

## Almost self-projective semimodules

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### Abstract

In this paper, the concept of almost self-projective semimodule has been introduced as a generalization of almost projective, and some characterization of this notion has been studied. A semimodule  $W$  is called almost self-projective if  $W$  is almost  $W$ -projective semimodule. Some properties and results of this concept have been obtained.

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### 1. Introduction

Numerous authors have developed some generalizations of projective modules. Particularly by Harada and his allies [1-4]. Since every module is a semimodule, these concepts can be converted to semimodules, considering the difference between the module and semimodule and trying to treat it. Huda studied Projective semimodule[5] and some projective semimodule generalizations [6]. Altaee and Alhossaini introduced  $\pi$ -projective semimodules[7]. Aljebory and Alhossaini introduced almost projective

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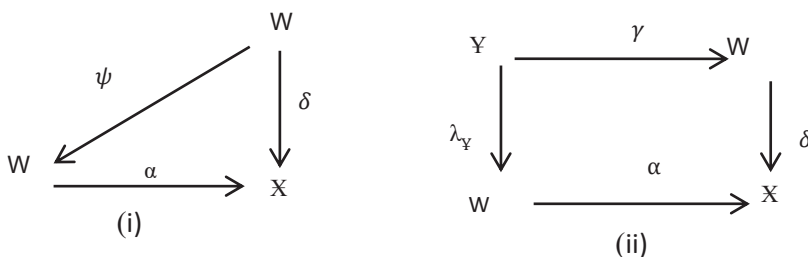
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semimodules[8]. In this paper, almost self-projective semimodules have been introduced as a generalization of almost projective, and some characterization of this concept has been introduced. A semimodule  $W$  is called almost self-projective if  $W$  is almost  $W$ -projective semimodule. Some properties of this concept have been presented.  $R$  will be a semiring with identity,  $W$  is a unitary  $R$ -semimodule. A map  $\phi: W \rightarrow W'$  is called homomorphism of  $R$ -semimodules if satisfies:  $\phi(b + b') = \phi(b) + \phi(b')$ ;  $\phi(rb) = r\phi(b)$  for all  $b, b' \in W$  and  $r \in R$ . The collection of all  $R$ -homomorphisms of  $W$  into  $W'$  is signified by  $\text{Hom}(W, W')$ , which is an  $R$ -semimodule. If  $\phi$  a homomorphism from  $R$ -semimodule  $W$  to itself  $\phi$  is called endomorphism of  $W$  and the collection of all endomorphism of  $W$ , indicated by  $\text{End}(W)$  can be expressed shortly,  $\dot{S} = \text{End}(W)$  which is a semiring. Hence it is an  $\dot{S}$ -semimodule and an  $R$ -semimodule [9].

## 2. Almost Self-Projective Semimodules

This section presented the notion of almost self-projective semimodules as well as investigated some properties of this concept. In this part, we will use  $\dot{S} = \text{End}(W)$

**Definition(2.1):** Let  $W$  and  $X$  be two semimodules.  $W$  is called almost self-projective if for each epimorphism  $\alpha: W \rightarrow X$ , and every homomorphism  $\delta: W \rightarrow X$ , either there is  $\psi: W \rightarrow W$  satisfying  $\alpha\psi = \delta$ , or there is  $\gamma: Y \rightarrow W$  where  $Y$  is a nonzero direct summand of  $W$  satisfying  $\delta\gamma = \alpha\lambda_Y$ . As the following diagrams:



### Remarks and Examples 2.2:

- (1) The trivial semimodule over any semiring is almost self-projective.
  - (2) A free semimodule over any semiring  $R$  is an almost self-projective.
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- (1) Every almost - projective semimodule and every projective semimodule is almost self- projective furthermore, any quasi projective semimodule is almost self-projective.
  - (2) whenever  $W$  is indecomposable almost self-projective semimodule and if for some  $X$ ,  $\alpha$  and  $\delta$ , the diagram (i) is not satisfied, then  $Y = W$  and  $\lambda_Y = 1_W$ , hence  $\delta\gamma = \alpha$  and  $\delta$  must be onto.

- (3) The fact in (2) can be rewritten in this way. Whenever  $W$  is indecomposable almost self-projective semimodule and for some  $X$ ,  $\alpha: W \rightarrow X$  is an epimorphism, but  $\delta: W \rightarrow X$  is not, then the diagram(i) must be satisfied and there is endomorphism  $\psi$  of  $W$  satisfying  $\alpha\psi = \delta$ .
- (4) The following implications hold for any semimodule:
- $$\text{Projective} \rightarrow \text{Almost-Projective} \rightarrow \text{Almost self-Projective}$$
- (5) Every simple semimodule is quasi-projective; hence it is almost-self projective.

**Proposition 2.3:** Suppose  $W \oplus W$  is almost a self-projective semimodule, then  $W$  is almost  $W \oplus W$ -projective semimodule.

**Proof:** Since  $W \oplus W$  is almost a self-projective semimodule, this means  $W \oplus W$  is almost  $W \oplus W$ -projective semimodule, and since  $W$  is a direct summand of  $W \oplus W$ , hence from Proposition(2.5, [8])  $W$  is almost  $W \oplus W$ -projective semimodule.

**Corollary 2.4:** For any integer  $n \geq 2$ , if  $W^n$  is almost a self-projective semimodule, then  $W$  is almost  $W^n$ -projective.

**Proposition 2.5:** Suppose that  $W$  is an indecomposable, almost self-projective semimodule,  $U, Y \leq W$ . If there is an epimorphism  $\alpha: W/U \rightarrow W/Y$ , then either  $U \subseteq Y$ , or there is  $\theta \in \dot{S}$  such that  $\theta(U) \subseteq Y$ .

**Proof:** Suppose  $W$  is almost a self-projective semimodule. Since  $W$  is almost self-projective, either there is  $\varphi: W \rightarrow W$  such that  $\alpha\pi_U\varphi = \pi_Y$  implies  $\alpha\pi_U\varphi(U) = 0$  and  $\pi_Y(U) = \frac{U+Y}{Y}$ , then  $\frac{U+Y}{Y} = 0$  implies  $U+Y=Y$ , then  $U \subseteq Y$ , or there is  $\theta: W \rightarrow W$  such that  $\pi_Y\theta = \alpha\pi_U1_W$ , then  $\pi_Y(\theta(U)) = \frac{\theta(U)+Y}{Y}$  and  $\alpha\pi_U1_W(U) = \alpha(0) = 0$  implies that  $\frac{\theta(U)+Y}{Y} = 0$ , then  $\theta(U) + Y = Y$  this means  $\theta(U) \subseteq Y$ .

**Proposition 2.6:** If  $W = X_1 \oplus X_2$  is almost a self-projective semimodule, where  $X_1$  and  $X_2$  are fully invariant sub-semimodules of  $W$ , then  $X_i$  is almost  $X_j$ -projective semimodule for  $i, j = \{1, 2\}$ .

**Proof:** Assume that  $W = X_1 \oplus X_2$  is almost a self-projective semimodule; hence  $W$  is almost a  $W$ -projective semimodule. Since  $X_i$  is a direct summand of  $W$ , then by Proposition(2.5, [8])  $X_i$  is almost  $W$ -projective semimodule and by Proposition (2.6, [8]), we get  $X_i$  is almost  $X_j$ -projective semimodule.

**Proposition 2.7:** If  $W = X_1 \oplus X_2$  is almost a self-projective semimodule, where  $X_1$  and  $X_2$  are fully invariant sub-semimodules of  $W$ ; hence  $W$  is almost  $X_j$ -projective semimodule for  $j = 1, 2$ .

**Proof:** Since  $W$  is almost a self-projective semimodule, therefore  $W$  is almost a  $W$ -projective semimodule by Proposition(2.6,[8])  $X_j$  is almost  $W$ -projective semimodule and from Proposition(2.6,[8])  $W$  is almost  $X_j$ -projective semimodule for  $j=1, 2$ .

**Proposition 2.8:** Suppose that  $W$  is an indecomposable, almost self-projective semimodule, for  $\alpha, \beta \in \dot{S}$  if  $\beta(W) \subseteq \alpha(W)$ , then  $\beta\dot{S} \subseteq \alpha\dot{S}$ .

**Proof:** Since  $\beta(W) \subseteq \alpha(W)$ ,  $\text{Hom}(W, \beta(W)) \subseteq \text{Hom}(W, \alpha(W))$  and since  $W$  is almost self-projective, then  $\beta\dot{S} = \text{Hom}(W, \beta(W))\theta \subseteq \text{Hom}(W, \alpha(W))\theta = \alpha\dot{S}$ , for some  $\theta \in \dot{S}$ , then  $\beta\dot{S} \subseteq \alpha\dot{S}$ .

**Proposition 2.9:** Suppose that  $W$  is almost a self-projective semimodule which is indecomposable and  $\beta, \alpha \in \dot{S}$ , then, If  $\alpha(W)$  is an image of  $\beta(W)$ , then  $\alpha\dot{S}$  is an image of  $\beta\dot{S}$ .

**Proof:** Let  $\vartheta: \beta(W) \rightarrow \alpha(W)$  be an epimorphism because  $W$  is almost self-projective semimodule, then either there is  $g \in \dot{S}$  such that  $\alpha g = \vartheta\beta$ , or there is  $\gamma \in \dot{S}$  such that  $\alpha = \vartheta\beta\gamma$ . Define  $\mu: \beta\dot{S} \rightarrow \alpha\dot{S}$  by  $\mu(\beta\varphi) = \alpha g\varphi$ ,  $\forall \varphi \in \dot{S}$ ,  $\mu$  is well defined since (if  $\beta\varphi = \beta\varphi'$ , then  $\alpha g\varphi(m) = \vartheta\beta\varphi(m) = \alpha\varphi(m) = \alpha\varphi'(m) = \vartheta\beta\varphi'(m) = \alpha g\varphi'(m)$  hence  $\mu$  is well defined. Since  $W$  is almost a self-projective semimodule, by Proposition(2.8)  $\vartheta(\beta(W)) \subseteq \alpha(W)$  implies  $\vartheta\beta\dot{S} \subseteq \alpha\dot{S} \Rightarrow \alpha g\dot{S} \subseteq \alpha\dot{S} \Rightarrow \mu(\beta\dot{S}) \subseteq \alpha\dot{S}$ ; hence  $\mu$  is an epimorphism.

A semimodule  $W$  is said to be principally a self-generator if, for each element  $b$  in  $W$ , there is epimorphism  $\beta: W \rightarrow Rb$ . [10]

**Proposition 2.10:** Let  $W$  be an indecomposable, almost self-projective semimodule,  $\beta \in \dot{S}$ , if  $\beta(W)$  is a simple sub-semimodule of  $W$ , then  $\beta\dot{S}$  is simple sub-semimodule of  $\dot{S}$ . Conversely is true if  $W$  is principally a self-generator semimodule.

**Proof:** Suppose that  $\beta(W)$  is a simple sub-semimodule of  $W$ ; if  $0 \neq \beta\alpha\dot{S} \neq \beta\dot{S}$  for some  $\alpha \in \dot{S}$ , then  $\beta\alpha(W)$  is nonzero proper sub-semimodule of  $\beta(W)$ , which is a contradiction with hypotheses, then  $\beta\dot{S}$  is  $\dot{S}$ -simple semimodule. Conversely, suppose  $W$  is principally self-generator semimodule, and  $\beta\dot{S}$  is simple. If  $0 \neq \beta(Rm) \neq \beta(W)$ , then for some  $\alpha \in \dot{S}$ ,  $\beta\alpha(W) = \beta(Rm) \neq \beta(W)$  hence by Proposition(2.8)  $0 \neq \beta\alpha\dot{S} \subseteq \beta\dot{S}$  that is a contradiction with the simplicity of  $\beta\dot{S}$ .

**Definition 2.11 [10]:** A sub-semimodule  $U$  is said to be an  $W$ -cyclic sub-semimodule of  $W$ , if it is isomorphic to  $W/Y$  for some sub-semimodule  $Y$  of  $W$ .

**Definition 2.12:** Let  $W$  and  $Z$  be two semimodules  $Z$  is said to be a  $W$ -mini-injective semimodule if for every homomorphism from simple  $W$ -cyclic sub-semimodule of  $W$  to  $Z$  can be expanded to  $W$ .

A semimodule  $W$  is said to be a mini-injective semimodule if it is a mini-injective for any semimodule  $Z$  and it is quasi mini-injective semimodule if it  $W$ -mini-injective.

**Proposition 2.13:** Let  $W$  be almost a self-projective semimodule if  $\text{Hom}(W, W')$  is mini-injective as right  $\dot{S}$ -semimodule; hence  $W'$  is  $W$ -mini-injective semimodule.

**Proof:** Let  $\alpha \in \dot{S}$  and  $\vartheta: \alpha(W) \rightarrow W'$  be homomorphism where  $\alpha(W)$  is a simple  $W$ -cyclic sub-semimodule of  $W$ , since  $W$  is almost self-projective semimodule,  $\alpha\dot{S}$  is simple, then  $\vartheta\alpha \in \text{Hom}(W, W')$ . Let  $\beta: \alpha\dot{S} \rightarrow \text{Hom}(W, W')$  be  $\dot{S}$ -homomorphism described by  $\beta(\alpha\gamma) = \vartheta\alpha\gamma, \gamma \in \dot{S}$ . Since  $\text{Hom}(W, W')$  is mini-injective, hence there is  $\beta': \dot{S} \rightarrow \text{Hom}(W, W')$  satisfying that  $\beta' i = \beta$ . Let  $f = \beta'(1)$ , then  $\vartheta\alpha = \beta\alpha = \beta'\alpha = \beta'(1)\alpha = f\alpha$ . Therefore  $W'$  is  $W$ -mini-injective semimodule. As the following diagrams in (c):

$$(c) \quad \begin{array}{ccc} \alpha\dot{S} & \xrightarrow{i} & \dot{S} \\ \beta \downarrow & \searrow \beta' & \\ \text{Hom}(W, W') & & \end{array} \quad \begin{array}{ccc} \alpha(W) & \xrightarrow{i} & W \\ \vartheta \downarrow & \searrow f & \\ W' & & \end{array}$$

**Proposition 2.14:** Suppose  $W$  is almost a self-projective semimodule which is a self-generator; if  $W'$  is  $W$ -mini-injective, hence  $\text{Hom}(W, W')$  is mini-injective as right  $\dot{S}$ -semimodule.

**Proof:** Let  $\alpha\dot{S}$  be a simple sub-semimodule of  $\dot{S}$ -semimodule  $\dot{S}$ . Let  $\beta: \alpha\dot{S} \rightarrow \text{Hom}(W, W')$  be  $\dot{S}$ -homomorphism and let  $\beta(\alpha) = \vartheta, \alpha \in \dot{S}$ , because  $W$  is self-generator,  $\ker \alpha = \sum_{\gamma \in I} \gamma(W)$  where  $I \subseteq \dot{S}$ . Pick any  $\gamma \in I$ , since  $\alpha\gamma = 0$ , then  $\beta(\alpha\gamma) = \beta(\alpha)\gamma = \vartheta\gamma = 0$  and then  $\text{Im } \gamma \subseteq \ker \vartheta$ . Thus, it follows  $\ker \alpha \subseteq \ker \vartheta$ , considers  $\alpha$  as an epimorphism  $\alpha': W \rightarrow \alpha(W)$ , then  $\ker \alpha' \subseteq \ker \alpha$ , therefore there is homomorphism  $\psi: \alpha(W) \rightarrow W$ , satisfying that  $\psi\alpha' = \vartheta$ , since  $W$  is almost a self-projective and self-generator semimodule by Proposition(2. 13), then  $\alpha(W)$

$$(d) \quad \begin{array}{ccccc} W & \xrightarrow{\alpha'} & \alpha(W) & \xrightarrow{i} & W \\ & \searrow \vartheta & & \searrow \vartheta & \\ & & W' & & \end{array} \quad \begin{array}{ccc} \alpha\dot{S} & \xrightarrow{i} & \dot{S} \\ \beta \downarrow & \searrow \beta' & \\ \text{Hom}(W, W') & & \end{array}$$

is simple subsemimodule of  $W$ . By hypothesis  $\psi$  can be expanded to homomorphism  $\phi: W \rightarrow W'$  satisfying  $\phi i \alpha' = \vartheta$  where  $i: \alpha(W) \rightarrow W$  and then  $\phi \alpha = \vartheta$ . Let  $\beta': \dot{S} \rightarrow \text{Hom}(W, W')$  defined by  $\beta'(\gamma) = \phi \gamma$ , then  $\beta'(\alpha) = \phi \alpha = \vartheta = \beta(\alpha)$ . Therefore  $\text{Hom}(W, W')$  is mini-injective.

**Proposition 2.15:** Suppose that  $W$  is an indecomposable almost self-projective semimodule and  $\alpha: W \rightarrow X$  be epimorphism, then there is  $\varphi \in \dot{S}$  such that  $\ker \alpha = \ker \alpha \varphi$ .

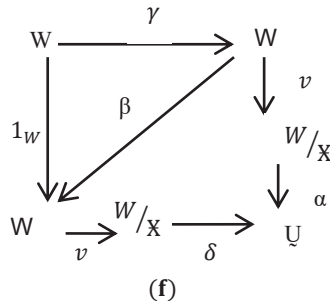
**Proof:** Since  $\alpha: W \rightarrow X$  epimorphism, the  $W/\ker \alpha \cong X$ . Consider the diagram (e):

$$\begin{array}{ccc}
 W & \xrightarrow{\gamma} & W \\
 1_W \downarrow & \nearrow \varphi & \downarrow \pi \\
 W & & W/\ker \alpha \\
 & \xrightarrow{\alpha} & \downarrow \delta \\
 & & X
 \end{array}
 \quad (e)$$

Let  $\delta: W/\ker \alpha \rightarrow X$  be an isomorphism and  $\pi: W \rightarrow W/\ker \alpha$  be the natural projection, because  $W$  is an indecomposable almost self-projective semimodule, then either there is  $\varphi: W \rightarrow W$  satisfying  $\alpha \varphi = \delta \pi$  or there is  $\gamma: W \rightarrow W$  such that  $\alpha 1_W = \delta \pi \gamma$ . The first case implies  $\ker \alpha \varphi = \ker \delta \pi = \pi^{-1} \ker \delta = \pi^{-1}(0) = \ker \alpha$ . In the second case,  $\ker \alpha = \ker \alpha 1_W = \ker \delta \pi \gamma$  can be consider  $\varphi = \gamma$ , then  $\ker \alpha \varphi = \ker \alpha$ .  $\square$

**Proposition 2.16:** Suppose that  $W$  is an indecomposable, almost self-projective semimodule and  $X, Y$  invariant sub-semimodules of  $W$ , then  $X \cap Y$  is fully invariant sub-semimodule of  $W$ .

**Proof:** Let  $\delta: W/X \rightarrow U$  be any epimorphism,  $\alpha: W/X \rightarrow U$  be homomorphism and  $v: W \rightarrow W/X$  be the natural projection. Now since  $W$  is almost self-projective semimodule, either there is  $\beta: W \rightarrow W$  such that  $\delta v \beta = \alpha v$  or there is  $\gamma: W \rightarrow W$  such that  $\delta v 1_W = \alpha v \gamma$ , since  $X \cap Y \subseteq \ker \delta v \cap \ker \alpha v$ , then  $\beta(X \cap Y) \subseteq \beta(X) \subseteq X$  and  $\beta(X \cap Y) \subseteq \beta(Y) \subseteq Y$ , then  $\beta(X \cap Y) \subseteq \beta(X) \cap \beta(Y)$ , also  $X \cap Y \subseteq \ker \delta v 1_W \cap \ker \alpha v$ , then  $\gamma(X \cap Y) \subseteq X \cap Y$ . This means  $X \cap Y$  is fully invariant subsemimodule of  $W$ . As the diagram (f).



A semimodule  $W$  is alleged to have the summand intersection property if the intersection of any two summands in  $W$  is again a direct summand in  $W$ . Shorty, SIP.[12]

**Proposition 2.17:** The direct sum of indecomposable semimodules is almost self-projective.

**Proof:** Similar to Proposition(2. 9, [8]).

**Proposition 2.18:** Suppose that  $W$  is an almost self-projective semimodule; if  $W$  has the SIP, then for any fully invariant direct summands  $X, Y$  of  $W$ ,  $X + Y$  is almost self-projective semimodule.

**Proof:** Let  $W$  be almost self-projective semimodule and has the SIP and let  $X, Y$  be any direct summands of  $W$ . Let  $W = (X \cap Y) \oplus U$  for some subsemimodule  $U$  of  $W$ . Then  $X = (X \cap Y) \oplus (X \cap U)$ ,  $Y = (X \cap Y) \oplus (Y \cap U)$ , then  $X + Y = (X \cap Y) \oplus (X \cap U) \oplus (Y \cap U)$ , by hypothesis  $(X \cap Y)$ ,  $(X \cap U)$  and  $(Y \cap U)$  are summands of  $W$  and so they are almost self-projective. Hence  $X + Y$  is almost self-projective semimodule as the direct sum of almost self-projective semimodules  $(X \cap Y)$ ,  $(X \cap U)$  and  $(Y \cap U)$ .

### 3. Conclusion

New generalizations of projective semimodules, namely almost self-projective semimodules, have been introduced. Some characterizations and results of this notion have been obtained in this paper. Some conditions have been added of semimodules to get desired results.

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