

The inclusion subsemimodule graph of a semimodule

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Abstract

Let R be an abelian semiring with unity, U be an R -semimodule. The inclusion subsemimodule graph of U , indicated $I_s(U)$, is a graph with nodes that all non-trivial subsemimodules of U and two different nodes N and L are adjacent if and only if $N \subset L$ or $L \subset N$. In this work, proved that if U is subtractive semimodule then $I_s(U)$ is not connected if is a direct sum of two simple R -semimodules. Besides, it has been proved that $I_s(U)$ is a complete graph if and only if U is a uniserial semimodule. girth, diameter, chromatic and clique numbers of $I_s(U)$ have been studied. and only if U .

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1. Introduction

In 2015, Akbari et. al. [1] defined the inclusion ideal ring graph. In 2020, the submodule inclusion graph of a module researched by Goswami in [9]. There are papers on transfer a graph to a module and topological space [12], [13]. Some papers transfer graphs to a semimodule have seemed lately, see [3], [4] and [11]. Inspired by preceding studies on the inclusion graph of algebraic constructions, in this paper, inclusion subsemimodule graphs of a semimodule. In this paper R is an abelian semiring with identity and U is a unitary left R -semimodule. A semiring R is definite as an algebraic structure $(R, +, \cdot)$ where $(R, +)$, $(R, \cdot)$ are comm. semigroups, joined by $a(b + c) = ab + ac$ for all $a, b, c \in R$, there exists $0, 1 \in R$ in which $d + 0 = d$, $d0 = 0d = 0$ for $d \in R$. We mean from a non-trivial subsemimodule of U that nonzero proper

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subsemimodule U . If $x, x + y \in N$, implies $y \in N$, then we say that N is a subtractive subsemimodule ($= k$ -subsemimodule) [2]. If each subsemimodule of U is subtractive, then U is named subtractive [10]. $M \subset U$ is named maximal in U if $M \not\subset Q$ for all $Q \subset U$. A semimodule U is named local if it has a single maximal subsemimodule. The intersection of all maximal subsemimodules U is named Jacobson radical of U , denoted $\text{Rad}(U)$. A semimodule U is Artinian if and only if every non-empty set of subsemimodules of U has a minimal element [10]. A semimodule U is named uniserial if every the subsemimodules U are linearly ordered by inclusion. If U is a direct sum of uniserial semimodules we say that U is serial. If $U \neq 0$ and it has no non-trivial subsemimodules then U is named simple. A semimodule U is named semisimple if the sum of its simple subsemimodules. The length of U is the length of all composition series of U , symbolized $l_R(U)$. References for graphs are [7]; for semirings see [8].

Let $V(G)$ and $E(G)$ symbolize the set of nodes and edges of a graph G , correspondingly. The induced subgraph G on $Y \subset V(G)$ indicated $\langle Y \rangle_G$. For two distinct nodes Q, P indicated by $Q - P$ (or $\{Q, P\} \in E(G)$) means that Q and P are adjacent. Any alternate sequence of edges and vertices is named a path in G . We indicate a trail with n nodes in G by P_n . A graph G is joined if a path among all two distinct nodes. A graph that is not joined is named disjointed graph. For two nodes S and D in G , the length of the minimal path from x to y indicated by $d(S, D)$. If no such trail exists, then we define $d(S, D) = \infty$. The diameter of G symbolized $\text{diam}(G)$ is definite by way of $\text{diam}(G) = \sup\{d(S, D) | S \text{ and } D \text{ two vertices of } G\}$. The graph with n nodes if every two nodes are adjacent, is named the complete graph of order n , symbolized by K_n . The smallest number of colors which can be assigned to the nodes of G such that all two adjacent nodes have dissimilar colors is named the chromatic number of G , symbolized by $\chi(G)$. A clique in G is a complete sub-graph. The number of nodes in the biggest clique of G , symbolized by $\omega(G)$, is named the clique number of G . Clearly, we have $\chi(G) \geq \omega(G)$. But, Beck [6] conjectured for a graph G , $\chi(G) = \omega(G)$. A cycle is a closed trail. is named The girth of G and is indicated by $\text{gr}(G)$, which is the length of the minimal cycle in G . $\text{gr}(G) = \infty$ if G has no cycle. A joined graph without any cycle is named a tree.

Section 2, it is established that if U is a subtractive semimodule, $I_S(U)$ is not connected if U is a direct sum of two simple R -semimodules. We characterize all semimodules for which $I_S(U)$ is not joined. Besides, the girth and the diameter of $I_S(U)$ are fixed. We revised some semimodules whose inclusion graphs are complete. In Section 3, we study the clique number of $I_S(U)$.

2. Some Basic Properties of $I_S(U)$

Definition 2.1: For an R -semimodule U , inclusion subsemimodule graph of U , $I_S(U)$, is an undirected graph whose nodes are all non-trivial subsemimodules of U and two distinct nodes N, L are adjacent if and only if $N \subset L$ or $L \subset N$.

Example 2.2: (1) If a semimodule U is simple, then $I_S(U)$ is an empty graph. (2) $I_S(\mathbb{Z}_4)$ contains a unique isolated vertex. (3) The graph $I_S(\mathbb{Z}_6)$ is as a disjoint union of two K_1 's.

For a semiring R , $I_S(R)$ is the ideal inclusion graph of R , where R is regarded as a left R -semimodule.

Theorem 2.3: *Let U be an R -semimodule. Then $I_S(U)$ is a complete graph if and only if U is a uniserial semimodule.*

Proof: Consider U is uniserial. Let N, L be a nontrivial subsemimodules of U . Then we have $N \subset L$ or $L \subset N$ which produces N and L are adjacent in $I_S(U)$ and we are done. Conversely, assume $I_S(U)$ is complete so any two subsemimodules of U are adjacent in $I_S(U)$. So, all subsemimodules of U are in one chain of inclusion.

Corollary 2.4: *Assume that U is a semimodule. If U is a serial semimodule, then $I_S(U)$ disjoint union of complete graphs.*

Proposition 2.5: Let U be a subtractive Artinian R -semimodule. Then $I_S(U)$ complete graph if and only if U has a unique minimal subsemimodule.

Proof: U has at least one minimal subsemimodule since U is Artinian. Also, all subsemimodule of U has a minimal subsemimodule. So, if U has a single minimal subsemimodule T , then T is contained in each non-trivial subsemimodule of U . Then $I_S(U)$ is complete. The converse is clear. $\square$

Proposition 2.6: Let U be a subtractive Noetherian R -semimodule. Then U has a unique maximal subsemimodule if and only if $I_S(U)$ is a complete graph.

A subtractive semimodule U is a direct sum of simple semimodules X^1, X^2 and indicated by $U = X^1 \oplus X^2$ iff $U = X^1 + X^2$ and $X^1 \cap X^2 = (0)$ [10]

Theorem 2.7: *Let U be a subtractive R -semimodule. Then the graph $I_S(U)$ is disjoint if and only if U is the direct sum of two simple R -semimodules.*

Proof: let $I_S(U)$ is disconnected. Assume Γ_1 and Γ_2 are two components of $I_S(U)$ such that $U_1 \in \Gamma_1$ and $U_2 \in \Gamma_2$ for a couple of non-trivial subsemimodules X and Y of U . Since there is no X, Y -path, X is not contained in Y , and Y is not contained in X . If $X \cap Y \neq (0)$, then there is an X, Y -path of the form $X - X \cap Y - Y$, a contradiction. Henceforth, assume $X \cap Y = (0)$. Now, if $U \neq X + Y$, then $X - X + Y - Y$ is trail from X to Y , an illogicality. Hence, we have $U = X + Y$. We show that X and Y are minimal subsemimodules of U . To see this, assume that Z be a subsemimodule of U that $(0) \neq Z \subseteq X$. If $Z \subset X$, so Z and X are adjacent vertices, which indicates that $Z \in \Gamma_1$. So there is no Z and Y -trail and by arguing as above, they are not comparable. If $U \neq Z + Y$, then there is a Z, Y -path of the form $Z - Z + Y - Y$, a contradiction. Otherwise, $U = Z +$

Y and by the semimodularity condition, since Z is a k -subsemimodule [5], we get $X = X \cap (Z + Y) = Z + X \cap Y = Z$, a contradiction. So X is a minimal subsemimodule of U . A similar argument show that Y is minimal in U . But, the minimality of X and Y implies that, they are simple R -semimodules and sine $U = X \oplus Y$, we are done according to [10].

Conversely, assume that U is a sum of two simple R -semimodules. Then any non-trivial subsemimodule U is minimal, so no couple of non-trivial subsemimodules are adjacent in $I_S(U)$. So $I_S(U)$ is a union of isolated nodes. $\square$

Corollary 2.8: *Let U be a subtractive semimodule. If $I_S(U)$ is not joined, then $I_S(U)$ is a null graph, and as any null graph is not joined, so the converse is correct.*

Theorem 2.9: *Assume that U be a subtractive R -semimodule with the connected graph $I_S(U)$. Then $\text{diam}(I_S(U)) \leq 3$.*

Proof: Assume that $I_S(U)$ is a joined graph. Then $U \neq X \oplus Y$ for any two simple R -semimodules X and Y . By the above argument in proof Theorem 2.7, for every two nontrivial subsemimodules N and L of U , we obtain an N and L -path $N - N + L - L$ or $N - N \cap L - L$ in $I_S(U)$. So $\text{diam}(I_S(U)) \leq 3$.

Proposition 2.10: *Let U be an R -semimodule. If $I_S(U)$ has a circuit of length 4 or 5, so $I_S(U)$ contains a triangle.*

Proof: let $C_1 : L_1 - L_2 - L_3 - L_4 - L_5 - L_1$ is a cycle of length 5 in $I_S(U)$. That means a chain $L_i \subset L_j \subset L_k$ in U , where $1 \leq i, j, k \leq 5$ and so $I_S(U)$ has a triangle. Again, let $C_2 : L_1 - L_2 - L_3 - L_4 - L_1$ be a cycle of length 4 in $I_S(U)$. Assume L_1 is not adjacent to L_3 and L_2 is not adjacent to L_4 . So, $L_2, L_4 \supset L_1 + L_3$ or $L_2, L_4 \subset L_1 \cap L_3$. So $L_1 \cap L_3, L_1 + L_3 \notin \{L_1, L_2, L_3, L_4\}$. So, $L_2 - L_1 - L_1 + L_3 - L_2$ or $L_2 - L_1 - L_1 \cap L_3 - L_2$ is a triangle in $I_S(U)$. $\square$

Lemma 2.11: [10, Theorem 3.10] *If U is a subtractive R -semimodule, then U is semisimple if and only if for every $K \subseteq U$, there exists $N \subseteq U$ such that $T = K + N, K \cap N = 0$.*

Corollary 2.12: *If U is a subtractive semisimple semimodule, then $\text{Rad}(U) = 0$.*

Lemma 2.13: [10, Th. 4.4] *For a subtractive semimodule U , U is semisimple and finitely generated iff U is Artinian and $\text{Rad}(U) = 0$. So, U is Artinian if U is finitely generated.*

Proposition 2.14: *Let U be a subtractive semimodule with $I_S(U)$ as a tree. Then the next hold:*

- (1) $I_S(U)$ caterpillar with $\text{diam}(I_S(U)) \leq 3$.
- (2) $\text{Rad}(U) \neq 0$.

(3) If U contains a single maximal subsemimodule, the graph $I_S(U)$ is a star.

Proof:

- (1) By Theorem 2.7.
- (2) First, we show that U is Artinian. If a chain of proper subsemimodules of U has at least three elements, then $I_S(U)$ has a triangle, an illogicality. So, U is Artinian. By contrary, assume $\text{Rad}(U) = 0$. By Lemmas 2.11 and 2.13, U is semisimple. By Th. 2.7, $I_S(U)$ is not joined. So $I_S(U)$ is not a tree, a illogicality. $\text{Rad}(U) \neq 0$.
- (3) If $K_1 \subset K_2 \subset K_3$ are non-trivial three subsemimodules of U , then $I_S(U)$ has a triangle, a conflict. So, one can suppose that every subsemimodule of U is maximal or minimal of U . If U has a single maximal subsemimodule then each subsemimodule of U will be adjacent to $\text{Rad}(U)$. Since $I_S(U)$ has no cycle, each subsemimodule ($\neq \text{Rad}(U)$) is minimal. $I_S(U)$ is a star.

Theorem 2.15: Let U be a subtractive R -semimodule. Then $\text{gr}(I_S(U)) \in \{3, 6, \infty\}$.

Proof: If there is K_1, K_2 and K_3 three non-trivial subsemimodules with $K_1 \subset K_2 \subset K_3$, then $I_S(U)$ has a triangle and thus $\text{gr}(I_S(U)) = 3$. Hence, let a non-trivial subsemimodule of U is maximal or minimal thus, U is Artinian. At this time, let us suppose $\text{Rad}(U) \neq 0$. By Corollary 2.12, U is not semisimple. By Theorem 2.7, $I_S(U)$ is joined. If U has a single maximal subsemimodule, then $\text{Rad}(U)$ is a unique maximal subsemimodule of U . So all other non-trivial subsemimodule of U is minimal. So, $I_S(U)$ is star and $\text{gr}(I_S(U)) = \infty$. If U has no single maximal subsemimodule, then $\text{Rad}(U)$ is a minimal subsemimodule of U . If $\text{Rad}(U)$ is the single minimal subsemimodule of U , then $I_S(U)$ is star and $\text{gr}(I_S(U)) = \infty$. So, assume $N \neq \text{Rad}(U)$ is a minimal subsemimodule of U . If F_1 and F_2 maximal subsemimodules of U containing N , then $F_1 = N + \text{Rad}(U) = F_2$, a conflict. So all minimal subsemimodule of U , except $\text{Rad}(U)$, is contained in a unique maximal subsemimodule of U . So $I_S(U)$ is tree, $\text{gr}(I_S(U)) = \infty$. Also, if there is a cycle of length 4 or 5 in $I_S(U)$, then by Proposition 2.10, $I_S(U)$ contains a triangle, then $\text{gr}(I_S(U)) = 3$. Also, if $I_S(U)$ has a circuit of length ≥ 6 , then $\text{gr}(I_S(U)) = 6$.

3. Chromatic Number and Clique Number of $I_S(U)$

Proposition 3.1: If U has a finite number of subsemimodules then $\chi(I_S(U)) = \omega(I_S(U))$.

Proof: show $I_S(U)$ and its complement graph has not induced an odd cyclic length of at least 5. Assume in contrary $N_1 - N_2 - \dots - N_{2n+1} - N_1$ is an induced subgraph of $I_S(U)$ and $n \geq 2$. There are N_i, N_j and N_k such that $N_i \subset N_j \subset N_k$, where $1 \leq i, j, k \leq 2n + 1$. So $\langle N_i, N_j, N_k \rangle_G$ Forms a cycle, an illogicality. Assume, now $H = \langle N_1, N_2, N_{2n+1} \rangle_G$ is a complement of the

above cycle. Take $N_1 - N_{n+1} - N_{2n+1} - N_n \cdots - N_2 - N_{n+2} - N_1$, as a subgraph of H . Similarly, there are N_i, N_j and N_k such that $N_i \subset N_j \subset N_k$, where $1 \leq i, j, k \leq 2n + 1$, $(j \equiv n + i) \bmod 2n + 1$ and $(k \equiv n + j \equiv i - 1) \bmod 2n + 1$. So, $\langle N_i, N_j, N_k \rangle_G$ is a triangle, $N_i - N_k$, a contradiction.

Remark 3.2: If U is a subtractive semimodule, then $\omega(I_s(U)) = l_R(U)$.

Remark 3.3: $\omega(I_s(U)) = 1$ if and only if $|I_s(U)| = 1$ or $|I_s(U)| \geq 2$ and $U = X^1 \oplus X^2$, X^1, X^2 simple semimodules. $\omega(I_s(U)) < \infty \rightarrow l_R(U) < \infty$

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