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Soft P^*–field and some of its properties

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Abstract

This article introduces the soft P^*–field. Characterization, examples, and different properties of the proposed concept are presented. The idea of the smallest soft P^*–field studies.

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1. Introduction

Algebra is a significant topic in the field of algebraic and real analysis [1]. Wang [2] and Ahmed et al.[3-7] have studied many types of classes of sets which are generalizations of σ–algebra such as σ–ring, ring, $algebra$, $\alpha-\sigma-algebra$, $\beta-\sigma-algebra$ and explained some important results of these

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notions. Endou [8] has been studied the σ–ring concept as a family $\mathcal{M}$ such that it is closed under countable union and difference between any two sets. In 2021 Ahmed et al [9] has been studied the $\sigma - algebra$ concept to define an outer measure. The idea of $\sigma - algebra$ and algebra are studied by [10]. The notion of soft set was studied by [11] where a universal set $\mathcal{U}$ and the set of parameters $\mathcal{E}$ with respect to $\mathcal{U}$. Let $\mathcal{A}$ be a nonempty subset of $\mathcal{E}$. Then the set of ordered pairs $\mathcal{F}_{\mathcal{A}} = \{(x, \mathfrak{f}_{\mathcal{A}}(x)): x \in \mathcal{E}, \mathfrak{f}_{\mathcal{A}}(x) \in \mathrm{P}(\mathcal{U})\}$ is called a soft set on $\mathcal{U}$, where $\mathfrak{f}_{\mathcal{A}}: \mathcal{E} \longrightarrow \mathrm{P}(\mathcal{U})$ such that $\mathfrak{f}_{\mathcal{A}}(x) = \Phi$ if $x \notin \mathcal{A}$. The soft set $\mathcal{F}_{\mathcal{A}}$ is called null soft set and denoted by $\mathcal{F}_{\Phi}$ or $\widetilde{\Phi}$, if $\mathfrak{f}_{\mathcal{A}}(x) = \Phi$ $\forall x \in \mathcal{A}$. A soft set $\mathcal{F}_{\mathcal{A}}$ over $\mathcal{U}$ is named $\mathcal{A}$ –universal soft set and denoted by $\mathcal{F}_{\tilde{\mathcal{A}}}$ or $\tilde{\mathcal{A}}$, if $\mathfrak{f}_{\mathcal{A}}(x) = \mathcal{U}$ $\forall x \in \mathcal{A}$.

If $\mathcal{A} = \mathcal{E}$, then the $\mathcal{A}$ –universal soft set is called a universal soft set and is denoted by $\mathcal{F}_{\tilde{\mathcal{E}}}$ or $\tilde{\mathcal{E}}$. The soft union of $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}$ is symbolized by $\mathcal{F}_{\mathcal{A}} \widetilde{\cup} \mathcal{F}_{\mathcal{B}}$ and defined by the approximate function $\mathfrak{f}_{\mathcal{A}}(x) \cup \mathfrak{f}_{\mathcal{B}}(x)$ $\forall x \in \mathcal{E}$. The soft intersection of $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}$ is symbolized by $\mathcal{F}_{\mathcal{A}} \widetilde{\cap} \mathcal{F}_{\mathcal{B}}$ and defined by the approximate function $\mathfrak{f}_{\mathcal{A}}(x) \cap \mathfrak{f}_{\mathcal{B}}(x)$ $\forall x \in \mathcal{E}$.

2. The main results

We will involve the definition of soft set in the field, so we introduced the definition of $soft\ \mathcal{P}^* - field$ and proof many propositions and give important examples

Definition 2.1: A universal set $\mathcal{U}$ and the set of parameters $\mathcal{E}$ with respect to $\mathcal{U}$ and let $\widetilde{\mathcal{M}}$ be a class of soft subsets of $\mathcal{F}_{\tilde{\mathcal{E}}}$ ($\widetilde{\mathcal{M}} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$). Then $\widetilde{\mathcal{M}}$ is called soft $\mathcal{P}^*$– field on $\mathcal{U}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ for short (soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$) if the following statement holds:

1- $\mathcal{F}_{\Phi} \in \widetilde{\mathcal{M}}$.

2- If $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2} \in \widetilde{\mathcal{M}}$, then $\mathcal{F}_{\mathrm{M}_1} \widetilde{\cap}\, \mathcal{F}_{\mathrm{M}_2} \in \widetilde{\mathcal{M}}$.

3- If $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}, \ldots \in \widetilde{\mathcal{M}}$, . then $\widetilde{\bigcup_{\mathrm{i}=1}^{\infty}}\ \mathcal{F}_{\mathrm{M}_\mathrm{i}} \in \widetilde{\mathcal{M}}$.

Proposition 2.2: Let $\{\widetilde{\mathcal{M}}_\alpha\}_{\alpha \in \mathrm{I}}$ be a collection of soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Then $\widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha$ also is soft $\mathcal{P}^*$– field.

Proof: If $\widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha = \{\mathcal{F}_{\Phi}\}$, we are done. So, assume $\widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha \neq \{\mathcal{F}_{\Phi}\}$. Since $\widetilde{\mathcal{M}}_\alpha$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ $\forall \alpha \in \mathrm{I}$, then $\mathcal{F}_{\Phi} \in \widetilde{\mathcal{M}}_\alpha$ $\forall \alpha \in \mathrm{I}$. Hence $\mathcal{F}_{\Phi} \in \widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha$.

Let $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2} \in \widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha$. Then $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2} \in \widetilde{\mathcal{M}}_\alpha$ $\forall \alpha \in \mathrm{I}$, since $\widetilde{\mathcal{M}}_\alpha$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ $\forall \alpha \in \mathrm{I}$, then $\mathcal{F}_{\mathrm{M}_1} \widetilde{\cap}\, \mathcal{F}_{\mathrm{M}_2} \in \widetilde{\mathcal{M}}_\alpha \forall \alpha \in \mathrm{I}$. But the soft intersection of $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}$ is defined by using an approximate function $\mathfrak{f}_{\mathrm{M}_1}(x) \cap \mathfrak{f}_{\mathrm{M}_2}(x) \forall x \in \mathcal{E}$. That is,

$\mathcal{F}_{\mathrm{M}_1} \widetilde{\cap}\, \mathcal{F}_{\mathrm{M}_2} = \{(x, \mathfrak{f}_{\mathrm{M}_1}(x) \cap \mathfrak{f}_{\mathrm{M}_2}(x)): x \in \mathcal{E}\}$.So,$\{(x, \mathfrak{f}_{\mathrm{M}_1}(x) \cap \mathfrak{f}_{\mathrm{M}_2}(x)): x \in \mathcal{E}\} \in \widetilde{\mathcal{M}}_\alpha$ $\forall \alpha \in \mathrm{I}$. Thus $\{(x, \mathfrak{f}_{\mathrm{M}_1}(x) \cap \mathfrak{f}_{\mathrm{M}_2}(x)) : x \in \mathcal{E}\} \in \widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha$, that is,

$\mathcal{F}_{\mathrm{M}_1} \widetilde{\cap}\, \mathcal{F}_{\mathrm{M}_2} \in \widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha$. Let $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}, \ldots \in \widetilde{\bigcap_{\alpha \in \mathrm{I}}}\ \widetilde{\mathcal{M}}_\alpha$.

Then $\mathcal{F}_{M_1}, \mathcal{F}_{M_2}, \ldots \in \widetilde{\mathcal{M}}_\alpha\ \forall\alpha\in I$, since $\widetilde{\mathcal{M}}_\alpha$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}\ \forall\alpha\in I$, then $\widetilde{\bigcup_{i=1}^{\infty}}\ \mathcal{F}_{M_i} \in \widetilde{\mathcal{M}}_\alpha \forall\alpha\in I$. But the soft union of $\mathcal{F}_{M_1}, \mathcal{F}_{M_2}$ is defined by using an approximate function $\bigcup_{i=1}^{\infty} \mathfrak{f}_{M_i}(x)\ \forall x\in\mathcal{E}$. $\widetilde{\bigcup_{i=1}^{\infty}}\ \mathcal{F}_{M_i} = \{(x, \bigcup_{i=1}^{\infty} \mathfrak{f}_{M_i}(x)) : x\in\mathcal{E}\}$.

Hence $\{(x, \bigcup_{i=1}^{\infty} \mathfrak{f}_{M_i}(x))\ : x\in\mathcal{E}\} \in \widetilde{\mathcal{M}}_\alpha \forall\alpha\in I$. So $\{(x, \bigcup_{i=1}^{\infty} \mathfrak{f}_{M_i}(x)) : x\in\mathcal{E}\} \in \widetilde{\bigcap_{\alpha\in I}}\ \widetilde{\mathcal{M}}_\alpha$, that is, $\widetilde{\bigcup_{i=1}^{\infty}}\ \mathcal{F}_{M_i}\ \in \widetilde{\bigcap_{\alpha\in I}}\ \widetilde{\mathcal{M}}_\alpha$. Therefore $\widetilde{\bigcap_{\alpha\in I}}\ \widetilde{\mathcal{M}}_\alpha$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Definition 2.3: A universal set $\mathcal{U}$ and the set of parameters $\mathcal{E}$ with respect to $\mathcal{U}$. If $\tilde{\mathcal{J}} \subseteq \tilde{P}(\mathcal{F}_{\tilde{\mathcal{E}}})$. Then the soft intersection of all soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, that contains $\tilde{\mathcal{J}}$ is named the soft $\mathcal{P}^*$– field generated by $\tilde{\mathcal{J}}$ and denoted by $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, that is,

$$\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} = \widetilde{\bigcap} \{\widetilde{\mathcal{M}}_\alpha : \widetilde{\mathcal{M}}_\alpha \text{is a soft } \mathcal{P}^*\text{– field on } \mathcal{F}_{\tilde{\mathcal{E}}} \text{ and } \tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{M}}_\alpha \forall\alpha \in I\}.$$

Proposition 2.4: $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is the smallest soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contains $\tilde{\mathcal{J}}$.

Proof: $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} = \widetilde{\bigcap} \{\widetilde{\mathcal{M}}_\alpha : \widetilde{\mathcal{M}}_\alpha$is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{M}}_\alpha\ \forall\alpha \in I\}$, then from Proposition 2.2 $\Rightarrow \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Let $\widetilde{\mathcal{M}}_\alpha$ be a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{M}}_\alpha \forall\alpha \in I$. Then $\tilde{\mathcal{J}}\ \subseteq \widetilde{\bigcap}\, \widetilde{\mathcal{M}}_\alpha$. Therefore, $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. Assume $\widetilde{\mathcal{M}^{\#}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ where $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{M}}^{\#}$. So $\widetilde{\bigcap} \{\widetilde{\mathcal{M}}_\alpha : \widetilde{\mathcal{M}}_\alpha$is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathcal{J}} \subseteq\ \widetilde{\mathcal{M}}_\alpha\ \ \forall\alpha \in I\} \subseteq \widetilde{\mathcal{M}}^{\#}$, hence $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \subseteq \widetilde{\mathcal{M}}^{\#}$. Therefore, $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is smallest soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}$.

Example 2.5: Let $\mathcal{U} = \{u_1, u_2\}$ and $\mathcal{E} = \{e_1, e_2\}$.

If $\tilde{\mathcal{J}}$={ $\{(e_1, \{u_1\})\}, \{(e_1, \{u_2\})\}\}$.

Then $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} = \{\, \mathcal{F}_\Phi, \{(e_1, \{u_1\})\}, \{(e_1, \{u_2\})\}, \{(e_1, \mathcal{U}\,)\}\}$ is the smallest soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}$.

Proposition 2.6: If $\tilde{\mathcal{J}} \subseteq \tilde{P}(\mathcal{F}_{\tilde{\mathcal{E}}})$, then $\tilde{\mathcal{J}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ if and only if $\tilde{\mathcal{J}} = \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$.

Proof: Assume $\tilde{\mathcal{J}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathcal{J}}\ \subseteq \tilde{\mathcal{J}}$. Since $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is the smallest soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, then $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \subseteq\ \tilde{\mathcal{J}}$. But $\tilde{\mathcal{J}}\ \subseteq\ \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ from definition of $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. Thus $\tilde{\mathcal{J}} =\ \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$.

Conversely: Let $\tilde{\mathcal{J}} = \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. From Proposition 4.1.8, we have $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, hence $\tilde{\mathcal{J}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Theorem 2.7: *If* $\tilde{\mathcal{J}} \subseteq \tilde{P}(\mathcal{F}_{\tilde{\mathcal{E}}})$ *and* $\mathcal{F}_C \mathrel{\tilde{\subseteq}} \mathcal{F}_{\tilde{\mathcal{E}}}$. *Then* $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \widetilde{\cap} \mathcal{F}_C = \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}}\widetilde{\cap}\mathcal{F}_C})$.

Proof: First, we show that $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}\widetilde{\cap}\mathcal{F}_C$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}\widetilde{\cap}\mathcal{F}_C$. We know that, $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}\widetilde{\cap}\mathcal{F}_C = \{\mathcal{F}_M = \mathcal{F}_N\widetilde{\cap}\mathcal{F}_C : \mathcal{F}_N \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}\}$. Since $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, then $\mathcal{F}_\Phi \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. Now, $\mathcal{F}_\Phi = \mathcal{F}_\Phi \tilde{\sqcap} \mathcal{F}_C$, then $\mathcal{F}_\Phi \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Let $\mathcal{F}_{M_1}, \mathcal{F}_{M_2} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Then $\mathcal{F}_{M_1} = \mathcal{F}_{N_1} \tilde{\sqcap} \mathcal{F}_C$ and

$\mathcal{F}_{M_2} = \mathcal{F}_{N_2} \tilde{\sqcap} \mathcal{F}_C$ where $\mathcal{F}_{N_1}, \mathcal{F}_{N_2} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. So, we have

$\mathcal{F}_{M_1} \tilde{\sqcap} \mathcal{F}_{M_2} = (\mathcal{F}_{N_1} \tilde{\sqcap} \mathcal{F}_C) \tilde{\sqcap} (\mathcal{F}_{N_2} \tilde{\sqcap} \mathcal{F}_C) = (\mathcal{F}_{N_1} \tilde{\sqcap} \mathcal{F}_{N_2}) \tilde{\sqcap} \mathcal{F}_C$.

Since $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{N_1}, \mathcal{F}_{N_2} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, then $\mathcal{F}_{N_1} \tilde{\sqcap} \mathcal{F}_{N_2} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, hence by the definition of $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$, we get $\mathcal{F}_{M_1} \tilde{\sqcap} \mathcal{F}_{M_2} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Let $\mathcal{F}_{M_1}, \mathcal{F}_{M_2}, \ldots \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Then $\mathcal{F}_{M_j} = \mathcal{F}_{N_j} \tilde{\sqcap} \mathcal{F}_C$ $\forall$ j = 1,2, ...

Where $\mathcal{F}_{N_j} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. So, $\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{M_j} = \widetilde{\bigcup_{j=1}^{\infty}} (\mathcal{F}_{N_j} \tilde{\sqcap} \mathcal{F}_C) = (\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{N_j}) \tilde{\sqcap} \mathcal{F}_C$

Since $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}, \ldots \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, then $\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{N_j} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ and hence $\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{M_j} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Therefore, $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$. To show $\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Let $\mathcal{F}_D \in \tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C$, then $\mathcal{F}_D = \mathcal{F}_Z \tilde{\sqcap} \mathcal{F}_C$ where $\mathcal{F}_Z \in \tilde{\mathcal{J}}$, but $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, thus $\mathcal{F}_Z \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ and hence $\mathcal{F}_Z \tilde{\sqcap} \mathcal{F}_C \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$, so $\mathcal{F}_D \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. Therefore, $\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$.

Now, since $\mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$ is smallest soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C$, then $\mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C}) \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C$. To prove $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\sqcap} \mathcal{F}_C \subseteq \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$. Define a class $\tilde{\mathfrak{R}}$ as follows: $\tilde{\mathfrak{R}} = \{\mathcal{F}_H : \mathcal{F}_H \tilde{\sqcap} \mathcal{F}_C \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C}) \text{ and } \mathcal{F}_H \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}\}$ and we claim that $\tilde{\mathfrak{R}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathfrak{R}}$ contain $\tilde{\mathcal{J}}$. Since $\mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C}), \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ are soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, then $\mathcal{F}_\Phi \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$ and $\mathcal{F}_\Phi \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$.

Now, $\mathcal{F}_\Phi = \mathcal{F}_\Phi \tilde{\sqcap} \mathcal{F}_C$, then $\mathcal{F}_\Phi \in \tilde{\mathfrak{R}}$. Let $\mathcal{F}_{H_1}, \mathcal{F}_{H_2} \in \tilde{\mathfrak{R}}$. Then $\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_C \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$, $\mathcal{F}_{H_1} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ and $\mathcal{F}_{H_2} \tilde{\sqcap} \mathcal{F}_C \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$, $\mathcal{F}_{H_2} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. So, we have

$(\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_{H_2}) \tilde{\sqcap} \mathcal{F}_C = (\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_C) \tilde{\sqcap} (\mathcal{F}_{H_2} \tilde{\sqcap} \mathcal{F}_C)$. Since $\mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C}), \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ are soft $\mathcal{P}^*-$field, then $(\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_{H_2}) \tilde{\sqcap} \mathcal{F}_C = (\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_C) \tilde{\sqcap} (\mathcal{F}_{H_2} \tilde{\sqcap} \mathcal{F}_C) \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$ and $(\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_{H_2}) \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, hence by the definition of $\tilde{\mathfrak{R}}$, we get $(\mathcal{F}_{H_1} \tilde{\sqcap} \mathcal{F}_{H_2}) \in \tilde{\mathfrak{R}}$.

Let $\mathcal{F}_{H_1}, \mathcal{F}_{H_2}, \ldots \in \tilde{\mathfrak{R}}$. Then $\mathcal{F}_{H_j} \tilde{\sqcap} \mathcal{F}_C \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$, $\mathcal{F}_{H_j} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ $\forall$ j = 1,2, ... now, $(\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{H_j}) \tilde{\sqcap} \mathcal{F}_C = \widetilde{\bigcup_{j=1}^{\infty}} (\mathcal{F}_{H_j} \tilde{\sqcap} \mathcal{F}_C)$. Since $\mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C}), \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ are soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, then $(\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{H_j}) \tilde{\sqcap} \mathcal{F}_C = \widetilde{\bigcup_{j=1}^{\infty}} (\mathcal{F}_{H_j} \tilde{\sqcap} \mathcal{F}_C) \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$ and

$(\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{H_j}) \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$, hence by the definition of $\tilde{\mathfrak{R}}$, we get $(\widetilde{\bigcup_{j=1}^{\infty}} \mathcal{F}_{H_j}) \in \tilde{\mathfrak{R}}$.

Therefore, $\tilde{\mathfrak{R}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Let $\mathcal{F}_N \in \tilde{\mathcal{J}}$. Then $\mathcal{F}_N \tilde{\sqcap} \mathcal{F}_C \in \tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C$. Since $\mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C}), \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ are soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contains $\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C$ and $\tilde{\mathcal{J}}$ respectively, then $\mathcal{F}_N \tilde{\sqcap} \mathcal{F}_C \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\sqcap} \mathcal{F}_C})$ and $\mathcal{F}_N \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$. Hence $\mathcal{F}_N \in \tilde{\mathfrak{R}}$, thus $\tilde{\mathcal{J}} \subseteq \tilde{\mathfrak{R}}$. Consequentially, $\tilde{\mathfrak{R}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathfrak{R}}$ contain $\tilde{\mathcal{J}}$.

But $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is a smallest soft $\mathcal{P}^*-$field that contains $\tilde{\mathcal{J}}$, so we get $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \subset \tilde{\mathfrak{R}}$.

Let $\mathcal{F}_M \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\cap} \mathcal{F}_C$, then $\mathcal{F}_M = \mathcal{F}_Z \tilde{\cap} \mathcal{F}_C$ where $\mathcal{F}_Z \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$,hence $\mathcal{F}_Z \in \tilde{\mathfrak{R}}$, thus $\mathcal{F}_Z \tilde{\cap} \mathcal{F}_C \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\cap} \mathcal{F}_C})$, that is, $\mathcal{F}_M \in \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\cap} \mathcal{F}_C})$. so, $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\cap} \mathcal{F}_C \subseteq \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\cap} \mathcal{F}_C})$. Therefore, $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \tilde{\cap} \mathcal{F}_C = \mathcal{P}^*(\widetilde{\tilde{\mathcal{J}} \tilde{\cap} \mathcal{F}_C})$.

Proposition 2.8: If $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_C \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$ and $\widetilde{\mathcal{M}}_* = \{\mathcal{F}_M \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_M \tilde{\supset} \mathcal{F}_C \tilde{\cap} \mathcal{F}_N, \mathcal{F}_N \in \widetilde{\mathcal{M}} \}$. Then $\widetilde{\mathcal{M}}_*$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Proof: Since $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, then $\mathcal{F}_\Phi \in \widetilde{\mathcal{M}}$, now $\mathcal{F}_\Phi \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_C \tilde{\cap} \mathcal{F}_\Phi \tilde{\subseteq} \mathcal{F}_\Phi$, hence $\mathcal{F}_\Phi \in \widetilde{\mathcal{M}}_*$. Let $\mathcal{F}_{M_1}, \mathcal{F}_{M_2} \in \widetilde{\mathcal{M}}_*$. Then there is $\mathcal{F}_{N_1}, \mathcal{F}_{N_2} \in \widetilde{\mathcal{M}}$ such that $\mathcal{F}_{M_1} \tilde{\supset} \mathcal{F}_C \tilde{\cap} \mathcal{F}_{N_1}$ and $\mathcal{F}_{M_2} \tilde{\supset} \mathcal{F}_C \tilde{\cap} \mathcal{F}_{N_2}$, which implies

$\mathcal{F}_{M_1} \tilde{\cap} \mathcal{F}_{M_2} \tilde{\supset} (\mathcal{F}_C \tilde{\cap} \mathcal{F}_{N_1}) \tilde{\cap} (\mathcal{F}_C \tilde{\cap} \mathcal{F}_{N_2}) = \mathcal{F}_C \tilde{\cap} (\mathcal{F}_{N_1} \tilde{\cap} \mathcal{F}_{N_2})$. But $\mathcal{F}_{N_1}, \mathcal{F}_{N_2} \in \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Then $\mathcal{F}_{N_1} \widetilde{\cap} \mathcal{F}_{N_2} \in \widetilde{\mathcal{M}}$. Therefore $\mathcal{F}_{M_1} \tilde{\cap} \mathcal{F}_{M_2} \in \widetilde{\mathcal{M}}_*$. Let $\mathcal{F}_{M_1}, \mathcal{F}_{M_2}, \ldots \in \widetilde{\mathcal{M}}_*$. Then there is $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}, \ldots . \in \widetilde{\mathcal{M}}$ such that $\mathcal{F}_{M_i} \tilde{\supset} \mathcal{F}_C \tilde{\cap} \mathcal{F}_{N_i}$ $\forall$ i=1,2,… hence $\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{M_i} \tilde{\supset} \mathcal{F}_C \tilde{\cap} (\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{N_i})$. Since, $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}, \ldots \in \widetilde{\mathcal{M}}$, then Then $\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{N_i} \in \widetilde{\mathcal{M}}$. Therefore $\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{M_i} \in \widetilde{\mathcal{M}}_*$. So, we conclude that $\widetilde{\mathcal{M}}_*$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Proposition 2.9: If $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_C \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$ and $\widetilde{\mathcal{M}}_* = \widetilde{\mathcal{M}} \tilde{\cap} \mathcal{F}_C = \{\mathcal{F}_M \tilde{\subseteq} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_M = \mathcal{F}_N \tilde{\cap} \mathcal{F}_C, \mathcal{F}_N \in \widetilde{\mathcal{M}} \}$. Then $\widetilde{\mathcal{M}}_*$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Proof: Since $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$, then $\mathcal{F}_\Phi \in \widetilde{\mathcal{M}}$, now $\mathcal{F}_\Phi \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_\Phi = \mathcal{F}_\Phi \tilde{\cap} \mathcal{F}_C$, hence $\mathcal{F}_\Phi \in \widetilde{\mathcal{M}}_*$. Let $\mathcal{F}_{M_1}, \mathcal{F}_{M_2} \in \widetilde{\mathcal{M}}_*$. Then $\mathcal{F}_{M_1} = \mathcal{F}_{N_1} \tilde{\cap} \mathcal{F}_C$ and $\mathcal{F}_{M_2} = \mathcal{F}_{N_2} \tilde{\cap} \mathcal{F}_C$ where $\mathcal{F}_{N_1}, \mathcal{F}_{N_2} \in \widetilde{\mathcal{M}}$, this implies that $\mathcal{F}_{M_1} \tilde{\cap} \mathcal{F}_{M_2} = (\mathcal{F}_{N_1} \tilde{\cap} \mathcal{F}_C) \tilde{\cap} (\mathcal{F}_{N_2} \tilde{\cap} \mathcal{F}_C) = (\mathcal{F}_{N_1} \tilde{\cap} \mathcal{F}_{N_2}) \tilde{\cap} \mathcal{F}_C$. But $\mathcal{F}_{N_1}, \mathcal{F}_{N_2} \in \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Then $\mathcal{F}_{N_1} \widetilde{\cap} \mathcal{F}_{N_2} \in \widetilde{\mathcal{M}}$. Therefore $\mathcal{F}_{M_1} \tilde{\cap} \mathcal{F}_{M_2} \in \widetilde{\mathcal{M}}_*$.

Let $\mathcal{F}_{M_1}, \mathcal{F}_{M_2}, \ldots \in \widetilde{\mathcal{M}}_*$. Then there are $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}, \ldots . \in \widetilde{\mathcal{M}}$ such that

$\mathcal{F}_{M_i} = \mathcal{F}_{N_i} \tilde{\cap} \mathcal{F}_C$ $\forall$ i=1,2,… which means $\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{M_i} = (\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{N_i}) \tilde{\cap} \mathcal{F}_C$.

Since, $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}, \ldots \in \widetilde{\mathcal{M}}$, Then $\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{N_i} \in \widetilde{\mathcal{M}}$. Therefore $\widetilde{\bigcup_{i=1}^{\infty}} \mathcal{F}_{M_i} \in \widetilde{\mathcal{M}}_*$. This completes the proof.

Proposition 2.10: If $\tilde{\mathcal{J}}_1, \tilde{\mathcal{J}}_2 \subseteq \tilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$ such that $\tilde{\mathcal{J}}_1 \subset \tilde{\mathcal{J}}_2$, then $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_1)} \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_2)}$.

Proof: To prove $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_1)} \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_2)}$. We must show that $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_2)}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}_1$. Let $\widetilde{\mathcal{M}} \in \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_2)}$. Then $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}_2$ but $\tilde{\mathcal{J}}_1 \subset \tilde{\mathcal{J}}_2$ by hypothesis, thus $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ which include $\tilde{\mathcal{J}}_1$ which implies that $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_2)}$ is a soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}_1$. But $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_1)}$ is the smallest soft $\mathcal{P}^*-$field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\tilde{\mathcal{J}}_1$. Therefore, $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_1)} \subseteq \widetilde{\mathcal{P}^*(\tilde{\mathcal{J}}_2)}$.

Proposition 2.11: If $\widetilde{\mathcal{J}_1}, \widetilde{\mathcal{J}_2} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$&$\widetilde{\mathcal{J}_1} \subset \widetilde{\mathcal{J}_2} \subset \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})}$, then $\widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})} = \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})}$.

Proof: From **Proposition 2.10**, we have $\widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})} \subseteq \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})}$.

To prove $\widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})} \subseteq \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})}$. Let $\widetilde{\mathcal{M}} \epsilon \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})}$. Then $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\widetilde{\mathcal{J}_2}$ but $\widetilde{\mathcal{J}_1} \subset \widetilde{\mathcal{J}_2}$, thus $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contain $\widetilde{\mathcal{J}_1} \Longrightarrow \widetilde{\mathcal{M}} \epsilon\, \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})}$. so, $\widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})} \subseteq \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})}$. Therefore, $\widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})} = \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})}$.

Proposition 2.12: Let $\mathcal{U}$ be an uncountable set and a set of parameters $\mathcal{E}$ with respect to $\mathcal{U}$ such that $\mathcal{E}$ is uncountable.

Define a class $\widetilde{\mathcal{M}}$ as follows :

$\widetilde{\mathcal{M}} = \{\mathcal{F}_M \mathrel{\tilde{\subseteq}} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_M$ is a countable soft set or $\mathcal{F}_M^{\tilde{c}}$ is countable soft set$\}$. Then $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Proof: Clearly, $\mathcal{F}_\Phi \epsilon \widetilde{\mathcal{M}}$. Let $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2} \epsilon \widetilde{\mathcal{M}}$. Then either $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}$ or $\mathcal{F}_{\mathrm{M}_1}^{\tilde{c}}, \mathcal{F}_{\mathrm{M}_2}^{\tilde{c}}$ are countable soft sets. If each of $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}$ is a countable soft set, then $\mathcal{F}_{\mathrm{M}_1} \tilde{\cap} \mathcal{F}_{\mathrm{M}_2}$ is a countable soft set. If $\mathcal{F}_{\mathrm{M}_1}^{\tilde{c}}, \mathcal{F}_{\mathrm{M}_2}^{\tilde{c}}$ are countable soft set, then

$\mathcal{F}_{\mathrm{M}_1}^{\tilde{c}} \tilde{\cup} \mathcal{F}_{\mathrm{M}_2}^{\tilde{c}}$ is a countable soft set. Since $\mathcal{F}_{\mathrm{M}_1}^{\tilde{c}} \tilde{\cup} \mathcal{F}_{\mathrm{M}_2}^{\tilde{c}} = (\mathcal{F}_{\mathrm{M}_1} \tilde{\cap} \mathcal{F}_{\mathrm{M}_2})^{\tilde{c}}$.

Consequentially, either $\mathcal{F}_{\mathrm{M}_1} \tilde{\cap} \mathcal{F}_{\mathrm{M}_2}$ or $(\mathcal{F}_{\mathrm{M}_1} \tilde{\cap} \mathcal{F}_{\mathrm{M}_2})^{\tilde{c}}$ are a countable soft sets. Thus $\mathcal{F}_{\mathrm{M}_1} \tilde{\cap} \mathcal{F}_{\mathrm{M}_2} \epsilon \widetilde{\mathcal{M}}$. Let $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}, \ldots \epsilon \widetilde{\mathcal{M}}$. Then $\forall k$=1,2,… either $\mathcal{F}_{\mathrm{M}_\mathrm{k}}$ or $\mathcal{F}_{\mathrm{M}_\mathrm{k}}^{\tilde{c}}$ is countable set. If $\mathcal{F}_{\mathrm{M}_\mathrm{k}}, \forall k$=1,2,… is a countable soft set, then $\widetilde{\bigcup_{\mathrm{k}=1}^{\infty}}\ \mathcal{F}_{\mathrm{M}_\mathrm{k}}$ is a countable soft set. If $\mathcal{F}_{\mathrm{M}_\mathrm{k}}^{\tilde{c}}$ is countable soft set $\forall k$=1,2,…, then

$\widetilde{\bigcap_{\mathrm{k}=1}^{\infty}}\ \mathcal{F}_{\mathrm{M}_\mathrm{k}}^{\tilde{c}}$ is a countable soft set. From De - Morgan Laws, we get: $\widetilde{\bigcap_{\mathrm{k}=1}^{\infty}}\ \mathcal{F}_{\mathrm{M}_\mathrm{k}}^{\tilde{c}} = (\widetilde{\bigcup_{\mathrm{k}=1}^{\infty}}\ \mathcal{F}_{\mathrm{M}_\mathrm{k}})^{\tilde{c}}$. So $\widetilde{\bigcup_{\mathrm{k}=1}^{\infty}}\ \mathcal{F}_{\mathrm{M}_\mathrm{k}} \epsilon \widetilde{\mathcal{M}}$. Therefore, $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Proposition 2.13: Let $\mathcal{U}$ be an uncountable set and a set of parameters $\mathcal{E}$ with respect to $\mathcal{U}$ such that $\mathcal{E}$ is uncountable. Define a class

$\tilde{\mathcal{J}} = \{\mathcal{F}_{\mathcal{A}} = \{(x, \mathfrak{f}_{\mathcal{A}}(x) = \{u\}) : x \epsilon \mathcal{E}, \mathfrak{f}_{\mathcal{A}}(x) \epsilon \mathrm{P}(\mathcal{U})\}\}$. Then

$\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \subset [\mathcal{F}_\Phi \tilde{\cup} \{\mathcal{F}_M \mathrel{\tilde{\subseteq}} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_M$ or $\mathcal{F}_M^{\tilde{c}}$ is countable soft set$\}]$.

Proof: Assume $\widetilde{\mathcal{M}} = [\mathcal{F}_\Phi \tilde{\cup} \{\mathcal{F}_M \mathrel{\tilde{\subseteq}} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_M$ or $\mathcal{F}_M^{\tilde{c}}$ is countable soft set$\}]$. To prove $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} = \widetilde{\mathcal{M}}$, we must prove that $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{M}}$.

From Proposition 4.21, we have $\widetilde{\mathcal{M}}$ is a soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Let $\mathcal{F}_{\mathcal{A}} = \{(x, \mathfrak{f}_{\mathcal{A}}(x) = \{u\}) : x \epsilon \mathcal{E}, \mathfrak{f}_{\mathcal{A}}(x) \epsilon \mathrm{P}(\mathcal{U})\} \epsilon \tilde{\mathcal{J}}$. Then $\mathfrak{f}_{\mathcal{A}}(x) = \{u\}$ is a singleton set; hence it is a countable set. So, we get $\mathcal{F}_{\mathcal{A}}$ is a countable soft set. Thus $\mathcal{F}_{\mathcal{A}} \epsilon \widetilde{\mathcal{M}}$. Therefore $\tilde{\mathcal{J}} \subseteq \widetilde{\mathcal{M}}$. From Proposition 4.8: we get $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is the smallest soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contains $\tilde{\mathcal{J}}$. So, we have $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \subset \widetilde{\mathcal{M}}$.

Conclusion

The main results obtained are $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})}$ is the smallest soft $\mathcal{P}^*$– field on $\mathcal{F}_{\tilde{\mathcal{E}}}$ that contains $\tilde{\mathcal{J}}$. and If $\tilde{\mathcal{J}} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$ and $\mathcal{F}_{\mathrm{C}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. Then $\widetilde{\mathcal{P}^*(\tilde{\mathcal{J}})} \widetilde{\bigcap} \mathcal{F}_{\mathrm{C}} = \mathcal{P}^* \widetilde{(\tilde{\mathcal{J}} \widetilde{\bigcap} \mathcal{F}_{\mathrm{C}})}$. Moreover, If $\widetilde{\mathcal{J}_1}, \widetilde{\mathcal{J}_2} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$ & $\widetilde{\mathcal{J}_1} \subset \widetilde{\mathcal{J}_2} \subset \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})}$, then $\widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_1})} = \widetilde{\mathcal{P}^*(\widetilde{\mathcal{J}_2})}$.

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