

Semi-essentially prime modules

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Abstract

Let R be a commutative ring with 1 and M be a (left) unitary R – module. This essay gives generalizations for the notions prime module and some concepts related to it. We termed an R – module M as semi-essentially prime if $\text{ann}_R(M) = \text{ann}_R(N)$ for every non-zero semi-essential submodules N of M . Given some of their advantages characterizations and examples, and we study the relation between these and some classes of modules.

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1. Introduction

Let R be a commutative ring with 1 and M be a (left) unitary R – module. A non-zero submodule K is termed an essential submodule of M if $K \cap L \neq 0$ for every non-zero submodule L of M [1]. In [2] Ali S. Mijbass and Nada K. Abdullah introduced and studied the notion of semi-essential submodules, where a non-zero submodule N of an R – module M is termed semi-essential submodule ($N \leq_{s.e} M$) if $N \cap P \neq 0$ for every non-zero prime submodule P of M . And a submodule P of M is termed prime if P is a proper and whenever $rx \in P$ for every $r \in R, x \in M$ up to either $x \in P$ or $r \in [P:M]$, where $[P:M] = \{r \in R : rM \subseteq P\}$ [3].

An R – module M is termed an R – module M is semi-essentially prime if $\text{ann}_R(M) = \text{ann}_R(N)$ for every non-zero semi-essential submodules N of

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M . This essay introduced and studied the notion of the semi-essentially prime module as a generalization of prime *module*, and given some of their advantages characterizations and examples. Also, we view that below definite terms semi-essentially prime, semi-uniform, essentially prime, and prime modules are equivalent. In this work we define semi-essentially prime, an R – module M is termed a semi-essentially prime module if $\text{ann}_R(M) = \text{ann}_R(N)$ for every non-zero semi-essential submodules N of M . Moreover studied the relations between semi-essentially prime *modules* and some of *modules*.

2. Semi-Essentially Prime Modules

An R – module M is termed *primemodule* if $\text{ann}_R(M) = \text{ann}_R(N)$, for every non-zero submodules N of M [4].

Definition 2.1: An R – module M is termed a semi-essentially prime module if $\text{ann}_R(M) = \text{ann}_R(N)$, for every non-zero semi-essentially submodules N of M .

Remarks and Examples 2.2:

1. Every prime *module* is semi-essentially prime *module*, conversely is not true in general for example Z_6 as Z – module is semi-essentially prime *module* because Z_6 is the only semi-essential submodule of Z_6 , but not prime.
2. Z and Q as Z – module is semi-essentially prime modules, because $\text{ann}_R(nZ) = \text{ann}_R(Z)$, for all $nZ \leq_{s.e} Z$, $n \in Z^+$. Similarly for Q .
3. Every semi-simple R – module is semi-essentially prime *module* but not conversely because Z as Z – module is a semi-essentially prime *module* but not simple.
4. Z_4 as Z – module is not semi-essentially prime *module*. (Because $\text{ann}_R(Z_2) \neq \text{ann}_R(\langle \bar{2} \rangle)$; $\langle \bar{2} \rangle \leq_{s.e} Z_4$).
5. A homomorphic image of a semi-essentially prime *module* need not be semi-essentially prime in general for example Z as Z – module is a semi-essentially compressible module and $\frac{Z}{8Z} \simeq Z_8$ is not a semi-essentially compressible module, by (4).
6. If M is a simple module, then M is a semi-essentially prime module because M is the only semi-essential submodule of M .
7. The direct sum of a semi-essentially prime *module* need not be semi-essentially prime. Consider the following example let $Z_4 \simeq Z_2 \oplus Z_2$ as Z – module. Z_2 is semi-essentially prime see remarks and examples (2.2)(1), but Z_4 is not semi-essentially prime see remarks and examples (2.2)(4).

Proposition 2.3: A semi-essential submodule of a semi- essentially prime *module* is also semi- essentially prime *module*.

Proof: Let $0 \neq K \leq_{s.e} M$ and M be a semi-essentially prime module. Let $0 \neq L \leq_{s.e} K \leq_{s.e} M$, by [5] $L \leq_{s.e} M$. Because M is semi-essentially prime module, then $\text{ann}_R(N) = \text{ann}_R(M) = \text{ann}_R(K)$. Therefore N is a semi-essentially prime module.

An R -module M is termed a semi-essentially compressible module if M can be embedded in every of it is a nonzero semi-essentially submodule. Equivalently, M is semi-essentially compressible if found a monomorphism $f: M \rightarrow N$ whenever N is a non-zero semi-essential submodule of $M[6]$.

Proposition 2.4: If M is a semi-essentially compressible R -module, then M is semi-essentially prime module

Proof: Let $0 \neq N \leq_{s.e} M$. We have to show that $\text{ann}_R(M) = \text{ann}_R(N)$. Let $r \in \text{ann}_R(N)$, then $rN = 0$, but M is a semi-essentially compressible R -module, $f: M \rightarrow N$ is a monomorphism, then $f(rM) = rf(M) \subseteq rN = 0$, so $rM = 0$, thus $r \in \text{ann}_R(M)$. Therefore $\text{ann}_R(M) = \text{ann}_R(N)$ and is semi-essentially prime module

Comparison with the proposition (2.15) [7]

Proposition 2.5: Let $M = M_1 \oplus M_2$ be an R -module such that $\text{ann}_R M_1 \oplus \text{ann}_R M_2 = R$. If M_1 and M_2 are semi-essentially compressible modules, then M is semi-essentially module

Proof: By proposition (2.15) [7] and above proposition, we have done.

An R -module M is called prime if, $\text{ann}_R(M) = \text{ann}_R(N)$ for each nonzero submodule N of M [8]

Remark 2.6: Every semi-essentially prime module is essentially prime module, conversely is not true in general

Proposition 2.7: Let M be a fully prime R -module, then M is a semi-essentially prime module if and only if M is an essentially prime module, where an R -module M is termed fully prime if every proper submodule of M is a prime submodule[9].

Proof: Since M is fully prime, then by[7] every semi-essential submodule is an essential submodule, hence every essential prime module is semi-essentially prime. Conversely, view remark (2.5).

The recollection that a non-zero R -module M is termed semi-uniform if every non-zero submodule of an R -module is semi-essential [2].

Proposition 2.8: If M is a semi-uniform R -module, then M is prime if and only if M is semi-essentially prime.

Proof: Suppose that M is semi-essentially prime *module*. Let N be a non-zero submodule of M because M is semi-uniform, then N is a semi-essential submodule in M . But M is semi-essentially prime *module*, then $\text{ann}_R(M) = \text{ann}_R(N)$ for all $0 \neq N \leq_{s.e} M$. Thus M is prime *module*. Conversely, it is obvious, from remarks and examples (2.2) (1).

Proposition 2.9: Let M_1 and M_2 be isomorphic R – *modules*. Then M_1 is a semi-essentially prime if and only if M_2 is a semi-essentially prime.

Proof: Suppose that M_1 is a semi-essentially prime *module* and let $0 \neq L \leq_{s.e} M_2$. We want to show that $\text{ann}_R(M_2) = \text{ann}_R(L)$ for all $0 \neq L \leq_{s.e} M$. By [7] $f^{-1}(L) \leq_{s.e} M_1$ and M_1 is a semi-essentially prime *module*, then $\text{ann}_R(M_1) = \text{ann}_R(f^{-1}(L))$. Let $r' \in \text{ann}_R(L)$, then $r'L = 0$, thus $f^{-1}(r'L) = 0$, then $f^{-1}(r')f^{-1}(L) = 0$, so $f^{-1}(r') \in \text{ann}_R(f^{-1}(L)) = \text{ann}_R(M_1)$, thus $f^{-1}(r')M_1 = 0$, then $r'M_2 = 0$. Therefore $\text{ann}_R(M_2) = \text{ann}_R(L)$ and M_2 is a semi-essentially prime *module*.

The recollection that an R – *module* M subisomorphic to an R – *module* M' if there exist R – monomorphism $\phi: M \rightarrow M'$ and $\theta: M' \rightarrow M$, in this case, we say the R – *module* M and M' are subisomorphic [10].

Proposition 2.10: The following statements are equivalent for R – *module* M :

1. M is a semi-essentially compressible *module*.
2. M is a semi-essentially prime *module*.
3. M is a subisomorphic to a semi-essentially prime *module*
4. M contains a semi-essentially prime submodule N such that $\exists$ a monomorphism $f: M \rightarrow N$.

Proof: (1) $\Rightarrow$ (2) It is clear by Proposition (2.4).

(2) $\Rightarrow$ (3) It is clear by Proposition (2.9).

(3) $\Rightarrow$ (4) Suppose that M is subisomorphic to a semi-essentially prime *module* C and found monomorphisms $\psi: M \rightarrow C$ and $\phi: C \rightarrow M$. Let $K = \psi(C) \leq_{s.e} M$, then by Proposition (2.3) K is a semi-essentially prime *submodule* of M , since by (3) $\exists$ a monomorphism $f: M \rightarrow K$.

(4) $\Rightarrow$ (1) Let $0 \neq L$ be any semi-essential *submodule* of R – *module* M , then $L \cap N \leq N$, so by (4) $\exists$ monomorphisms $f: N \rightarrow L \cap N$ and $g: M \rightarrow N$ also $i: L \cap N \rightarrow L$ is the inclusion monomorphism, thus $i \circ f \circ g: M \rightarrow L$ is a monomorphism. Therefore M is a semi-essentially compressible *module*.

Definition 2.11: An R – *module* M is termed semi-essentially uniform if every nonzero semi-essential *submodule* of M is essential in M .

Remark 2.12: Every uniform is semi-essentially uniform, but the converse need not be true in general for example Z_6 as Z – *module* is semi-essentially uniform *module*, but not uniform.

Proposition 2.13: If M is a semi-essentially compressible R – *module* , then M is a semi-essentially uniform *module*.

Proof: Let $0 \neq N \leq_{s,e} M$ and $0 \neq x \in N \leq M$, then by Proposition (2.5)[6], $\exists f: M \rightarrow R_x$ by $f(x) = rx \in N$, $\forall x \in M$, therefore N is an essential *submodule* in M .

Proposition 2.14: Let M be an R – *module*. M is a semi-essentially compressible *module* if and only if there exists a monomorphism $\varphi \in \text{End}_R(M)$ such that $\text{Im}\varphi \subseteq N$ for each a nonzero semi-essential *submodule* N of M .

Proof: Suppose that M is a semi-essentially compressible *module*. Let $0 \neq N \leq_{s,e} M$, then $f: M \rightarrow N$ is a monomorphism, so there exists $\varphi = i \circ f \in \text{End}_R(M)$ where $i: N \rightarrow M$ is the inclusion monomorphism and $\text{Im}\varphi = i \circ f(M) = f(M) \subseteq N$. Conversely, let $0 \neq N \leq_{s,e} M$, by assumption there exists a monomorphism $\varphi \in \text{End}_R(M)$ and $\varphi(M) \subseteq N$. Thus $\varphi: M \rightarrow N$ is a monomorphism. Therefore M is a semi-essentially compressible *dule* .

Corollary 2.15: Let M be a semi-essentially compressible R – *module*, then a homomorphic image is a subisomorphic to a semi-essentially compressible *submodule* N of M .

Proof: Since M is a semi-essentially compressible R – *module*, then by above proposition $f(M) \subseteq N$ for each nonzero semi-essential *submodule* N of M and by proposition (2.7)[6]. Then N is a semi-essentially compressible *submodule* of M . $i: f(M) \rightarrow N$, $j: N \rightarrow M$ are the inclusion monomorphisms and $f: M \rightarrow f(M)$, thus $i: f(M) \rightarrow N$ and $f \circ j: N \rightarrow f(M)$ are monomorphism. Therefore $f(M)$ is a subisomorphic to a semi-essentially compressible *submodule* N of M .

Lemma: Let M be a semi-essentially prime R – *module* , then $\text{ann}_R(N)$ is a prime ideal of R for each nonzero semi-essential *submodule* N of M .

Proof: Let $a, b \in R$ such that $.b \in \text{ann}_R(N)$, then $abN = 0$. Suppose $bN \neq 0$. But $0 \neq N \leq_{s,e} M$ and $bN \subseteq N$, then $bN \leq_{s,e} M$, since M is a semi-essentially prime and $a \in \text{ann}_R(N)$ and hence $\text{ann}_R(N)$ is a prime ideal of R .

3. Conclusion

The present work provides new information. A semi-essentially compressible module is semi-essentially prime. It is subisomorphic to a semi-essentially prime module. Also if a module is semi-essentially prime then the annihilator of any nonzero semi-essential submodule is a prime ideal of a ring R .

References

- [1] Muna, A. Ahmed, "Weak Essential Submodules", University of Baghdad, *Ibn Al-Haitham J. for Pure & Appl. Um-Salama Science Journal* Vol. 6(1) (2009).
- [2] Ali. S. Mijbass and Nada. K. Abdullah, Semi-Essentially Submodule and Semi-Uniform Modules. *J. of Kirkuk University-Scientific studies*; 4 (1), 48-58 (2009).
- [3] Muna, A. Ahmed, "Some Results on Weak Essential Submodules", University of Baghdad, *Ibn Al-Haitham J. for Pure & Appl. Baghdad Science Journal* Vol.13 (3) (2016).
- [4] Haithab A.Shahad and Nuhad S.Al-Muthafar, "On essential-prime submodules" *Journal of Discrete Mathematical Sciences and Cryptography*, Dol:10.1080/09720529.2021.1968572
- [5] Israa. H. Muslem, "Some types of Retractable and Compressible Modules" M.Sc.thesis. College of Education for pure Science, University of Baghdad (2016).
- [6] Mohammed B. H. Alhakeem and Nuhad S. Al-Mothafar, "Semi-Essentially Compressible Modules and Semi-Essentially Retractable Modules" *Iraqi Journal of Science*, Too appear.
- [7] Muna, A. Ahmed and maysaa R. Abbas, "On Semi-Essentially Submodules", University of Baghdad, *Ibn Al-Haitham J. for Pure & Appl. Sci*; Vol. 28 (1) (2015).
- [8] Inaam Mohammed Ali and Tha'ar Younis Ghawi, "Essentially Prime Modules and Related Modules" *Journal of Basrah Researches ((Sciences))* Volume 37. Number 4. C (2011).
- [9] R. L. McCasland and P. F. Smith, "Prime Submodules of Noetherian Modules", *Rocky Mt. Math.* 23, 1041-1062 (1993).
- [10] Omar M Al-Ragab & Nuhad S. Al-Mothafar, Radical semiprime submodules, *Journal of Discrete Mathematical Sciences and Cryptography*, 24:7, 1895-1899 (2021), DOI: 10.1080/09720529.2021.1957180.

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