \dots \in \mathcal{M}$, hence $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$.Therefore, $\mathcal{M}$ is $\mathcal{P}^*-$ $field$ on $\mathcal{U}$.

Example 2.37: Let $\mathcal{U}=\{m_1,m_2,m_3,m_4\}$ and $\mathcal{M} =\{ \Phi,\{ m_2\},\{ m_1,m_2\},\{m_2, m_3\},\{m_1,m_2,m_3\} \}$. So $\mathcal{M}$ is a $\mathcal{P}^*-$ $field$, but it is not $\mathcal{P}-$ $field$, since $\mathcal{U} \notin \mathcal{M}$.

The following examples show that the relation among $\mathcal{P}-$ $field$ and $field$ are independent.

Example 2.38: Suppose that $\mathcal{U}=\{ m_1,m_2,m_3\}$ and $\mathcal{M} =\{ \Phi,\{ m_1\},\{m_3\},\{m_1, m_3\},\mathcal{U} \}$. So $\mathcal{M}$ is a $\mathcal{P}$-field of a set $\mathcal{U}$. On the other hand, $\mathcal{M}$ is not a field since $\{m_1, m_3\} \in \mathcal{M}$ but $\{m_1, m_3\}^c = \{m_2\} \notin \mathcal{M}$.

Example 2.39: Let $\mathcal{U} = \mathbb{R}$ and $\mathcal{M}$ consist of all finite unions of disjoint right – semi-closed intervals. Then $\mathcal{M}$ is a field, but $\mathcal{M}$ is not a $\mathcal{P}$-field. Because if we take $M_n = (0, 1- (1/n)]$,n=1,2,… then $M_n \in \mathcal{M}$, but $\bigcup_{n=1}^{\infty} M_n = (0,1) \notin \mathcal{M}$.

The example below indicates that the relation between $\mathcal{P}$-field and σ- ring are independent.

Example 2.40: Let $\mathcal{U}=\{m_1,m_2,m_3\}$ and $\mathcal{M} =\{ \Phi,\{ m_1\},\{m_3\},\{m_1, m_3\}\}$. Then it is clear that $\mathcal{M}$ is σ-ring but not $\mathcal{P}-$ $field$, because $\mathcal{U} \notin \mathcal{M}$.

Example 2.41: Let $\mathcal{U}=\{m_1,m_2,m_3\}$ and $\mathcal{M} =\{\Phi,\{ m_1\},\{m_3\},\{m_1, m_3\}, \mathcal{U} \}$. Then it is clear that $\mathcal{M}$ is $\mathcal{P}-$ $field$, but $\mathcal{M}$ is not a σ-ring because $\mathcal{U}, \{m_3\} \in \mathcal{M}$, but $\mathcal{U} \setminus \{m_3\} =\{m_1, m_2\} \notin \mathcal{M}$.

Proposition 2.42: Every σ-ring containing $\mathcal{U}$ is a $\mathcal{P}-$ $field$, while the converse need not be true, as shown in the above example.

Proof: Assume $\mathcal{M}$ is a σ-ring containing $\mathcal{U}$ and $M \in \mathcal{M}$, then $M \setminus M \in \mathcal{M}$. Hence $\Phi, \mathcal{U} \in \mathcal{M}$. Let $M_1, M_2, \dots \in \mathcal{M}$. From the definition of σ-ring, we have $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$. Let $M_1, M_2, \dots \in \mathcal{M}$. Then $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$. Consider $M =\bigcup_{i=1}^{\infty} M_i$.

Since $\bigcap_{i=1}^{\infty} M_i = M \setminus \bigcup_{i=1}^{\infty}(M \setminus M_i)$ and $M \setminus \bigcup_{i=1}^{\infty}(M \setminus M_i) \in \mathcal{M}$. Then, $\bigcap_{i=1}^{\infty} M_i \in \mathcal{M}$. Therefore, $\mathcal{M}$ is a $\mathcal{P} - field$ on $\mathcal{U}$.

The examples below indicate that the relation between $\mathcal{P}$-field and ring are independent.

Example 2.43: Let $\mathcal{U} = \{m_1, m_2, m_3\}$ and $\mathcal{M} = \{\Phi, \{m_1\}, \{m_2\}, \{m_1, m_2\}\}$. Then it is clear that $\mathcal{M}$ is a ring but not $\mathcal{P} - field$, because $\mathcal{U} \notin \mathcal{M}$.

Example 2.44: Let $\mathcal{U} = \{m_1, m_2, m_3\}$ and $\mathcal{M} = \{\Phi, \{m_1\}, \{m_2\}, \{m_1, m_2\}, \mathcal{U}\}$. Then it is clear that $\mathcal{M}$ is $\mathcal{P} - field$ but not ring because $\mathcal{U}, \{m_2\} \in \mathcal{M}$, but $\mathcal{U} \setminus \{m_2\} = \{m_1, m_3\} \notin \mathcal{M}$.

Definition 2.45 [11]: Let $\mathcal{M}$ be a non-empty family of subsets of a set $\mathcal{U}$. Then $\mathcal{M}$ is named a $\boldsymbol{\Gamma}$ –algebra if,

1. $\Phi, \mathcal{U} \in \mathcal{M}$.
2. If $D \in \mathcal{M}$ and there exist $\Phi \neq E_i \subset D \subset \mathcal{U}$, then at least one of E_i's$\in \mathcal{M}$.
3. If $A_1, A_2, \ldots \in \mathcal{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

The relation between $\mathcal{P} - field$ and Γ–algebra are independent, as shown below.

Example 2.46: If $\mathcal{U} = \{a, b, c, d\}$ and

$$\mathcal{M} = \begin{Bmatrix} \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{b, c\}, \\ \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \Phi, \mathcal{U} \end{Bmatrix},$$ then $\mathcal{M}$ is a Γ–algebra but not $\mathcal{P}$–field, since $\{a, d\}, \{b, d\} \in \mathcal{M}$, but $\{a, d\} \cap \{b, d\} = \{d\} \notin \mathcal{M}$.

Example 2.47: If $\mathcal{U} = \{a, b, c, d\}$ and $\mathcal{M} = \{\{a, b, c\}, \Phi, \mathcal{U}\}$, then $\mathcal{M}$ is $\mathcal{P}$–field but not Γ–algebra because any subsets of $\{a, b, c\}$ does not belong to $\mathcal{M}$.

Conclusion

The main results obtained are every $\sigma - field$ is $\mathcal{P} - field$ and if $\mathcal{I} \subseteq P(\mathcal{U})$, then $\mathcal{P}(\mathcal{I}) \subseteq \sigma(\mathcal{I})$ and $\mathcal{P}(\mathcal{I}) \subseteq \delta(\mathcal{I})$. Moreover, every $\mathcal{P} - field$ is a monotone class and a relation between $\mathcal{P} - field$ and σ- the ring is independent.

References

[1] A. S. Mohammed, S. H. Asaad, and I. S. Ahmed, "New properties of strongly regular left almost semi-group," *J. Discret. Math. Sci. Cryptogr.*, vol. 25, no. 2, pp. 615–622 (2022), doi: 10.1080/09720529.2021.1982491.

[2] Z. Wang and G. Klir, Generalized Measure Theory, 1st ed. Springer Science and Business Media, LLC, New York (2009).

[3] I. S. Ahmed and H. H. Ebrahim, "Generalizations of σ-field and new collections of sets noted by δ-field," AIP Conf. Proc., vol. 2096, no. 20019, p. (020019-1)-(020019-6) (2019), doi: 10.1063/1.5097816.

[4] I. S. Ahmed and H. H. Ebrahim, "On α-field and β-field," *J. Phys. Conf. Ser.,* vol. 1294, no. 3 (2019), doi: 10.1088/1742-6596/1294/3/032015.

[5] S. H. Asaad, I. S. Ahmed, and H. H. Ebrahim, "On Finitely Null-additive and Finitely Weakly Null-additive Relative to the σ–ring," *Baghdad Sci. J.,* no. March, pp. 1148–1154 (2022), doi: 10.21123/bsj.2022.5771.

[6] N. Endou, K. Nakasho, and Y. Shidama, "σ-ring and σ-algebra of Sets," *Formaliz. Math.*, vol. 23, no. 1, pp. 51–57 (2015), doi: 10.2478/forma-2015-0004.

[7] I. S. Ahmed, S. H. Asaad, and H. H. Ebrahim, "Some new properties of an outer measure on a σ–field," *J. Interdiscip. Math.*, vol. 24, no. 4, pp. 947–952 (2021), doi: 10.1080/09720502.2021.1884386.

[8] R. B. ASH, Real Analysis and Probability, 1st ed. Academic Press, Inc. (London) Ltd. (1972).

[9] H. H. Ebrahim and R. Ali, "λ–Algebra With Some Of Their Properties," *Ibn AL- Haitham J. Pure Appl. Sci.*, vol. 33, no. 2, pp. 72–80 (2020).

[10] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties, 1st ed. CRC Press, New York (2015).

[11] M. M. Abbood, A. Al-fayadh, and H. H. Ebrahim, "On studying bi- Γ -algebra and some related concepts," vol. 13, no. October 2021, pp. 1565–1572 (2022).

[12] Kareem, F. S., & Shlaka, H. M. : Jordan-Lie Inner Ideals of the Orthogonal Lie Algebras. *Wasit Journal of Computer and Mathematics Sciences,* 1(2), 34-54 (2022).

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