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Associate graph of a commutative ring

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Abstract

In this study, a brand-new graph definition known Associate graph of a ring R denote $Ass(R)$ is present, where the graph's vertices stand in for R's elements s.t, any two vertices α and β merage by an edge if and only if $\alpha = r\beta$ and $\beta = s\alpha$. In this paper, we investigated some new property of $Ass(R)$ are studied., the complement of $Ass(R)$ is finally defined and a few of its characteristics are researched.

Subject Classification: *05C07, 05C09.*

Keywords: *Associate graph, Associate element, Commutative ring, Graph theory.*

1. Introduction

There is a lot of research relating algebraic ring and graph theories. Sangmin Chun and D.D.Anderson. Studied Associate elements in commutative

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rings [1]. the Prime Graph of a Ring was defined by Bhavanari S. et al. [3]. the clean graph of a ring was investigated by Mohammad Habibi et al.[7]. Nil clean graph of rings was stuided by Dhiren K.Basnet and Jayanta Bhattacharyya [4], while the Pseudo-VonNeumann Regular Graph of a Commutative Ring was investigated by Naeel E. Arif et al. [6], A graph is defined as an ordered pair ($V(G)$, $E(G)$), where $V(G)$ is a set of distinct vertices that are not empty and $E(G)$ is denote to a set of unordered pairs of those same vertices that may or may not be empty. Edges of the graph G are the elements of $E(G)$. An edge between two vertices α and β end vertices is represented by $\overline{\alpha\beta}$ [8]. In this paper, The term Associate graph of a commutative ring, abbreviated $Ass(R)$, and some of its features are introduced in this study.

Basic concepts :

Definition 1.1: [1] Let $a, b \; in \; R$, then a and b are said to be associates, denoted $(a{\sim}b)$, if there exist $r, s \; \in R$ with $ra = b \; and \; sb = a$.

Definition 1.2: [2] Assume that H be a graph; if the following condition holds the set S ⊆ V(H) is referred to as a dominant set; α ∈ V(H), then either α ∈ S or ∃ ℵ ∈ S such that α and ℵ are contiguous.

Definition 1.3: [8] By linking a path with two end vertices ,a cycle graph $C_n$was created with n vertices. The circle's vertex offsets are then each two degrees.

Definition 1.4: [5] empty graph is a graph of n isolated vertices with no edges.

2. Primary Results:

Definition 2.1: Make R a ring. Associate graph of ring R, indicated as *Ass (R)* is a graph $G(V, E)$ where $V(G) = R$ and $E(G) = \{ \overline{xy} / x = ry \; and y = sx \; and \; x \neq y \;, for \; any \; r, s \; \in R \}$, and Each vertex in *Ass* (*R*) has a neighboring 0 as well

Examples: let $R = Z_n$

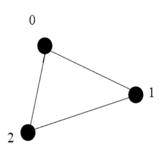

Figure 1
$(\boldsymbol{Ass(Z_3)})$

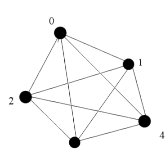

Figure 2
$(\boldsymbol{Ass(Z_5)})$

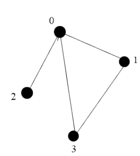

Figure 3
$(\boldsymbol{Ass(Z_4)})$

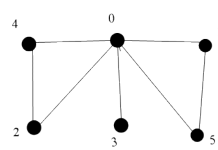

Figure 4
$(\boldsymbol{Ass(Z_6)})$

Remarks 2.2: Assume that *Ass(R)* be associate graph of *R* with *R* =Z_n .Then

1- Since A vertex 0 is adjacent to a residual vertices in *Ass* (*R*) then it is a connected simple graph, so degree (0) = $|R| - 1$.

2- The distance between any two vertices $\alpha \; and \; \beta$ that are not zero is $d(\alpha, \beta) \leq 2$.

3- Let *a,b* $\neq 0 \in R$ such that $a = sb$ and $b = ra$ (for any s and r in R), and observe that the subgraph formed by by *a*, *b* and *0* is K_3 graph, the graph *Ass(R)* where *R*= Z_6 as[fig 4] the subgraph that is created by {5,1,0} is K_3 graph.

Since p3 is a prime integer and R=Z_p, Ass(R) is K_p.

Theorem 2.4: *Let* $p \geq 3$ *is prime integer and R*= Z_p, *then Ass(R) is* K_p.

Proof: By definition of *Ass(R)*, 0 is adjacent to all residual vertices. Let *a* and *b* be any two nonzero vertex in *Ass(R)*. Since *R* is a field and every element in *R* has an inverse, then $\exists \; r, s \in R \; such \; that \; rs = 1 \; (\, r \; inverse \; s \,)$ now *a*= *rsa* and let *b*=*sa* then *a*=*rb* and *b*= *sa*, that is main every element in *R* are adjacent and *Ass(R)* is a complete graph.

Corollary 2.5: *Let* $R = Z_\eta, and \; \eta > 3$ *(η=2p , where p is a prime number), then the vertex* $\eta/2$ *adjacent only to the vertex 0 in Ass(R).*

Corollary 2.6: *Let p be a prime integr whree* $R = Z_p$, *then* $Ass(Z_p)$ *is a Hamiltongraph.*

Proof*:* By above theorem 2.4 $Ass(Z_p)$ is complete graph and so every complete graph is Hamilton graph then $Ass(Z_p)$ is Hamilton graph .

Theorem 2.7: *Let* $p \geq 3$ *is prime integer and R*= Z_p, *then Ass(R) has* $\binom{p}{\lambda}\frac{(\lambda-1)!}{2}$ *of length* λ *and the number of all cycles in Ass(R) is* $\sum_{\lambda=3}^{p}\binom{p}{\lambda}\frac{(\lambda-1)!}{2}$.

Proof: We proof *Ass(R)* has $\binom{p}{\lambda}\frac{(\lambda-1)!}{2}$ of length λ by induction it is true where λ=3 that is *Ass(R)* has one cycle where *p=3* that is $\binom{3}{3}\frac{(3-1)!}{2} = 1$ now for *p*=5 we have $\binom{5}{3}\frac{(3-1)!}{2} = 10$ that is for any *p* we have $\binom{p}{3}\frac{(3-1)!}{2}$ of cycle 3 in *Ass(R)*

Now we claim that $\binom{p}{\lambda-1}\frac{(\lambda-2)!}{2}$ is true for $\lambda - 1$ and proof it is true for $\lambda \leq p$

$$\binom{p}{\lambda}\frac{(\lambda-1)!}{2} = \frac{p!}{(\lambda)!\,(p-\lambda)!}\frac{(\lambda-1)(\lambda-2)!}{2}$$
$$= \frac{(\lambda-1)(p-\lambda+1)}{(\lambda)}\binom{p}{\lambda-1}\frac{(\lambda-2)!}{2}$$

That is for any *p*, *Ass(R) has* $\binom{p}{\lambda}\frac{(\lambda-1)!}{2}$ of length λ , that is the number of all cycles in *Ass(R)* equal to $\sum_{\lambda=3}^{p}\binom{p}{\lambda}\frac{(\lambda-1)!}{2}$.

3. Associate graph Invariants:

This section, related to some results to the invariances of graph theory are studied. The following theorem computes the perimeter of *Ass*(R).

3.1 *Girth of Ass (R)*

The duration of the shortest cycle in graph G determines the graph's girth. These outcomes pertain to girth ass(R).

Theorem 3.1.1: *Let* $n \geq 3$ *and R=* Z_n *, A Girth for a graph Ass(R) equal to 3.*

Proof: By definition of *Ass(R),* a vertex 0 is adjacent to all remaining vertices; that is, we have a shortest cycle with a length of 3 $\{ 0, \alpha , \beta \}$ for any nonzero vertices α andβ

3.2 *Dominating set of Ass* (*R*)

The set $\mathcal{S} \subseteq V(H)$ is known as the dominant set of *Ass* (R) if the condition is met; $a \in V(H)$ then either $\alpha \in \mathcal{S}$ or $\exists \aleph \in \mathcal{S}$ such that α and $\aleph$ are nearby. The calculations below demonstrate that for a finite commutative ring, the dominating number is 1, where the dominating number is the carnality of the smallest dominating set.

Theorem 3.2.1: *Let* $n \geq 3$ *and R=* Z_n *then adominating number for agraph Ass (*Z_n *) is 1.*

Proof: Since 0 is adjacent to all of the vertices in *Ass*(Z_n) graph , the smallest dominating set is {0} hence, 1 is the dominant number.

4. A complement for a graph *Ass(R).*

Definition 4.1: A graph $Ass^c(R)$ where *R* be a ring is said to be a complement for a graph *Ass(R)* such that $V(Ass^c(R)) = R$ and $E(Ass^c(\mathrm{R}))$={ $\overline{\alpha\beta}$/ $\nexists r$ and *s* in *R* s.t $\alpha = r\beta$ and $\beta = s\alpha$ and $\alpha \neq \beta$ }.

Note: Since 0 is adjacent to every vertex in $Ass\,(\mathrm{Z_n})$ Then $Ass^c(\mathrm{Z_n})$ is a disconnected graph and it has at least two components.

Theorem 4.2: *Let p be a prime integr then angraph* $Ass^c(Z_p)$ *is empty graph.*

Theorem 4.3: *Let n=2p*>3*, where p is a prime integer where R=* Z_n *then deg (*$\frac{n}{2}$*) = n-2 in* Ass^c*(R)*

Proof: By above theorem 2.4, a vertex $n/2$ adjacent only to 0 in $Ass\,(\mathrm{Z_n})$ then it most adjacent to all remaining vertices in $Ass^c(\mathrm{Z_n})$ except 0 that, is *deg* ($\frac{n}{2}$) $= [n-1] - 1 = n-2$.

5. Conclusion

An explanation of an associate graph *Ass*(R) and the number of cycles C_n in *Ass* (R) which was introduced in this study was discovered When $R = Z_p$. Additionally, a girth and the dominating number of *Ass*(R) are provided; along with a specification for the associate graph's complement, represented by $Ass^c(R)$.

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